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Egoroff, σ, and convergence properties in some archimedean vector lattices

by

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Abstract. An archimedean vector lattice A might have the following properties:

(1) the sigma property (σ): For each $\{a_n\}_{n\in\mathbb{N}} \subseteq A^+$ there are $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ and $a \in A$ with $\lambda_n a_n \le a$ for each n;

(2) order convergence and relative uniform convergence are equivalent, denoted (OC $\Rightarrow$ RUC): if $a_n \downarrow 0$ then $a_n \to 0$ r.u.

The conjunction of these two is called *strongly Egoroff*.

We consider vector lattices of the form $D(X)$ (all extended real continuous functions on the compact space X) showing that (σ) and (OC $\Rightarrow$ RUC) are equivalent, and equivalent to this property of X: (E) the intersection of any sequence of dense cozero-sets contains another. (In case X is zero-dimensional, (E) holds iff the clopen algebra $\operatorname{clop} X$ of X is a 'Egoroff Boolean algebra'.)

A crucial part of the proof is this theorem about any compact X: For any countable intersection of dense cozero-sets U, there is $u_n \downarrow 0$ in $C(X)$ with $\{x \in X : u_n(x) \downarrow 0\} = U$. Then, we make a construction of many new X with (E) (thus, dually, strongly Egoroff $D(X)$), which can be F-spaces, connected, or zero-dimensional, depending on the input to the construction. This results in many new Egoroff Boolean algebras which are also weakly countably complete.

1. Preliminaries.

We list the numerous relevant definitions, with some commentary.

All vector lattices (Riesz spaces) will be archimedean (see [16]) and all topological spaces will be Tychonoff ([6], [9]).

Let A be a vector lattice.

In A, for a (countable) sequence $(u_n)_{n\in\mathbb{N}}$ in A: $u_n \downarrow 0$ means $u_n\!\downarrow$, i.e., $u_1 \ge u_2 \ge \cdots$, and $\bigwedge^A u_n = 0$ ($\bigwedge^A$ is the infimum in A);

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$u_n \to 0$ r.u. (relatively uniformly) means there is $q \in A$ with $u_n \to 0$ (q), which means that for every $\varepsilon > 0$ there exists $n(\varepsilon)$ for which $n \geq n(\varepsilon) \Rightarrow |u_n| \leq \varepsilon q$. (This 'r.u. convergence' was introduced for $\mathbb{R}^{\mathbb{R}}$ in [20].)

(σ) and (OC $\Rightarrow$ RUC) are defined in the Abstract. The origins of these conditions are discussed in [16, §16 and Chap. 10]. There, (OC $\Rightarrow$ RUC) is also called 'order convergence is stable' and 'order convergence and r.u. convergence are equivalent'. (We add: (σ) in $\mathbb{R}^{\mathbb{R}}$ seems to have been introduced in the remarkable [20].)

A is called *strongly Egoroff* (s.E.) if in A a certain double sequence condition holds, and [16] shows (archimedean) A is s.E. iff A has (σ) and (OC $\Rightarrow$ RUC). We use this as the definition of s.E. ('Egoroff' is another double sequence condition, which we need not mention.)

Now let X be a topological space (usually compact). Much of the following is explained in [9] and [6].

$C(X)$ is the vector lattice (and ring) $\{f \in \mathbb{R}^X : f \text{ continuous}\}$. For $f \in C(X)$ the *cozero-set* of f is

$$\operatorname{coz} f = \{x \in X : f(x) \neq 0\}$$

(and $\mathrm{Z}f = X \setminus \operatorname{coz} f = \{x \in X : f(x) = 0\}$). Moreover,

$$\operatorname{coz} X = \{\operatorname{coz} f : f \in C(X)\}, \qquad \operatorname{dcoz} X = \{S \in \operatorname{coz} X : S \text{ dense}\}.$$

Generally, for X any set and $\mathcal{A} \subseteq \mathcal{P}(X)$, the power set of X,

$$\mathcal{A}_\delta = \Big\{ \bigcap_{n \in \mathbb{N}} A_n : (\forall n \in \mathbb{N})(A_n \in \mathcal{A}) \Big\}.$$

Thus we have $\operatorname{dcoz}_\delta X$ (which we write for $(\operatorname{dcoz} X)_\delta$).

Various properties of an X will be involved.

X is called (or has the property)

- F if each $S \in \operatorname{coz} X$ is C^*-embedded;
- QF (quasi-F) if each $S \in \operatorname{dcoz} X$ is C^*-embedded;
- *almost* P if $\operatorname{dcoz} X = \{X\}$, equivalently, each nonempty G_δ has nonempty interior;
- P if each G_δ is open;
- BD (*basically disconnected*) if each $S \in \operatorname{coz} X$ has $\overline{S}$ open;
- ED (*extremally disconnected*) if each open S has $\overline{S}$ open;
- ZD (*zero-dimensional*) if $\operatorname{clop} X$ is a base for the topology;
- ccc (*countable chain condition*) if each pairwise disjoint family of nonempty open sets is countable (or finite);
- (E) if $\operatorname{dcoz} X$ is co-initial in $\operatorname{dcoz}_\delta$, in the inclusion order (i.e., for any $S_1, S_2, \ldots \in \operatorname{dcoz} X$, there exists $S_0 \in \operatorname{dcoz} X$ with $S_0 \subseteq \bigcap_{n \in \mathbb{N}} S_n$).

These properties are related as follows:

$$\begin{tikzcd}
 & \mathrm{ZD} & \mathrm{ED} \arrow[l] \arrow[d] & \\
(E) & \mathrm{P} \arrow[l] \arrow[u] \arrow[r] \arrow[d, shift left] & \mathrm{BD} \arrow[ul] \arrow[r] & \mathrm{F} \arrow[ul, "+\,\mathrm{ccc}"'] \arrow[dl] \\
 & \text{almost P} \arrow[u, shift left, "+\,\mathrm{BD}"] \arrow[r] & \mathrm{QF} &
\end{tikzcd}$$

And (almost P) $\cap$ BD = P [2], BD $\cap$ ccc $\subseteq$ ED [22]; X is QF (resp. BD, ED) iff the Čech–Stone compactification βX is [5, 9]; X almost P $\Rightarrow \beta X$ is QF (simply observe that for every $S \in \operatorname{dcoz} \beta X$ we have $X \subseteq S$).

In case X is compact ZD, the Boolean algebra $\operatorname{clop} X$ is a base, and X is ED (resp. BD, F, (E)) iff $\operatorname{clop} X$ *qua* Boolean algebra (BA) is complete (resp. σ-complete, weakly countably complete, Egoroff).

Important examples of Egoroff BAs are those BAs associated with the M/N mentioned in the second paragraph below, and any Maharam algebra. See [22, 8] and §9 below.

Now let $D(X)$ be the set

$$\{f \in C(X, [-\infty, +\infty]) : f^{-1}(-\infty, +\infty) \text{ dense}\}.$$

(Here $(-\infty, +\infty)$ is the reals $\mathbb{R}$, $[-\infty, +\infty]$ its two-point compactification $\mathbb{R} \cup \{\pm\infty\}$ with the obvious order. $D(X)$ is denoted $C^\infty(X)$ sometimes.)

In the pointwise order $D(X)$ is a lattice, and is closed under scalar multiplication. In $D(X)$, $f + g = h$ means $f(x) + g(x) = h(x)$ when all three are real. This $+$ (and the analogous $\cdot$) is only partially defined (given f, g there may be no h). The $+$ (or the $\cdot$) is fully defined iff X is QF; then $D(X)$ is an archimedean vector lattice. See [12] and [5].

An important example of a strongly Egoroff vector lattice is M/N (measurable functions modulo null functions) for σ-finite measures, as discussed in [13] ('the Egoroff Theorem holds') and [16] (it is strongly Egoroff). Here $M/N \approx D(X)$, with X ED and ccc, as a consequence of the Yosida Representation Theorem, which we now describe. (See §6 below for further discussion of these M/N.)

Suppose that A is an archimedean vector lattice with a distinguished positive weak unit e_A (which means $e_A \wedge |a| = 0$ implies $a = 0$); we write '$A \in W$'.

The *Yosida representation* of $A \in W$ is: There is a compact Y_A and an injection $A \xrightarrow{\eta} D(Y_A)$ such that $\eta(e_A) =$ the constant function 1, $\eta(A)$ separates the points of Y_A, $\eta(A)$ is closed under the operations in $D(Y_A)$ requisite for being a vector lattice, and $A \overset{\eta}{\approx} \eta(A)$ as vector lattices. We will usually view A as a sublattice of $D(Y_A)$. Then $A^{-1}\mathbb{R}$ denotes $\{a^{-1}(\mathbb{R}) : a \in A \leq D(Y_A)\}$. Of course, $A^{-1}\mathbb{R} \subseteq \operatorname{dcoz} Y_A$.

The (usual) Yosida representation of A being any $C(X)$, or any $D(X)$ with X QF, uses e_A = the constant function 1, has $Y_A = \beta X$, with $\eta(a) = \beta a$, the Čech–Stone extension, and $\eta(C^*(X)) = C(\beta X)$, $\eta(D(X)) = D(\beta X)$. Here, $C(X)^{-1}\mathbb{R} = \{S \in \operatorname{dcoz}\beta X : S \supseteq X\}$ and $D(X)^{-1}\mathbb{R} = \operatorname{dcoz}\beta A$.

Evidently, a general Y_A need not be QF. The $Y = M/N$ mentioned above is BD, since M/N is σ-complete, has ccc because of the measure, and hence is ED. Then $M/N = D(Y_{M/N})$ by using the lateral σ-completeness [3, 3.3].

2. $\mathbb{R}^{\mathbb{N}}$

THEOREM 2.1. *The vector lattice* $\mathbb{R}^{\mathbb{N}}$ *has the properties* (σ) *and* (OC $\Rightarrow$ RUC). *That is,* $\mathbb{R}^{\mathbb{N}}$ *is strongly Egoroff.*

Proof. (σ) Denote $n \in \mathbb{N}$ as x_n. Given $\{b_n\} \subseteq \mathbb{R}^{\mathbb{N}+}$, replace b_n by $\overline{b}_n$ defined as $\overline{b}_n(x) = \bigvee\{b_n(y) : y \le x\}$, then replace $\overline{b}_n$ by $\overline{\overline{b}}_n = \bigvee\{\overline{b}_k : k \le n\} \vee 1$. Then

(i) $\overline{\overline{b}}_n$ is an increasing function of x,
(ii) $1 \le \overline{\overline{b}}_n \le \overline{\overline{b}}_{n+1}$ for all n.

Clearly, if $\{\overline{\overline{b}}_n\}$ 'has the σ-property', so does the original $\{b_n\}$.

Simplify the notation back to $\{b_n\}$, assuming the features (i) and (ii).

Now set $\lambda_n \equiv 1/b_n(x_n)$, and define b as $b(x_n) \equiv b_n(x_n)$. It is easy to verify that $\lambda_n b_n \le b$ for all n.

(OC $\Rightarrow$ RUC) Note that, in $\mathbb{R}^{\mathbb{N}}$, $u_n \downarrow 0$ iff $u_n(x) \downarrow 0$ for all $x \in \mathbb{N}$.

So, suppose the latter. Then, for all $[0,k] \subseteq \mathbb{N}$, $u_n \to 0$ uniformly on $[0,k]$ (since that set is finite). Thus, for every k there is $n(k)$ for which $u_{n(k)} \le 1/k^2$ on $[0,k]$. We can suppose that $n(1) < n(2) < \cdots$. Note that $k \le x$ implies $u_{n(k)} \le x$.

For $x \in \mathbb{N}$ define, $s(x) \equiv \bigvee_{i \le x} u_{n(i)}(x) \vee 1$, and then $g(x) \equiv xs(x)$. Then (we claim) for all k, $ku_{n(k)} \le g$. This will prove $u_k \to 0$ (g).

To prove the claim, take $x \in \mathbb{N}$. If $x \le k$, then $u_{n(k)}(x) \le 1/k^2$, so $ku_{n(k)}(x) \le 1/k \le 1 \le g(x)$. If $k < x$, then $u_{n(k)}(x) \le s(x)$, so $ku_{n(k)}(x) \le xs(x) = g(x)$. ∎

REMARKS 2.2. (i) 2.1 is a special case of [16, 71.5 and 71.4].

(ii) 2.1(σ) is a special case of [10, 2.1]: $\mathbb{R}^I$ has (σ) iff $|I| < \mathfrak{b}$ (the bounding number). With our main Theorem 5.1, it follows that $\mathbb{R}^I$ is strongly Egoroff iff $|I| < \mathfrak{b}$. See also our remarks on Boolean algebras in §8.

3. $C(S)$, S locally compact and σ-compact. Properties of $\mathbb{R}^{\mathbb{N}}$ will imply properties of such $C(S)$ (but not (OC $\Rightarrow$ RUC)) *via* the following.

LEMMA 3.1. *Suppose S is locally compact and σ-compact.*

(a) *If X is compact and S is dense in X, then $S = \operatorname{coz} w$ for some (various) $w \in C(X)^+$.*

Using $X = \beta S$ in (a), *set $g = 1/w \in C(S)$, and $X_n = g^{-1}[0, k+1]$ for $k \in \mathbb{N}$. Then*

(b) *each X_k is compact, $X_k \subseteq \operatorname{Int} X_{k+1}$, and $S = \bigcup_{k\in\mathbb{N}} X_k$;*
(c) *for each $\{r_k\} \subseteq (0, +\infty)$, there is $f \in C(S)$ such that, for all k, $x \in X_k \setminus X_{k-1} \Rightarrow f(x) \geq r_k$.*

Proof. (a) See [6].

(b) is obvious.

(c) Let $Z_k \equiv g^{-1}[k-1, k+1] \subseteq g^{-1}(k-2, k+2) \equiv U_k$. There is $v_k \in C(S, [0,1])$ with

$$v_k = [1 \text{ on } Z_k; 0 \text{ on } S \setminus U_k]$$

(because Z_k and $S \setminus U_k$ are disjoint zero-sets [9]). Now $\{U_k\}$ is a locally finite cozero cover of S, and so $f \equiv \sum_{k\in\mathbb{N}} r_r v_k \in C(S)$. Evidently, $f(x) \geq r_k$ for $x \in Z_k$, and $X_k \setminus X_{k-1} \subseteq Z_k$. ∎

THEOREM 3.2. *Suppose S is locally compact and σ-compact. The vector lattice $C(S)$ has the following properties:*

(a) *$C(S)$ has (σ).*
(b) *$C(S)$ has* (PWC $\Rightarrow$ RUC). *That is, if $u_n(x) \downarrow 0$ for all $x \in S$, then $u_n \downarrow 0$ r.u.*
(c) *Suppose S is dense in a compact X. Then there is $u_n \downarrow 0$ in $C(X)$ with $S = \{x \in X : \bigwedge_{n\in\mathbb{N}} u_n(x) = 0\}$.*

Proof. For (a) and (b), write $S = \bigcup_{k\in\mathbb{N}} X_k$ as in 3.1(b), (c). For $\kappa \in C(S)^+$ define $\kappa^* \in \mathbb{R}^{\mathbb{N}}$ as $\kappa^*(k) \equiv \bigvee\{\kappa(x) : x \in X_k\}$.

(a) Suppose $\{f_n\} \subseteq C(S)^+$. Then $\{f_n^*\} \subseteq \mathbb{R}^{\mathbb{N}+}$, so by 2.1, there are $\{\lambda_n\}$ and b with $\lambda_n f_n^* \leq b$ for all n. Use $r_k = b(k)$ in 3.1(c), finding $f \in C(S)$ with $f(x) \geq b(k)$ for $x \in X_k \setminus X_{k-1}$. It follows that $\lambda_n f_n \leq f$ for all n.

(b) Suppose in $C(S)$ that $f_n \downarrow 0$ and $\bigwedge_{n\in\mathbb{N}} f_n(x) = 0$ for all $x \in S$. Then $f_n(x) \downarrow 0$ for all $x \in X_k$, so by Dini's Theorem [21, 7.13], $f_n \to 0$ uniformly on each X_k. It follows that for the $\{f_n^*\} \subseteq \mathbb{R}^{\mathbb{N}}$, we have $f_n^* \downarrow 0$, and thus by 2.1, there is $g \in \mathbb{R}^{\mathbb{N}}$ for which $f_n^* \to 0$ (g).

Now by 3.1(c) with $r_k = g(k)$, there is $f \in C(S)$ for which we have $x \in X_k \setminus X_{k-1} \Rightarrow f(x) \geq g(k)$. We claim that for every $p > 0$ there is $n(p)$ with $pf_{n(p)} \leq f$, which means $f_n \to 0$ (f).

So fix $p > 0$. There is $n(p)$ for which $pf^*_{n(p)} \leq g$ in $\mathbb{R}^{\mathbb{N}}$, which means $pf^*_{n(p)} \leq f(x)$ for all $x \in S$, because given x and letting k be the first index with $x \in X_k \setminus X_{k-1}$, we have

$$pf_{n(p)}(x) \le pf^*_{n(p)}(k) \le g(k) \le f(x).$$

(c) (This is very easy.) From 3.1, $S \equiv \operatorname{coz} w$ for $w \in C(X)^+$, Then, for each n, Zw and $\{x : w(x) \ge 1/n\} \equiv Z_n$ are disjoint zero-sets and there is $u_k \in C(X, [0,1])$ with $u_k^{-1}\{1\} = Zw$ and $u_k^{-1}\{0\} = Z_n$. We can arrange $u_1 \ge u_2 \ge \cdots$, and then $S = \{x \in X : \bigwedge_{n\in\mathbb{N}} u_n(x) = 0\}$. ∎

REMARKS 3.3. (a) 3.2(a) is a simpler case of [10, 1.2], which says that $C(S)$ has (σ) if S is locally compact and paracompact with Lindelöf number (see [6]) $\le \mathfrak{b}$.

(b) If S is compact, then in $C(S)$ all functions are bounded, r.u. convergence is ordinary uniform convergence (regulated by the constant function 1), and 3.2(b) is Dini's Theorem.

(c) In 3.2(b) we cannot conclude (OC $\Rightarrow$ RUC). That property of $C(S)$ requires that S be almost P; see 7.3 below.

(d) The very simple 3.2(c) is a special case of the not-so-simple 4.3 below, which says (in effect) that S Lindelöf and Čech-complete suffices.

4. Sets of pointwise convergence. We make some observations necessary for our main Theorem 5.1.

PROPOSITION 4.1. *Suppose $G \in W$, viewing $G \le D(Y_G)$. Suppose $\{a_i\}_{i\in\mathbb{N}} \subseteq G$, and set $Z \equiv \{x \in Y_G : \bigwedge_{i\in\mathbb{N}} u_i(x) = 0\}$. Then*

(a) *Z is* $\operatorname{coz}_\delta Y_G$.

(b) *$\bigwedge^G u_i = 0$ iff Z is dense in Y_G (i.e., $Z \in \operatorname{dcoz}_\delta Y_G$).*

COROLLARY 4.2. *Suppose X is compact. If $\bigwedge^{C(X)}_{i\in\mathbb{N}} u_i = 0$, then $\{x \in X : \bigwedge_{i\in\mathbb{N}} u_i(x) = 0\} \in \operatorname{dcoz} X$.*

A crucial point of our main Theorem 5.1 requires the converse of 4.1.

THEOREM 4.3. *Suppose X is compact. If $S \in \operatorname{dcoz}_\delta X$, then there is $u_i \downarrow 0$ in $C(X)$ for which $\{x \in X : \bigwedge_{i\in\mathbb{N}} u_i(x) = 0\} = S$.*

Proof of 4.1. $S_{ni} = \{x \in X : u_i(x) < 1/n\}$ is $\operatorname{coz} Y_G$ (even $\operatorname{coz} G$) and $S_n \equiv \bigcup_{i\in\mathbb{N}} S_{ni}$ is $\operatorname{coz} Y_G$ (perhaps not $\operatorname{coz} G$). So $\bigcap_{n\in\mathbb{N}} S_n$ is $\operatorname{coz}_\delta Y_G$ and evidently $\bigcap_{n\in\mathbb{N}} S_n = Z$.

Now, if Z is dense then $\bigwedge^G_{i\in\mathbb{N}} u_i = 0$ (in fact $\bigwedge^{D(Y_G)}_{i\in\mathbb{N}} u_i = 0$ by continuity).

Suppose $\bigwedge^{D(Y_G)}_{i\in\mathbb{N}} u_i = 0$. Then S_n is dense for each n and therefore $\bigcap_{n\in\mathbb{N}} S_n$ is dense by the Baire Category Theorem. For suppose some S_n is not dense. Then there is an open $V \ne \emptyset$ with $V \cap S_n = \emptyset$ and thus $V \cap S_{ni} = \emptyset$ for all i, i.e., $x \in V \Rightarrow u_i(x) \ge 1/n$ for all i. Take $g \in G^+$ with $0 \le g \le 1/n$, $\{x \in X : g(x) = 1/n\} \subseteq V$ and $x \notin V \Rightarrow g(x) = 0$. Then

$u_i \geq g > 0$ for all i so $\bigwedge_{i\in\mathbb{N}}^G u_i \neq 0$. ($G$ 0-1 separates disjoint compact sets in Y_G, since G separates points and G is a vector lattice.) ∎

Proof of 4.2. The presentation of $C(X)$ is its Yosida representation. Apply 4.1. ∎

Let $Q = \prod_{n\in\mathbb{N}}[0,1]_n$ denote the Hilbert cube. For every n, let $\pi_n\colon Q \to [0,1]_n$ denote the projection. For every $0 < t < 1$, set $K(t) = \prod_{n\in\mathbb{N}}[t,1]_n$.

For every n, define $u_n\colon Q \to \mathbb{I}$ by

$$u_n(x) = \min\{x_1, \ldots, x_n\}.$$

Then $u_n \in C^+(Q)$, and $u_{n+1} \leq u_n$ for every n. Moreover, define $u\colon Q \to \mathbb{I}$ by

$$u(x) = \inf\{x_1, x_2, \ldots\}.$$

Observe that $u = \bigwedge_{n\in\mathbb{N}} u_n$, that u is not continuous, and that

$$P = u^{-1}(\{0\}) = \{x \in Q : \inf\{x_1, x_2, \ldots\} = 0\}$$

is a dense G_δ-subset of Q. Observe that $Q \setminus u^{-1}(\{0\}) = \bigcup_{n\in\mathbb{N}} K(1/n)$.

We now come to the proof of 4.3. We present two proofs; one is based on infinite-dimensional topology and the other one is direct and only uses standard facts. We will first present a reduction to compact metrizable spaces.

Let X be any compact space, and for every n, let U_n be a dense cozero-subset of X. Let $\alpha_n\colon X \to [0,1]$ be a continuous function such that $\alpha_n^{-1}(\{1\}) = X \setminus U_n$. Let $\alpha\colon X \to Q$ be defined by

$$\alpha(x) = (\alpha_1(x), \alpha_2(x), \ldots).$$

Set $Y = \alpha(X)$. For every n, set $V_n = \pi_n^{-1}([0,1)) \cap Y$. Then $\alpha^{-1}(V_n) = U_n$, hence V_n is a dense open subset of Y. Consequently, $S = \bigcap_{n\in\mathbb{N}} V_n$ is a dense G_δ-subset of Y such that $\alpha^{-1}(S) = X \setminus \bigcap_{n\in\mathbb{N}} U_n$.

We will show that we can re-embed Y in Q in such a way that $Y \cap P = S$. Assume for a moment that Y has this property. Let $w_n\colon X \to \mathbb{I}$ be the composition $u_n \circ \alpha$. Then clearly $w_{n+1} \leq w_n$ for every n. Let $f \in C^+(X)$ be such that $f \leq w_n$ for every n. If $x \in \bigcap_n U_n$, then $\alpha(x) \in S \subseteq P$ and so $\bigwedge_n (u_n \circ \alpha)(x) = 0$. Hence we conclude that $\bigwedge_n w_n(x) = 0$, and so $f(x) = 0$. This implies that f is identically 0 on the dense set $\bigcap_n U_n$, hence has to be identically 0 everywhere. Finally, if $x \notin \bigcap_n U_n$, then $\alpha(x) \notin P$, and thus $\bigwedge_n w_n(x) > 0$. This means that if we indeed succeed in re-embedding Y in the way we described, we are done.

First proof of 4.3. By [18, Proposition 6.5.4], $\Sigma' = A \setminus P$ contains the skeletoid Σ (defined in [18, p. 284]). Since it is clearly a countable union of Z-sets in Q, it is an absorber [18, Corollary 6.5.3]. Hence by [18, Corollary 6.5.3], there is a homeomorphic $\beta\colon Q \to Q$ such that

$$\alpha(\Sigma' \cup S) = \Sigma'.$$

Then $\beta(Y)$ is a copy of Y such that $\beta(Y) \cap \Sigma' = \beta(S)$. ∎

Second proof of 4.3. Write $Y \setminus S$ as $\bigcup_{n\in\mathbb{N}} A_n$, where each A_n is compact and $A_1 \subseteq A_2 \subseteq \cdots$. Write $Y \setminus A_n$ as $\bigcup_{i\in\mathbb{N}} B_{n,i}$, where each $B_{n,i}$ is compact.

For every n, let Q_n be a copy of Q. For $0 < t < 1$ let $K_n(t)$ denote the copy of $K(t)$ in Q_n.

We may assume that $Y \subseteq K_1(1/2)$. For every $n, i \in \mathbb{N}$, let $f_{n,i}\colon Y \to [1/(n+2), 1]$ be an Urysohn function such that $f_{n,i}(B_{n,i}) \subseteq \{1/(n+2)\}$ and $f_{n,i}(A_n) \subseteq \{1\}$. Now define $\alpha\colon Y \to \prod_{n\in\mathbb{N}} Q_n$ by

$$\alpha(z) = (z, (f_{1,i}(z))_i, (f_{2,i}(z))_i, \ldots, (f_{n,i}(z))_i, \ldots).$$

Then α is clearly an embedding.

Fact 1. *For every n, $\alpha(Y) \cap \prod_{i\in\mathbb{N}} K_i(1/(n+1)) = \alpha(A_n)$.*

Indeed, let $z \in A_n$. Observe that $\alpha(z)_1 = z$ and for every $k \in \mathbb{N}$, $z_k \geq 1/2 \geq 1/(n+1)$. For $k < n$ and $i \in \mathbb{N}$ we clearly have

$$f_{k,i}(z) \geq 1/(k+2) \geq 1/(n+1).$$

Moreover, for $k \geq n$ and $i \in \mathbb{N}$ we have $z \in A_n \subseteq A_k$ and so $f_{k,i}(z) = 1 \geq 1/(n+1)$. We conclude that $\alpha(z) \in \prod_{i\in\mathbb{N}} K_i(1/(n+1))$.

Conversely, assume that $z \in Y$ has $\alpha(z) \in \prod_{i\in\mathbb{N}} K_i(1/(n+1))$ but $z \notin A_n$. There exists $i \in \mathbb{N}$ such that $z \in B_{n,i}$. Then $f_{n,i}(z) = 1/(n+2) < 1/(n+1)$, which is a contradiction.

There is a natural homeomorphism between Q and $\prod_{n\in\mathbb{N}} Q_n$ by simply rearranging coordinates. This homeomorphism sends every $K(t)$ for $0<t<1$ onto $\prod_{n\in\mathbb{N}} K_n(t)$. Hence we are done. ∎

Remarks 4.4. (a) 4.3 for just $S \in \operatorname{dcoz} X$ is the very easy 3.2(c).

(b) 4.3 (for $S \in \operatorname{dcoz}_\delta X$) appears to sharpen results of Hahn, Sierpiński, and perhaps Hausdorff; see [11, pp. 307, 308].

5. D[QF]. The following is the main theorem of the paper. The proof will use almost everything we have said so far. Further commentary appears in 5.2, 5.3 and §7 below.

Theorem 5.1. *Suppose X is* QF. *The following are equivalent:*

(1) *X has* (E).
(2) *$D(X)$ has* (σ).
(3) *$D(X)$ has* (OC $\Rightarrow$ RUC).
(4) *If $S \in \operatorname{dcoz}_\delta X$, then there is $\{u_n\}_{n\in\mathbb{N}} \subseteq C(X)$ with $u_n \downarrow 0$ r.u. in $D(X)$ for which $S \supseteq \{x \in X : \bigwedge_{n\in\mathbb{N}} u_n = 0\}$.*
(5) *$D(X)$ is strongly Egoroff* (*i.e.,* (2) *and* (3) *hold*).

Proof. (5) is just '(2) and (3)'. Everything revolves around (1): we shall prove that each of (2), (3), (4) is equivalent to (1). This is probably not the most efficient, but perhaps reveals more. Towards 'revealing more', for each of our implications, we shall write $(x) \overset{\text{m.n}}{\Longrightarrow} (y)$ to indicate that Proposition/Theorem m.n is an/the essential ingredient in the proof that (x) implies (y).

At several points in these proofs, we use the fact (see §1) that

$$(\dagger) \qquad \text{for } A = D(X), X \text{ compact QF}, \quad A^{-1}\mathbb{R} = \operatorname{dcoz} X.$$

(2)⇒(1). Suppose $D(X)$ has (σ), and let $\{S_n\}_{n\in\mathbb{N}} \subseteq \operatorname{dcoz} X$; so for each n, we have $S_n = a_n^{-1}\mathbb{R}$ for some $a_n \in D(X)$. By (σ), there are $\{\lambda_n\}_{n\in\mathbb{N}}$ and a with $\lambda_n a_n \leq a$ for all n. Then $(\lambda_n a_n)^{-1}\mathbb{R} = a_n^{-1}\mathbb{R} \supseteq a^{-1}\mathbb{R}$.

$(1)\overset{3.2}{\Longrightarrow}(2)$. Suppose X has (E), and $\{a_n\}_{n\in\mathbb{N}} \subseteq D(X)^+$. There is $S \in \operatorname{dcoz} X$ with $S \subseteq \bigcap_{n\in\mathbb{N}} a_n^{-1}\mathbb{R}$, and $b \in D(X)$ with $b^{-1}\mathbb{R} = S$. Let $\bar{a}_n$ and $\bar{b}$ denote the restrictions to S, which lie in $C(S)$. Now, S is locally compact and σ-compact, so $C(S)$ has (σ) (by 3.2), so there are $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\bar{a} \in C(S)$ with

$$(\ddagger) \qquad \lambda_n \bar{a}_n \leq \bar{a} \quad \text{for all } n \text{ (pointwise on } S).$$

Then $\beta S = X$ (because X is QF), and $a = \beta\bar{a} \in D(X)$, and of course $\beta\bar{a}_n = a_n$. Since S is dense in X, the inequalities $(\ddagger)$ entail $\lambda_n a_n \leq a$ for all n.

$(3)\overset{4.2}{\Longrightarrow}(1)$. This is the hardest part, because of 4.3. Toward (E), take $S \in \operatorname{dcoz}_\delta X$. By 4.3, take $u_n \downarrow 0$ in $C(X)$ with $S = \{x \in X : \bigwedge_{n\in\mathbb{N}} u_n(x) = 0\}$ (actually, '$\supseteq$' suffices for the proof). Now $u_n \downarrow 0$ in $D(X)$ also, since the inclusion $C(X) \leq D(X)$ preserves arbitrary infima (exercise). By (3), there is $g \in D(X)$ with $u_n \to 0$ (g). But $u_n \to 0$ (g) implies pointwise convergence on $g^{-1}\mathbb{R}$, i.e., $S \supseteq g^{-1}\mathbb{R}$. Thus we have (E).

(4)⇒(1). Toward (E), take $S \in \operatorname{dcoz}_\delta X$. Apply (4) to get $\{a_n\} \subseteq C(X)$ and $g \in D(X)$ with $u_n \downarrow 0$ (g), and $S \supseteq \{x \in X : \bigwedge_{n\in\mathbb{N}} u_n(x) = 0\}$. Again, $u_n \to 0$ (g) implies pointwise convergence on $g^{-1}\mathbb{R}$, so $S \supseteq g^{-1}\mathbb{R}$, and we have (E).

$(1)\overset{3.2}{\Longrightarrow}(4)$. Toward (4), take $S \in \operatorname{dcoz}_\delta X$. By (E), there is $S_0 \in \operatorname{dcoz} X$ with $S_0 \subseteq S$. By 3.2(c), there is $u_n \downarrow 0$ in $C(X)$ for which $S_0 = \{x \in X : \bigwedge_{n\in\mathbb{N}} u_n(x) = 0\}$. By 3.2(b), there is $g \in C(S_0)$ for which the restrictions $u_n|S_0$ have $u_n|S_0 \to 0$ (g) in $C(S_0)$. Since X is QF, we have $\beta S_0 = X$, and $\beta g \in D(X)$. It follows that $u_n \to 0$ (βg) in $D(X)$.

$(1)\overset{4.1\&3.1}{\Longrightarrow}(3)$. Suppose $u_n \downarrow 0$ in $D(X)$. By 4.1, $S \equiv \{x \in X : \bigwedge_{n\in\mathbb{N}} u_n(x) = 0\} \in \operatorname{dcoz}_\delta X$. We have $u_1 \geq u_2 \geq \cdots$, and $S_0 = u_1^{-1}\mathbb{R} \cap S \in \operatorname{dcoz}_\delta X$ also, and $\bigwedge_{n\in\mathbb{N}} u_n(x) = 0$ for all $x \in S_0$. By (E), there is $S_1 \in \operatorname{dcoz} X$ with $S_1 \subseteq S_0$, so $\bigwedge_{n\in\mathbb{N}} u_n(x) = 0$ for all $x \in S_1$. Now, S_1 is locally compact and σ-compact, and the restrictions $u_n|S_1$ are in $C(S_1)$ (since $u_1|S_1 \in C(S_1)$ and $u_1 \geq u_n$ for all n). By 3.1(b), there is $g \in C(S_1)$ with $u_n|S_1 \to 0$ (g) in $C(S_1)$. As in '(1)⇒(4)', it follows that $u_n \to 0$ (βg) in $D(X)$. ∎

Remarks 5.2. One may wonder to what extent 5.1, or a particular condition (m) in 5.1, or a particular implication (m)⇒(n) in 5.1, generalizes to wider classes of vector lattices. We ignore condition (4).

(i) [16, 15.19] shows $C_k(\mathbb{N})$ (the functions of compact (finite) support on $\mathbb{N}$) has (OC ⇒ RUC). But obviously (σ) fails. Here there is no weak unit.

(ii) While for $D(X)$, (1) iff (2), for $C(X)$, neither implies the other. For any compact X, $C(X)$ has (σ), but $X = [0,1]$ fails (E). On the other hand, if X contains densely (a copy of) $\mathbb{N}$, then X will have (E) (because $\mathbb{N}$ is the minimum member of a dcoz X). Here is such an X with $C(X)$ failing (σ) (see of [10, 1.1(b)]). Let $X = \sum_{n\in\mathbb{N}} \mathbb{N}_n \cup \{\rho\}$, where a neighborhood of ρ contains $\sum_{n\geq k} \mathbb{N}_n$ for some k. Define $b_n \in C(X)$ as: if $x \notin \mathbb{N}_n$, then $b_n(x) = 0$; if $x \in \mathbb{N}_n$, then $b_n(x) = x$. This $\{b_n\}_{n\in\mathbb{N}}$ witnesses $C(X)$ failing (σ): if $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0, +\infty)$, choose $x_n \in \mathbb{N}_n$ with $x_n \geq n/\lambda_n$. Note that $x_n \to \rho$. The inequalities $\lambda_n b_n \leq b$ for all n would force $b(\rho) = +\infty$, so $b \notin C(X)$.

(iii) The proof of 5.1((1)⇒(3)) comes very close to requiring being in a $D(X)$, X QF.

More Remarks 5.3. (a) We do not know what to make of condition 5.1(4) (nor whether the inclusion $\supseteq$ there can be equality).

(b) For compact X not necessarily QF, one can write down the property '$D(X)$ *qua* lattice has (σ)'. One sees that in 5.1, (2)⇒(1) does not require X QF, while (2)⇐(1) seems to.

(c) As noted in §1, a vector lattice 'Measurable mod Null' for a σ-finite measure, or just 'M/N', has $M/N \approx D(X)$ for X extremally disconnected with ccc. [16, 71.4] (resp. [16, 71.5]) proves separately that such M/N has (OC ⇒ RUC) (resp. (σ)). The proof of the former is rather complicated, using the classical Egoroff Theorem. 5.1 shows these complications are in some sense avoidable. Of course, this derivation uses $M/N \approx D(X)$, which is a representation theorem, and the proofs alluded to just use the given presentation of the M/N (and [16] avoids representation theorems wherever possible).

(d) An example: In view of (c), one might ask if $D(X)$ satisfies 5.1 whenever X is compact ED with ccc. The answer is 'no'. Let Y be the irrationals, and $\beta Y \overset{\pi}{\twoheadleftarrow} a\beta Y = X$ the absolute (projective cover, Gleason cover) of βY with irreducible map π. Here, Y is $\mathrm{dcoz}_\delta\, \beta Y$, and it follows that $\pi^{-1}Y$ is $\mathrm{dcoz}_\delta\, X$, since irreducible maps inversely preserve density. If X had (E), there would be $S \in \mathrm{dcoz}\, X$ with $S \subseteq \pi^{-1}Y$, so $\pi(S) \subseteq Y$. But irreducible maps carry open sets to sets with dense interior, so Y would contain densely an open set in βY, which it does not.

(e) Another example: One might ask whether $D(X)$ satisfying 5.1 implies X has ccc, or says anything about the Souslin number of X. First, $\mathbb{R}^I$ for $|I| < \mathfrak{b}$ satisfies 5.1 (see 2.2; $\mathbb{R}^I \approx D(\beta I)$) and $\aleph_0 < |I|$ means βI fails ccc.

Second, the familar space $\lambda D = D \cup \{\lambda\}$, D discrete and neighborhoods of λ with countable complement, is a Lindelöf P-space, $\beta\lambda D$ is BD, $D(\beta\lambda D) \approx C(\lambda D)$ and the latter has (σ) [10, §3]. But the Souslin number of $\beta\lambda D$ is $|D|$.

6. C(almost P): convergence properties. This section is ground-clearing for §7.

PROPOSITION 6.1. *Suppose Y is almost* P. *Then βY is* QF, *$C(Y) \approx D(\beta Y)$, and for $C(Y)$, the properties* (σ), (OC $\Rightarrow$ RUC), *and* (E) *are equivalent.*

Proof. If $S \in \operatorname{dcoz} \beta Y$, then $S \cap Y \in \operatorname{dcoz} Y$ so $S \cap Y = Y$ (since Y is almost P), so $S \supseteq Y$ and therefore S is C^*-embedded in βY.

$C(Y) \ni f \mapsto \beta f \in D(\beta Y)$ is an injection, and is onto because Y is almost P.

The last assertion is 5.1 for our $C(Y)$. ∎

THEOREM 6.2. *$C(Y)$ has* (OC $\Rightarrow$ PWC) (*i.e., $u_n \downarrow 0$ implies $\bigwedge_{n\in\mathbb{N}} u_n(y) = 0$ for all $y \in Y$*) *iff Y is almost* P.

Proof. Suppose Y is almost P, and $u_n \downarrow 0$ in $C(Y)$. Applying 5.1 to $C(Y) \leq D(\beta Y)$, one finds

$$\Big\{x \in \beta Y : \bigwedge_{n\in\mathbb{N}} \beta u_n(x) = 0\Big\} \equiv T \in \operatorname{dcoz}_\delta \beta Y.$$

Since Y is almost P, $T \supseteq Y$ and therefore $u_n(y) \downarrow 0$ for all $y \in Y$.

Suppose Y is not almost P, and $f \in C(Y)^+$ has $Zf \neq \emptyset$, nowhere dense. Then, for all n, Zf and $\{y \in Y : f(y) \leq 1/n\} \equiv Z_n$ are disjoint zero-sets, so there is $v_n \in C(Y, [0,1])$ with $v_n = [1 \text{ on } Zf; 0 \text{ on } Z_n]$. Then $u_n = \bigwedge_{i\leq n} v_i$ has $u_n \downarrow 0$ in $C(Y)$, but $\bigwedge_{n\in\mathbb{N}} u_n(x) = 1$ for $x \in Zf$. ∎

THEOREM 6.3. *If $C(Y)$ has* (OC $\Rightarrow$ RUC), *then Y is almost* P.

Proof. Any $C(Y)$ has (RUC $\Rightarrow$ PWC) (because $|u_n - u| \leq \varepsilon g$ implies $|u_n(y) - u(y)| \leq \varepsilon g(y)$ for all $y \in Y$). So, if $C(Y)$ has (OC $\Rightarrow$ RUC), it also has (OC $\Rightarrow$ RUC), and 6.2 applies. ∎

The converse of 6.3 fails: see 6.5 below.

THEOREM 6.4. *Suppose $G \in W^*$* (*which means the unit is strong*). *Then G has* (σ), *and the following are equivalent:*

(a) *G has* (OC $\Rightarrow$ RUC) (*or, G is strongly Egoroff*),
(b) *G has* (OC $\Rightarrow$ PWC) (*'pwc' means pointwise on Y_G*),
(c) *Y_G is almost P.*

Proof. $G \in W^*$ means all $g \in G$ are bounded functions on Y_G. This implies that G has (σ), thus in (a) we have the '(or, $\cdots$)'. Also, $g_n \to 0$ r.u. in G iff $g_n \to 0$ uniformly on Y_G. This shows (a)⇒(b), and (b)⇒(a) by Dini's Theorem [21, 7.13] on Y_A. Finally, (b)$\Leftrightarrow$(c) is proved exactly as 6.2; the v_n there can be chosen from G because G separates compact sets in Y_G. ∎

Remarks 6.5. (a) Veksler [23] asserts (without proof) 6.2 and 6.3 for compact Y, and 6.4 for $G = C(Y)$, Y compact. (We interpret his phrase—in translation from Russian—'double sequence Theorem' to be the definition of 'strongly Egoroff' according to [16, p. 68], which for an archimedean vector lattice is equivalent to what we are using, namely '(σ) and (OC $\Rightarrow$ RUC)' [16, 68.8 and 70.2].)

(b) The converse of 6.3 is false; in fact, Y having P does not imply $C(Y)$ has (OC $\Rightarrow$ RUC). As noted in 2.1, $\mathbb{R}^I$ ($= C(I) \approx D(\beta I)$, I discrete) has (σ) (iff (OC $\Rightarrow$ RUC), by 5.1) iff $|I| < (\mathfrak{b})$. So $|I| \geq \mathfrak{b}$ (e.g., $|I| = 2^{\aleph_0}$) has the discrete I a P-space and $C(I)$ failing (OC $\Rightarrow$ RUC).

(c) Standard examples of compact almost P spaces are: one-point compactifications of uncountable discrete spaces, and $\beta X \setminus X$ for X locally compact and realcompact [7].

(d) Let Y be compact almost P. If $G \leq C(Y)$ is any point-separating vector sublattice, then G is strongly Egoroff (by 6.4, because $Y = Y_G$). Thus, if S is any subset of $C(Y)$ which separates points, then the vector lattice G generated in $C(Y)$ by S is strongly Egoroff.

7. Examples in C(almost P). We now construct many examples of the main Theorem 5.1, compact QF X with (E).

These spaces will be $X = \beta T$ for T almost P so that $D(X) = C(T)$ (6.1). Varying the input to the construction results in various properties of X: an F-space, connected or zero-dimensional. When X is zero-dimensional, there is the Boolean algebra $\operatorname{clop} X$, which is a 'Egoroff Boolean algebra'; see §8 for that discussion.

In the following, a *P-set* is a subset such that each G_δ containing it is a neighborhood of it; and Y^* denotes $\beta Y \setminus Y$.

Lemma 7.1. *Suppose $T = \bigcup_{n\in\mathbb{N}} T_n$, with each T_n a closed P-set in T, and $T_n \subseteq T_{n+1}$. Then:*

(a) *T has the weak topology with respect to $\{T_n\}_{n\in\mathbb{N}}$.*
(b) *If each T_n is compact, then T^* is strongly ω-bounded (each σ-compact subset has compact closure).*

Proof. (a) This means that $A \subseteq T$ is closed in T if for each n, $A \cap T_n$ is closed in T_n. Suppose A satisfies this latter condition, and suppose $x \in T\setminus A$, say $x \in T_1 \setminus A$. Let U be a cozero-set in T with $x \in U$ and $U \cap (A \cap T_1) = \emptyset$.

(The family $\{T_1 \cap U : U \in \operatorname{coz} T\}$ is a base in T_1.) Observe that

$$A \cap U = \bigcup_{n\in\mathbb{N}} (A \cap T_n \cap U)$$

is an F_σ in T which misses T_1. Since T_1 is a P-set in T, we have $\overline{A \cap U} \cap T_1 = \emptyset$, so $U \setminus \overline{A \cap U}$ is a neighborhood of x that misses A.

(b) is an immediate consequence of (a) and van Douwen's Lemma ([14, 3.5], [1, 3.8]). (Other proofs are possible.) ■

THEOREM 7.2. *Suppose that $T = \bigcup_{n\in\mathbb{N}} T_n$ with each T_n almost* P *and a compact P-set in T and $T_n \subseteq T_{n+1}$. Then T is σ-compact almost* P*, and βT is* QF *with* (E).

Proof. We will first show that T is almost P. To this end, let S be a nonempty G_δ in T. We may assume that $S \cap T_1 \neq \emptyset$. For every n, let U_n be the interior of $S \cap T_n$ in T_n. Clearly, U_n is nonempty and open in T_n, and $U_{n+1} \cap T_n \subseteq U_n$. Hence $U = \bigcup_{n\in\mathbb{N}} U_n$ has the property that $U \cap T_n$ is open in T_n for all n. But this implies by 7.1 that U is open in T. So T is almost P, and βT is QF (6.1).

Toward (E), suppose $\{S_n\}_{n\in\mathbb{N}} \subseteq \operatorname{dcoz} \beta T$. Then $\bigcap_{n\in\mathbb{N}} S_n \supseteq T$ (because T is almost P), so $F \equiv \bigcup_{n\in\mathbb{N}} (\beta T \setminus S_n) \subseteq T^*$, so by 7.1(b), $\overline{F}$ (closure in T^*) is compact. So $\overline{F}$ is closed in βT and misses T, and since T is Lindelöf, Smirnov's Theorem [6, 3.12.25] yields a zero-set Z in βT with $\overline{F} \subseteq Z \subseteq T^*$. Thus $\beta T \setminus Z \subseteq \bigcap_{n\in\mathbb{N}} S_n$, as desired. ■

The simplest examples of this situation: for every n, let K_n be compact almost P, and set $T_n \equiv \sum_{i\le n} K_i$ and $T \equiv \sum_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} T_n$. In this case (E) is obvious because T is almost P, and being locally compact and σ-compact, already a cozero-set in βT. In the construction which follows, the T is not locally compact and the βT is even F. See also §9 below.

LEMMA 7.3. *Suppose S is locally compact and σ-compact.*

(a) ([17, 1.25]; *also* [7], [9, 14.27]) *S^* is almost* P*, and has every σ-compact subspace C^*-embedded, and hence is* F.
(b) ([19, proof of 5.1]) *If A is closed in S, then $\overline{A} \cap S^*$ is a P-set in S^** (*the closure in βS*).

THEOREM 7.4. *Suppose K is compact, and for all n, K_n is a compact subset of K with $K_n \subseteq K_{n+1}$. Suppose J is locally compact, σ-compact, not compact. Let*

$$Z = K \times J, \qquad S_n = K_n \times J, \qquad T_n = S_n^*.$$

Then

$$S_n^* \subseteq Z^*, \qquad S_n^* \subseteq S_{n+1}^* \quad \forall n,$$

and $T = \bigcup_{n\in\mathbb{N}} T_n$ *(union in* Z^**) has the properties:* T *is almost* P *and* F, *and* βT *is* F *with property* (E).

Proof. We verify the hypotheses in 7.1, use 7.3 and apply 7.2.

Z is locally compact and σ-compact, and each S_n is closed in Z.

First, Z is normal and S_n is C^*-embedded (Tietze–Urysohn). Thus by a well-known argument, βS_n is (equivalent to) $\overline{S}_n$ (closure in βZ), and by 7.3(b), $S_n^* = Z^* \cap \overline{S}_n$, and is a P-set in Z^*. This also shows that $S_n^* \subseteq S_{n+1}^*$ for all n, and we have the union in Z^*, $T = \bigcup_{n\in\mathbb{N}} T_n$. Next, each S_n is also locally compact and σ-compact, hence ($S_n^* =$) T_n is almost P, and so is T by 7.2.

Now, T is a σ-compact subset of Z^*, hence T is C^*-embedded in Z^* by 7.3(a). It follows that T is F [9, 14.26]. We finally claim that T_n is a P-set in T. To see this, let $\{U_m\}_m$ be any family of open neighborhoods of T_n in T. Then $T \setminus U_m$ is σ-compact for every m, hence $E = T \setminus \bigcup_{m\in\mathbb{N}} U_m$ is σ-compact. But then $T_n \cap \overline{E} = \emptyset$, since T_n is a P-set in Z^*.

By 7.2, βT has (E). ∎

Examples 7.5. (a) In 7.4, use $K_n = \prod_{m\in\mathbb{N}}[1/n, 2-1/n]_m \subseteq \prod_{m\in\mathbb{N}}[0,2]_m = K$, and $J = [0,1)$. Then βT is a connected F-space with (E).

(b) In 7.4, use K and J zero-dimensional. Then βT is a zero-dimensional F-space with (E) (and the Boolean algebra $\operatorname{clop}\beta T$ is 'weakly countably complete' and 'Egoroff'—see §8).

Proof. Everything is obvious from 7.4, except perhaps that in (a), βT is connected. Here, each T_n is connected (by an easy argument like [9, p. 92]). And any union of an increasing sequence of connected spaces is again connected. ∎

Remark 7.6. In 7.5(b), the Boolean algebras $\operatorname{clop}\beta T$ are never σ-complete, in contrast to the Egoroff BAs mentioned in §8 below. This is because (almost P)$\cap$BD $=$ P (see §1), and a compact P-space is finite [9, 4K]. Thus, in 7.2, if βT were BD, T would be, and then T_n would be also (a P-set in a BD space is BD), and thus finite. But in 7.4, the T_n ($= S_n^*$) are not finite.

8. Boolean algebras. Let A be a Boolean algebra (BA) with zero-dimensional (ZD) Stone space SA (see [22] if necessary).

In [15], the 'Egoroff property' of a BA A is formulated in Boolean terms, and attributed to 'Nakano, though in a different form'. In [8, §316] the property is reformulated, and dualized to SA, where it becomes the topological property (E). (Our (E) does not assume ZD, §7 here has connected X with (E).) One might also compare the closely related discussion in [22, §§19, 20, 30], where (E) is almost defined.

If a compact QF X is also ZD, we have the BA $\operatorname{clop} X$ with $S(\operatorname{clop} X) = X$, and 5.1 says $D(X)$ is strongly Egoroff iff X has (E).

Note that in §7 we have some compact F-spaces X which are ZD, so the $\operatorname{clop} X$ are Egoroff BAs, and also 'weakly countably complete' (equivalent to $SA = X$ being F). See [17].

For the $M/N \approx D(X)$ mentioned in 5.3(c), which have X ED (thus ZD) with ccc, [13] shows that $\operatorname{clop} X$ is a Egoroff BA.

Also, the Maharam algebras discussed in [8, §393] can be seen to be Egoroff from the result of Todorčević [8, 393S].

[16] shows $\mathbb{R}^I$ has the Egoroff property for vector lattices (which we have not defined) iff $\mathcal{P}(I)$, the power set BA ($\approx \operatorname{clop} \beta I$), is Egoroff (see also [13]), and that for $|I| = \aleph_0$ this holds, and conversely under CH ($\aleph_1 = 2^{\aleph_0}$).

[4] shows $\mathcal{P}(I)$ is Egoroff iff $|I| < \mathfrak{b}$ (the bounding number). We noted in 2.2 that [10] shows $\mathbb{R}^I$ has (σ) (iff $\mathbb{R}^I$ is strongly Egoroff, by 5.1) iff $|I| < \mathfrak{b}$.

9. Some new examples from old. We have exhibited many compact QF X with (E) (with their dual $D(X)$, which are strongly Egoroff): the compact almost P from §6, the βT from §7; the Stone spaces SA from §8.

New examples are constructed (perhaps mixing the above types) as certain $X = \sum_{i\in I} X_i$, the X_i having (E) (hence the dual $D(X) = \prod_{i\in I} D(X_i)$).

There is certainly a restriction on $|I|$ here, as evidenced by the fact mentioned earlier several times that βI has (E) (i.e., $\mathbb{R}^I$ is strongly Egoroff) iff $|I| < \mathfrak{b}$. What we can say easily goes as follows; in the discussion we always refer to $X = \sum_{i\in I} X_i$, and assume that $|X_i| \geq 2$ for each i.

Lemma 9.1. *X is almost* P (*resp.* QF) *iff each X_i is almost* P (*resp.* QF). *In each case, βX is* QF.

(This is easily proved.)

Lemma 9.2 ([10, §3]). *Suppose all X_i are compact. Then $C(X)$ has (σ) iff each $C(X_i)$ has (σ) and $|I| < \mathfrak{b}$.*

Corollary 9.3. *Suppose all X_i are compact almost P. Then βX has* (E) *iff $|I| < \mathfrak{b}$.*

Proof. Here, $C(X) \approx D(\beta X)$ (by 9.1 and 6.1), so this vector lattice has (σ) iff βX has (E) (by 5.1). Each $C(X_i)$ has (σ) of course, so $C(X)$ has (σ) iff $|I| < \mathfrak{b}$ (by 9.2). ∎

Now, analogous to 9.2, one would like to have

Conjecture 9.4. Suppose all X_i are compact QF. Then $D(\beta X)$ has (σ) (i.e., βX has (E)) iff each $D(X_i)$ has (σ) (i.e., X_i has (E)) and $|I| < \mathfrak{b}$.

But we have proved neither implication (and similar issues arise in [10, §3]). However, we have

PROPOSITION 9.5. *Suppose all X_i are compact* QF.

(a) *If $D(\beta X)$ has* (σ)*, then each $D(X_i)$ has* (σ) *(i.e., if βX has* (E)*, then each X_i does).*

(b) *If each X_i has* (E) *and $|I| \le \aleph_0$, then βX has* (E)*.*

Proof. (a) The restriction map $D(X) \ni f \mapsto f|X_i \in D(X_i)$ is a surjective vector lattice homomorphism and such a map preserves (σ).

(b) Let $I = \mathbb{N}$. If $T \in \operatorname{dcoz} \beta X$, then $T_n \equiv T \cap X_n \in \operatorname{dcoz} X_n$, so by (E), there is $S_n \in \operatorname{dcoz} X_n$ with $S_n \subseteq T_n$, so $S \in \operatorname{dcoz} \beta X$ (because $X \in \operatorname{dcoz} \beta X$, since $|I| = \aleph_0$). ∎

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