On nowhere first-countable compact spaces with countable π-weight

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Abstract. The minimum weight of a nowhere first-countable compact space of countable π-weight is shown to be κ_B, the least cardinal κ for which the real line $\mathbb{R}$ can be covered by κ many nowhere dense sets.

Keywords: π-weight; nowhere first-countable; κ_B; compact space

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1. Introduction

All spaces under discussion are Tychonoff.

In [4], the author showed that there is a (naturally defined) compact space X which is (topologically) homogeneous under $\mathsf{MA}+\neg\mathsf{CH}$ but not under CH. This space has countable π-weight, character ω_1 and weight $\mathfrak{c}$. It is an open problem whether there can be a compact nowhere first-countable homogeneous space of countable π-weight and weight less than $\mathfrak{c}$. This cannot be done by a straightforward modification of the method in [4] since from Juhász [2, Theorem 5] it follows that under MA, every compact space of countable π-weight and weight less than $\mathfrak{c}$ is somewhere first-countable. Hence a homogeneous compactum of countable π-weight and weight less than $\mathfrak{c}$ is first-countable under MA ([4, Theorem 1.5]). Let λ be the minimum weight of a nowhere first-countable compact space of countable π-weight. Clearly, $\omega_1 \leq \lambda \leq \mathfrak{c}$. The aim of this note is to show that λ is equal to κ_B, the least cardinal κ for which the real line $\mathbb{R}$ can be covered by κ many nowhere dense sets. Hence there exists a nowhere first-countable compact space of weight κ_B and countable π-weight. Whether such a space can be homogeneous while $\kappa_B < \mathfrak{c}$ remains an open problem.

2. Preliminaries

Our basic references are Miller [5], Juhász [1] and Kunen [3].

For every space X, define $\kappa_B(X)$ to be the least cardinal κ such that X can be covered by κ many nowhere dense (in X) subsets of X. In Miller [5, Lemma 1] it is shown that for every crowded Polish space X we have $\kappa_B(X) = \kappa_B$.

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Let MA_κ(countable) denote the statement that for any countable partial order $\mathbb{P}$ and family $\mathscr{F}$ of dense subsets of $\mathbb{P}$, if $|\mathscr{F}| < \kappa$, then there exists a $\mathbb{P}$-generic filter G over $\mathscr{F}$. It is well-known, see Miller [5, Lemma 2], that κ_B is the greatest κ for which MA_κ(countable) holds.

The proof of the following result is standard and is included for the sake of completeness.

Lemma 2.1 (MA_{κ^+}(countable)). *Let X be a crowded space of weight at most κ and of countable π-weight. Assume that D is a nowhere dense subset of X. Then there exist disjoint open sets U and V in X such that $D \subseteq \overline{U} \cap \overline{V}$.*

Proof: Let $\mathscr{U}$ be a countable π-base for X. Put

$$\mathbb{P} = \{\langle p, q\rangle : (p, q \in [\mathscr{U}]^{<\omega}) \,\&\, (\bigcup p \cap \bigcup q = \emptyset) \,\&\, (\overline{\bigcup p} \cup \overline{\bigcup q} \subseteq X \setminus \overline{D})\}.$$

Order $\mathscr{P}$ in the natural way by $\langle p_0, q_0\rangle \leq \langle p_1, q_1\rangle$ iff $\bigcup p_1 \subseteq \bigcup p_0$ and $\bigcup q_1 \subseteq \bigcup q_0$. Let $\mathscr{V}$ be an open base for X such that $|\mathscr{V}| \leq \kappa$. Let $\mathscr{W} = \{V \in \mathscr{V} : V \cap D \neq \emptyset\}$. For every $W \in \mathscr{W}$, put

$$W^* = \{\langle p, q\rangle \in \mathscr{P} : (\bigcup p \cap W \neq \emptyset) \,\&\, (\bigcup q \cap W \neq \emptyset)\}.$$

We claim that W^* is dense in $\mathscr{P}$. To prove this, take an arbitrary $\langle p, q\rangle \in \mathscr{P}$. By assumption, $(\overline{\bigcup p} \cup \overline{\bigcup q}) \cap \overline{D} = \emptyset$ and $W \cap D \neq \emptyset$. Since X is crowded, there exist $U, V \in \mathscr{U}$ such that

$$\overline{U} \cup \overline{V} \subseteq W \setminus (\overline{D} \cup \overline{p} \cup \overline{q}).$$

Hence $p' = p \cup U$ and $q' = q \cup V$ belong to $\mathscr{P}$ and, clearly, $\langle p', q'\rangle \leq \langle p, q\rangle$. By our assumptions, there is a filter F in $\mathbb{P}$ such that for every $W \in \mathscr{W}$ we have $W^* \cap F \neq \emptyset$. Put

$$U = \bigcup\{p : (\exists q \in [\mathscr{U}]^{<\omega})(\langle p, q\rangle \in F)\},$$

and

$$V = \bigcup\{q : (\exists p \in [\mathscr{U}]^{<\omega})(\langle p, q\rangle \in F)\},$$

respectively. Then U and V are clearly as required. $\square$

It was shown in Miller [5, Theorem 1] that κ_B has uncountable cofinality. (Interestingly, Shelah [6] showed that the measure analogue of this may fail.)

3. Proofs

Theorem 5 and Lemma 4 in Juhász [2] imply that if X is countably compact, nowhere first-countable, and has a dense set of points of countable π-character, then $w(X) \geq \kappa_B$. For completeness sake, we include a simple proof of a weaker result which suffices for our purposes.

Lemma 3.1 (Juhász [2]). *Let κ be a cardinal for which there exists a compact nowhere first-countable space X with countable π-weight and weight κ. Then $\kappa_B \le \kappa$.*

Proof: Let $\mathscr{B}$ be an open base for X such that $|\mathscr{B}| = \kappa$. Moreover, let $\mathscr{U}$ be a countable π-base for X. For every $B \in \mathscr{B}$, put

$$S(B) = \overline{B} \setminus \bigcup \{U \in \mathscr{U} : U \subseteq B\}.$$

Since $\mathscr{U}$ is a π-base, it is clear that for every $B \in \mathscr{B}$ the set $S(B)$ is a nowhere dense closed subset of X.

We claim that $\bigcup_{B\in\mathscr{B}} S(B) = X$. To this end, pick an arbitrary $x \in X$. The collection $\mathscr{V} = \{U \in \mathscr{U} : x \in U\}$ is countable. Since $\chi(x, X) > \omega$, there exists $B \in \mathscr{B}$ which contains no $U \in \mathscr{V}$. Hence for every $U \in \mathscr{U}$ which is contained in B it follows that $x \notin U$, i.e., $x \in S(B)$.

There is an irreducible continuous surjection $f\colon X \to Y$, where the weight of Y is countable. Hence Y is covered by the collection of nowhere dense closed sets

$$\{f(S(B)) : B \in \mathscr{B}\}.$$

Clearly Y is crowded since X is. From this we conclude that $\kappa_B \le \kappa$, as required. $\square$

If X is a compact space and A and B are closed subsets of X such that $A \cup B = X$, then $X(A, B)$ denotes the topological sum $(\{0\} \times A) \cup (\{1\} \times B)$ of A and B and $\pi_{A,B}\colon X(A, B) \to X$ is defined by

$$\pi_{A,B}(t) = \begin{cases} a & (t = \langle 0, a\rangle, a \in A), \\ b & (t = \langle 1, b\rangle, b \in B). \end{cases}$$

Observe that $t \in A\cap B$ if and only if $|\pi^{-1}_{A,B}(\{t\})| \ge 2$ if and only if $|\pi^{-1}_{A,B}(\{t\})| = 2$.

Lemma 3.2. *$\pi_{A,B}\colon X(A, B) \to X$ is irreducible if and only if $A \setminus B$ is dense in A and $B \setminus A$ is dense in B.*

Proof: It will be convenient to denote $\{0\} \times A$ and $\{1\} \times B$ by A' and B', respectively. Assume first that $C \subsetneq X(A, B)$ is a proper closed set such that $\pi_{A,B}(C) = X$. We may assume without loss of generality that $U = A' \setminus C$ is nonempty. Put $V = \pi_{A,B}(U)$. Then V is a nonempty relatively open subset of A. Moreover, if $x \in V$, then there exists $\langle 1, b\rangle \in B'$ such that $B \ni b = \pi_{A,B}(\langle 1, b\rangle) = x$. As a consequence, $V \subseteq B$. There is an open subset W in X such that $W \cap A = V$. Since $V \subseteq B$, obviously $W \subseteq B$. Hence $A \setminus B$ is not dense in A.

For the converse implication, assume without loss of generality that $A \setminus B$ is not dense in A. Then $(\{0\} \times \overline{A \setminus B}) \cup (\{1\} \times B)$ is a proper closed subset of $X_{A,B}$ which is mapped onto X by $\pi_{A,B}$. $\square$

Lemma 3.3. *There is a nowhere first-countable compact space of weight κ_B and countable π-weight.*

Proof: Let $\tau\colon \kappa_B \to \kappa_B$ be a surjection every fiber of which has size κ_B. Moreover, let $\{D_\alpha : \alpha < \kappa_B\}$ be a family of closed and nowhere dense subsets of 2^ω covering 2^ω. Our space will be the inverse limit X_{κ_B} of a continuous inverse system $\{X_\alpha, \beta \le \alpha < \kappa_B, f^\alpha_\beta\}$ such that $X_0 = 2^\omega$ and for every $\alpha < \kappa_B$ and $\beta \le \alpha$,

(1) X_α is a compact space of weight at most $|\alpha|\cdot\omega$,
(2) $f^\alpha_\beta\colon X_\alpha \to X_\beta$ is a continuous, irreducible surjection,
(3) there are closed sets A_α and B_α in X_α such that
   (a) $A_\alpha \cup B_\alpha = X_\alpha$,
   (b) $A_\alpha \cap B_\alpha \supseteq (f^\alpha_0)^{-1}(D_{\tau(\alpha)})$,
   (c) $A_\alpha \setminus B_\alpha$ and $B_\alpha \setminus A_\alpha$ are dense in A_α respectively B_α,
   (d) $X_{\alpha+1} = X_\alpha(A_\alpha, B_\alpha)$ and $f^{\alpha+1}_\alpha = \pi_{A_\alpha,B_\alpha}$.

The construction of this inverse sequence is a triviality by a repeated application of Lemmas 2.1 and 3.2. The only thing left to verify is that X_{κ_B} has weight κ_B and is nowhere first-countable.

Striving for a contradiction, assume that X_{κ_B} is first-countable at t. Since κ_B has uncountable cofinality (see §2), there exists $\beta < \kappa_B$ such that

$$(\dagger) \qquad (f^{\kappa_B}_\beta)^{-1}(\{f^{\kappa_B}_\beta(t)\}) = \{t\}.$$

Let $\xi < \kappa_B$ be such that $f^{\kappa_B}_0(t) \in D_\xi$. Pick $\alpha > \beta$ so large that $\tau(\alpha) = \xi$. Then clearly

$$|(f^{\alpha+1}_\alpha)^{-1}(\{f^{\kappa_B}_\alpha(t)\})| = 2,$$

which contradicts ($\dagger$).

That the weight of X_{κ_B} is at most κ_B follows by construction. And that it has weight at least κ_B is a consequence of Lemma 3.1 and the fact that it is nowhere first-countable. Observe that X_0 has countable weight, and that X_{κ_B} admits a continuous, irreducible map onto X_0. Hence X_{κ_B} has countable π-weight. □

4. Questions

(1) Is there in ZFC a homogeneous nowhere first-countable compact space of countable π-weight and weight κ_B?
(2) What are the cardinals of the form $\mathrm{w}(X)$, where X is a nowhere first-countable compactum of countable π-weight?
(Let Π denote this set of cardinals. We showed that $\kappa_B \in \Pi$. Moreover, $\mathfrak{c} \in \Pi$. To check this, let X be the absolute of the unit interval. Then X has countable π-weight, is nowhere first-countable, and has weight $\mathfrak{c}$ (since it contains a copy of $\beta\omega$). We do not know whether there can be a cardinal $\kappa \in \Pi \setminus \{\kappa_B, \mathfrak{c}\}$.)

A natural question is whether there can be a κ in Π of countable cofinality. This question may have a very simple answer. Indeed, assume that there is a sequence

$$\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$$

in Π. For every n let X_n be a witness of the fact that $\kappa_n \in \Pi$. Then $X = \prod_{n<\omega} X_n$ is a witness that $\kappa = \sup_{n<\omega} \kappa_n \in \Pi$.

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