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Topological groups with a *bc*-base

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ABSTRACT

Approximately 10 years ago, Zambakhidze asked whether every non-zero-dimensional topological group with a *bc*-base is locally compact. Below we show that the small inductive dimension ind of any non-locally compact group with such a base doesn't exceed 1. We prove, however, that a σ-compact non-locally compact topological group with a *bc*-base is zero-dimensional. Two more results in this paper are worth mentioning: 1) if the free topological group $F(X)$ of a Tychonoff space X has a *bc*-base, then $\operatorname{ind}(X) \leq 0$, and 2) a topological group G has a *bc*-base if and only if G can be compactified by a zero-dimensional remainder.

1. Introduction

We will call a base $\mathscr{B}$ of a space X a *bc-base* if the boundary $B(U) = \overline{U} \setminus U$ of every member U of $\mathscr{B}$ is compact. Spaces with a *bc*-base are also called *rimcompact*. A separable metrizable space is rimcompact if and only if it can be compactified by a zero-dimensional remainder (de Groot [10], Freudenthal [8,9]; see also [1]). Here and everywhere below we call a non-empty space X *zero-dimensional* if X has a base consisting of clopen subsets, that is, if $\operatorname{ind}(X) = 0$. We also assume all spaces considered in this article to be Tychonoff.

Clearly, if a space X is zero-dimensional or locally compact, then X has a *bc*-base.

Approximately 10 years ago, L.G. Zambakhidze asked whether every non-zero-dimensional topological group with a *bc*-base is locally compact. As far as we know, no progress has been made on this problem. In

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this note we will show that the small inductive dimension ind of non-locally compact groups with a *bc*-base is not greater than 1. To do this, we establish a statement of independent interest: every non-empty compact subspace of any non-locally compact topological group with a *bc*-base is zero-dimensional. Moreover, we show that the free topological group $F(X)$ of a space X (see [3]) has a *bc*-base if and only if X is zero-dimensional. We also formulate and prove some applications of these results.

We also have to warn the reader that, besides the small inductive dimension ind, we occasionally consider below the large inductive dimension Ind and the covering dimension dim (see [5,6]). This allows to sharpen certain of the results obtained.

2. Topological groups with a *bcs*-base

We will call a base $\mathscr{B}$ of a space X a *bcs-base* if the boundary $B(U) = \overline{U} \setminus U$ of every member U of $\mathscr{B}$ is σ-compact.

Theorem 2.1. *Suppose that G is a non-σ-compact topological group with a bcs-base, and that $G = \bigcup\{Y_i : i \in \omega\}$, where each Y_i is a separable metrizable F_σ-subspace of G. Then*

(1) *G can be written as $A \cup B$, where A and B are zero-dimensional and A is σ-compact,*
(2) $\operatorname{ind}(G) = \operatorname{Ind}(G) = \dim(G) \le 1$,
(3) *any σ-compact subspace of G is zero-dimensional.*

Proof. Let us first observe that the group G is hereditarily Lindelöf. This implies that G is strongly hereditarily normal ([6, Theorem 2.1.4]) and strongly paracompact ([6, §2.4]). Hence $\operatorname{ind} G = \operatorname{Ind} G$ ([6, Theorem 2.4.4]).

Let F be an arbitrary σ-compact subspace of G, and let $\mathscr{B}$ be a *bcs*-base for G.

For any $i \in \omega$ and $U \in \mathscr{B}$, put $q_i(U) = U \cap Y_i$.

The family $\{q_i(U) : U \in \mathscr{B}\}$ is a base of the space Y_i. Since Y_i has a countable base, it follows that there exists a countable subfamily $\mathscr{B}_i$ of the base $\mathscr{B}$ such that the family $\eta_i = \{q_i(U) : U \in \mathscr{B}_i\}$ is also a base for Y_i.

Put $E_i = \bigcup\{B(U) : U \in \mathscr{B}_i\}$, and $E = \bigcup\{E_i : i \in \omega\} \cup F$. Clearly, E is a σ-compact subspace of G. Therefore, the subgroup H of G algebraically generated in G by E is also σ-compact. Since G is not σ-compact, it follows that $G \setminus H \neq \emptyset$. Let us fix $a \in G \setminus H$.

Claim 1. *The subspace $Z = G \setminus H$ of the space G is zero-dimensional.*

To justify this claim, we invoke a few simple facts:

Fact 1. The subspace $Z_i = Z \cap Y_i = Y_i \setminus H$ is zero-dimensional.

Indeed, $\{V \cap Z_i : V \in \eta_i\}$ is a base of Z_i, and each member of this base is an open and closed subset of Z_i.

Fact 2. Each Z_i is the union of a countable family of closed separable metrizable subspaces of Z.

This is so, since Y_i is an F_σ-subspace of G, and $Z_i = Z \cap Y_i$.

Fact 3. The space Z can be represented as the union of a countable family $\{P_i : i \in \omega\}$ of closed zero-dimensional separable metrizable subspaces P_i of Z.

This follows from Facts 1 and 2, since every subspace of a zero-dimensional space is, obviously, zero-dimensional.

Since the spaces P_i in Fact 3 are separable metrizable, we have:

$$\dim(P_i) = \operatorname{ind}(P_i) = \operatorname{Ind}(P_i) \leq 0,$$

for every $i \in \omega$.

Observe that the spaces G and Z are strongly hereditarily normal. It follows that $\operatorname{Ind}(Z_i) \leq 0$, for every $i \in \omega$ ([6, Theorem 2.3.8]). Now Claim 1 follows from Fact 3.

We have: $aH \subseteq Z$. It follows that aH is also zero-dimensional. Since H is homeomorphic to aH, it follows that H is zero-dimensional. This obviously implies that $\operatorname{ind}(G) \leq 1$ and that F is zero-dimensional.

Assume first that $\dim(G) = 0$. Then $\operatorname{ind}(G) = \operatorname{Ind}(G) = 0$ since G is strongly paracompact ([6, Theorem 3.1.30]). Assume next that $\dim(G) = 1$. Then $1 = \dim(G) \leq \operatorname{ind}(G) \leq 1$ again since G is strongly paracompact ([6, Theorem 3.1.29]). From this we conclude that $\dim(G) = \operatorname{ind}(G) = \operatorname{Ind}(G) = 1$ since we already observed that $\operatorname{ind}(G) = \operatorname{Ind}(G)$. Since H is σ-compact, we are done. □

Corollary 2.2. *Let X be a non-σ-compact separable metrizable space such that the free topological group $F(X)$ has a bcs-base. Then every σ-compact subspace of $F(X)$ is zero-dimensional, and* $\dim(F(X)) = \operatorname{ind}(F(X)) = \operatorname{Ind}(F(X)) \leq 1$. *In particular,* $\dim(X) \leq 1$, *and every σ-compact subspace of X is zero-dimensional.*

Proof. It is well known that $F(X)$ can be represented in the form:

$$F(X) = \bigcup\{Y_i : i \in \omega\},$$

where each Y_i is a separable metrizable F_σ-subspace of $F(X)$. Indeed, it is enough to put $Y_i = F_{i+1}(X) \setminus F_i(X)$, where $F_i(X)$ is the subspace of $F(X)$ consisting of "words" of length $\leq i$ (see [3, Theorem 7.1.13]). The subspace of "words" of length 1 is a closed homeomorph of the topological sum of two copies of X in $F(X)$, hence $F(X)$ is non-σ-compact. Thus, $F(X)$ in the role of G satisfies the assumptions in Theorem 2.1 which is clearly as required since we already observed that X is homeomorphic to a closed subspace of $F(X)$. □

Corollary 2.3. *Erdős space E does not have a bcs-base.*

Proof. By Dijkstra and van Mill [4, Corollary 12], $E \setminus A$ and E are homeomorphic for every σ-compact subspace A of E. Since $\dim(E) = 1$ (Erdős [7]), this shows that E does not satisfy the conclusions of Theorem 2.1. □

3. Translation-disjoint sets

Subsets A and B of a topological group G will be called *translation-disjoint* if for any open neighbourhood O of the neutral element e of G there exists $c \in O$ such that cA and B are disjoint.

Proposition 3.1. *Suppose that G is a topological group, and K and Z are non-empty subspaces of G. Then at least one of the following conditions hold:*

(1) *Z and K are translation-disjoint.*
(2) *There exists an open neighbourhood O of the neutral element e of G such that*

$$O \subseteq KZ^{-1}.$$

Proof. Assume that Z and K are not translation-disjoint. Then we can fix an open neighbourhood O of e such that $(yZ) \cap K \neq \emptyset$, for each $y \in O$. Thus, the next condition holds: (3) For each $y \in O$, there exist $z \in Z$ and $x \in K$ such that $yz = x$. Then $y = xz^{-1} \in KZ^{-1}$, that is, $O \subseteq KZ^{-1}$. We conclude that (2) holds. □

The next statement obviously follows from the preceding one, since KZ^{-1} is compact whenever K and Z are compact.

Lemma 3.2. *Suppose that G is a non-locally compact topological group. Then any two compact subsets A and B of G are translation-disjoint.*

3.1. Translation-disjointness and total disconnectedness

It is convenient to generalize the concept of translation-disjointness to topological spaces. In fact, several such generalizations, introduced below, might turn out to be useful.

Suppose that X is a topological space, and A, B are subsets of X. We will say that A and B are $\langle p, 1\rangle$-*disjoint* if, for any $x, y \in A$ and any open neighbourhoods U, V of x and y, respectively, there exists a continuous mapping $f\colon A \to X$ such that $f(x) \in U$, $f(y) \in V$, and $B \cap f(A) = \emptyset$. If, in addition, we can always choose f to be a homeomorphism of A onto $f(A)$, then we say that A and B are $\langle p, 2\rangle$-*disjoint*.

Proposition 3.3. *Any two translation-disjoint subsets of an arbitrary topological group G are $\langle p, 2\rangle$-disjoint.*

We say that a space X is separated by a compact subset F of X between points p and q of X if there are disjoint open subsets U and V such that $p \in U$, $q \in V$, and $U \cup V = X \setminus F$. A space X is *separated by compacta* if for any two distinct points $p, q \in X$, the space X is separated between p and q by some compact subspace of X.

A basic fact concerning $\langle p, 1\rangle$-disjoint sets is described in the following statement:

Proposition 3.4. *Suppose that X is a topological space, and A is a subspace of X such that A is $\langle p, 1\rangle$-disjoint with any compact subspace B of X. Furthermore, suppose that X can be separated by a compact subset of X between any two distinct points of A. Then A is totally disconnected.*

Proof. Fix any two distinct points p, q in A. By the assumption, there exists a compact subset $B \subseteq X$ such that $X \setminus B = U \cup V$, where U, V are disjoint open subsets of X, and $p \in U$, $q \in V$. Since the sets A and B are $\langle p, 1\rangle$-disjoint, there exists a continuous mapping $f\colon A \to X$ such that $f(p) \in U$, $f(q) \in V$, and the sets $f(A)$ and B are disjoint. It follows from the last condition that $f(A) \subseteq U \cup V$. Therefore, $A \subseteq f^{-1}(U) \cup f^{-1}(V)$, where $A_1 = A \cap f^{-1}(U)$ and $A_2 = A \cap f^{-1}(V)$ are disjoint open subsets of A, and $p \in A_1$, $q \in A_2$. Hence, A is totally disconnected. □

3.2. Translation-disjointness in free topological groups

Suppose now that $F(X)$ is the free topological group of a non-discrete space X. For $n \in \omega$, we denote by A_n the subspace of $F(X)$ consisting of all reduced words on X with length not greater than n ([3, p. 417]). We use this notation below.

Proposition 3.5. *For any $n, m \in \omega$, the subspaces A_n and A_m are translation-disjoint in $F(X)$.*

Proof. Fix $k \in \omega$ such that $k - n > m$. Take any open neighbourhood O of the neutral element e of $F(X)$. Since $O \setminus A_k$ is non-empty, we can fix $c \in O \setminus A_k$. Then, clearly, $A_m \cap cA_n = \emptyset$. Hence, the subspaces A_n and A_m are translation-disjoint. □

Since every compact subspace B of $F(X)$ is contained in some A_m ([3, Theorem 7.5.3]), the above statement implies the next one:

Corollary 3.6. *For each $n \in \omega$, the subspace A_n is translation-disjoint with any compact subspace of $F(X)$.*

4. Translation-disjointness and zero-dimensionality

We now come to our main results.

Theorem 4.1. *Suppose that G is a topological group and that $X \subseteq G$. Furthermore, suppose that $e \in X$, and the next condition is satisfied:*

(ad) *For every open neighbourhood U of e (in G) there exists an open neighbourhood Oe of e in G such that $Oe \subseteq U$ and the boundary $\overline{Oe} \setminus Oe$ and X are translation-disjoint in G.*

Then X is zero-dimensional at e.

Proof. Take any open neighbourhood W of e in G. We have to show that there exists an open and closed neighbourhood of e in X contained in W.

There exists a symmetric open neighbourhood U of e in G such that $U^2 \subseteq W$. By the assumption, we can take an open neighbourhood V of e in G such that $V \subseteq U$ and the boundary $K = \overline{V} \setminus V$ and X are translation-disjoint in G. Clearly, $V^2 \subseteq W$. Put $H = G \setminus \overline{V}$. Obviously, the sets V, K, H are pairwise disjoint, and $G = V \cup K \cup H$. We also put $P = X \setminus W$. It is clear that $P \subseteq H$. In fact, the next statement holds:

Claim 1. *$VP \subseteq H$.*

Indeed, if $vp \in \overline{V}$ for certain $v \in V$ and $p \in P$, then $Vp \cap V \neq \emptyset$, hence p can be written in the form $v_0^{-1}v_1$ for certain $v_0, v_1 \in V$. Hence $p \in V^{-1}V \subseteq U^{-1}U = U^2 \subseteq W$, which is a contradiction.

Since X and K are translation-disjoint in G, we can find $c \in V$ such that $cX \cap K = \emptyset$. Then $cX \subseteq V \cup H$, $ce = c \in V$, and $cP \subseteq H$, by Claim 1. It follows that the set $V_c = V \cap cX$ is open and closed in cX, $c \in V_c$, and $V_c \cap cP = \emptyset$. Hence, the set $M = c^{-1}V_c = c^{-1}(V \cap cX)$ is a clopen neighbourhood of e in X such that $M \cap P = \emptyset$, i.e., $M \subseteq W$. □

Corollary 4.2. *Let G be a topological group with a bc-base, and let X be a subset of G which is translation-disjoint with every compact subset of G. Then X is zero-dimensional.*

Hence, in the light of Corollary 3.6, we get:

Corollary 4.3. *Let X be a space such that $F(X)$ has a bc-base. Then X is zero-dimensional, that is,* $\mathrm{ind}(X) \leq 0$.

Theorem 4.4. *Suppose that G is a non-locally compact topological group with a bc-base. Then every compact subspace of G is zero-dimensional.*

In fact, a slightly more general statement holds:

Theorem 4.5. *Suppose that G is a non-locally compact topological group such that any two distinct points of G can be separated by a compactum. Then every σ-compact subspace of G is zero-dimensional.*

Proof. Take any compact subset A of G. Clearly, it is enough to show that A is zero-dimensional. By Lemma 3.2, A is translation-disjoint with any compact subset of G. It follows from Propositions 3.3 and 3.4 that A is totally disconnected. Since A is compact, we conclude that $\dim(A) = 0$. Therefore, every σ-compact subspace of G is zero-dimensional by the Countable Closed Sum Theorem ([6, 3.1.8]). $\square$

Corollary 4.6. *Every σ-compact non-locally compact topological group with a bc-base is zero-dimensional.*

The next basic result immediately follows from Theorem 4.4 and the definition of small inductive dimension:

Theorem 4.7. *If G is any non-locally compact topological group with a bc-base, then* $\operatorname{ind}(G) \leq 1$.

A subset K of a topological group G will be called *k-nowhere dense in* G if the interior of $K \cdot F$ is empty, for every compact subspace F of G.

The next statement obviously follows from the results we have already obtained above.

Theorem 4.8. *If G is a topological group with a bc-base, then every k-nowhere dense subspace of G is zero-dimensional.*

Now we can improve Corollary 4.3 as follows:

Corollary 4.9. *Let X be a space such that the free topological group $F(X)$ of X has a bc-base. Then the subspace A_n of $F(X)$ consisting of reduced words of length $\leq n$ is zero-dimensional.*

However, we do not know the answer to the next question:

Problem 4.10. Suppose that the free topological group $F(X)$ of a space X has a *bc*-base. Is then $F(X)$ zero-dimensional?

In connection with the last question and Corollary 4.9 we should mention that Shakhmatov [11] constructed an example of a normal space X such that $\operatorname{ind} X = 0$ but $F(X)$ is not zero-dimensional.

4.1. Topological groups with a zero-dimensional remainder

In this part we investigate when a topological group has a zero-dimensional remainder.

Theorem 4.11. *Suppose that a non-locally compact topological group G has a zero-dimensional remainder in a compactification $b(G)$. Then*

(a) *G is rimcompact, that is, G has a bc-base;*
(b) $\operatorname{ind}(G) \leq 1$;
(c) $\operatorname{ind}(b(G)) \leq 2$.

Proof. This theorem immediately follows from Theorem 4.7, the following two obvious lemmas, and some well-known basic facts of dimension theory. $\square$

A family γ of open subsets of a space X will be called *boundary-compact* (in X) if the boundary of every member of γ is compact.

Lemma 4.12. *If a nowhere locally compact space X is zero-dimensional, then in every remainder Y of X there exists a boundary-compact (in Y) π-base.*

Lemma 4.13. *If a topological group G has a boundary-compact π-base, then G has a bc-base as well, i.e. G is rimcompact.*

Thus, Theorem 4.11 is proved.

As an application, let us consider compactifications of the space $\mathbb{Q}$ of rational numbers. The 1-dimensional sphere S^1 can be interpreted as a compactification of $\mathbb{Q}$. The remainder $S^1 \setminus \mathbb{Q}$ of $\mathbb{Q}$ in this compactification is homeomorphic to the space $\mathbb{J}$ of irrational numbers. Notice that $\operatorname{ind}(\mathbb{J}) = 0$, and $\mathbb{J}$ is homeomorphic to a topological group. In this connection we mention the next easy to establish but curious fact:

Proposition 4.14. *If a zero-dimensional remainder Y of $\mathbb{Q}$ is homeomorphic to a topological group, then Y is homeomorphic to the space $\mathbb{J}$ of irrational numbers.*

We also obtain from Theorem 4.11 the following:

Corollary 4.15. *If $b(\mathbb{Q})$ is any compactification of $\mathbb{Q}$ such that the remainder $Y = b(\mathbb{Q}) \setminus \mathbb{Q}$ satisfies the condition $\operatorname{ind}(Y) \geq 2$, then Y is not homeomorphic to any topological group.*

Problem 4.16. Does there exist a compactification $b\mathbb{Q}$ of $\mathbb{Q}$ such that the remainder $Y = b(\mathbb{Q}) \setminus \mathbb{Q}$ is homeomorphic to a 1-dimensional topological group?

4.2. Translation-disjointness and some local properties

Recall that a space X is of *countable type* if every compact subspace of X is contained in a compact subspace with a countable base of open neighbourhoods in X.

Theorem 4.17. *Suppose that G is a topological group with a bc-base. Then at least one of the following conditions holds:*

(i) *Every closed subspace Z of G of countable type is zero-dimensional.*
(ii) *G is a paracompact p-space.*

Proof. Assume that (i) does not hold. Then, by Corollary 4.2, Z is not translation-disjoint with some compact subspace K of G. By Proposition 3.1, there is an open neighbourhood O of the neutral element e of G such that O is contained in the subspace $Y = KZ^{-1}$ of G. The natural mapping f of the space $K \times Z$ onto Y is perfect (Arhangel'skii [2, Corollary 5]). Since $K \times Z$ is of countable type, it follows that Y is of countable type as well. Since $O \subseteq Y$, we conclude that G is locally of countable type. Hence, G is of pointwise countable type. Therefore, G is a paracompact p-space, since it is a topological group. □

The next statement has a similar proof:

Theorem 4.18. *Suppose that G is a topological group, and A, B are any two subspaces of G with a countable network.. Then either A and B are translation-disjoint, or G has locally a countable network.*

Clearly, the following statement also holds:

Theorem 4.19. *Suppose that G is a topological group with a base $\mathscr{B}$ such that the boundary of every member U of $\mathscr{B}$ has a countable network. Then G satisfies at least one of the following conditions:*

(α) *G has locally a countable network.*
(β) *If a subspace Y of G has a countable network, then Y is totally disconnected.*

4.3. Some open questions

The next question of L.G. Zambakhidze remains the main open problem in the field:

Problem 4.20. Is every non-locally compact topological group with a *bc*-base zero-dimensional?

Problem 4.21. Is every metrizable non-locally compact topological group with a *bc*-base zero-dimensional?

A similar question can be formulated about topological groups that are paracompact p-spaces.

Theorem 4.5 suggests that the answer to the next question may be in the affirmative.

Problem 4.22. Is every non-locally compact topological group with a *bc*-base totally disconnected?

Observe that the proof of Corollary 4.2 does not provide any information on the spaces such that their free topological group has a *bcs*-base. Indeed, $F(X)$ is σ-compact provided X is.

The next open questions point in a somewhat different direction than all other questions in the paper as well as the results obtained in it.

Problem 4.23. Let $F(X)$ be the free topological group of a space X. Then is it possible to find a zero-dimensional subspace Y of $F(X)$ such that $KY = F(X)$, for some compact subspace K of $F(X)$?

Problem 4.24. Let $F(X)$ be the free topological group of a compact (metrizable) space X. Then is it possible to find a zero-dimensional subspace Y of $F(X)$ such that $KY = F(X)$, for some compact subspace K of $F(X)$?

Problem 4.25. Given a topological group G, when is it possible to find a zero-dimensional subspace Y of G such that $KY = G$, for some compact subspace K of G?

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