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UNIONS OF F-SPACES

KLAAS PIETER HART, LEON LUO, AND JAN VAN MILL

Abstract. We show that every space that is the union of a 'small' family consisting of special P-sets that are F-spaces, is an F-space. We also comment on the sharpness of our results.

Introduction

We assume that every space is Tychonoff unless specified otherwise, and βX and X^* stand for the Čech-Stone compactification and the Čech-Stone remainder of X respectively. A space is an F-space if disjoint cozero-subsets are contained in disjoint zero-subsets. Equivalently, X is an F-space if every cozero-subset of X is C^*-embedded in X. The study of F-spaces has a long history since the late 1950's [5]. For basic information on F-spaces, see [5], [6] and [8].

It is proven in [4] that each union of ω_1 many cozero-subsets of an F-space is again an F-space. Hence under the Continuum Hypothesis (abbreviated CH) each open subspace of an F-space of weight $\mathfrak{c}$ is again an F-space. In [1] an example was constructed of a compact F-space with weight $\omega_2 \cdot \mathfrak{c}$ that has an open subspace that is not an F-space. Hence CH is equivalent to the statement that each open subspace of an F-space with weight $\mathfrak{c}$ is again an F-space. See [1], [2] and [3] for more related results.

These results have motivated us to study the question "when is the union of F-subspaces again an F-space" more closely. In this note it is shown that if a space can be covered by a family of ω_1 many special P-sets, then it is an F-space. We shall also use the examples in [1, 2] to discuss the sharpness of our result.

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1. Preliminaries

A closed subset A of a space X is called a P-set if the intersection of any countable family of neighborhoods of A is again a neighborhood of A. If A is a singleton subset of X, then the point in it is usually referred to as a P-point.

Definition 1.1. A space is a P-space if every point is a P-point.

Definition 1.2. A closed subset A of a space X is called *nicely placed* in X if for every open neighborhood U of A there is a cozero-subset V of X such that $A \subseteq V \subseteq U$.

Definition 1.3. A subset A of a space X is said to be C^*-embedded in X if for each continuous function $f : A \to \mathbb{I}$, there is a continuous extension $\bar{f} : X \to \mathbb{I}$ of f.

Proposition 1.4 ([8, 1.61])**.** *A C^*-embedded subspace of an F-space is an F-space.*

If X is a set, and κ is a cardinal number, then $[X]^\kappa$ denotes $\{A \subseteq X : |A| = \kappa\}$.

2. Unions of F-spaces

In this section we present our main result on unions of F-subspaces. In the next sections we will comment on its sharpness.

Theorem 2.1. *Let X be a space with a cover $\mathcal{F}$ that consists of not more than ω_1 many P-subsets, each of which is a nicely placed C^*-embedded F-subspace of X. Then X is an F-space.*

Proof. Let U be a cozero-subset of X, and let $f : U \to \mathbb{I}$ be continuous. Enumerate $\mathcal{F} \cup \{\emptyset\}$ as $\{F_\alpha : \alpha < \omega_1\}$ where $F_0 = \emptyset$. We shall construct, by transfinite recursion, for each $\alpha < \omega_1$, a cozero subset V_α of X and a continuous function $f_\alpha : V_\alpha \to \mathbb{I}$ such that

(1) $V_0 = U$ and $f_0 = f$;
(2) $F_\alpha \subseteq V_\alpha$;
(3) if $\beta < \alpha$ then $V_\beta \subseteq V_\alpha$ and $f_\alpha \restriction V_\beta = f_\beta$.

Suppose that we have constructed V_β and f_β for all $\beta < \alpha$ where $\alpha < \omega_1$. Put $V = \bigcup_{\beta<\alpha} V_\beta$ and $g = \bigcup_{\beta<\alpha} f_\beta$. Clearly, V is a cozero-subset of X and g is continuous on V. Let $h = g \restriction F_\alpha$. Since $V \cap F_\alpha$ is a cozero-subset of F_α and F_α is an F-space, we can extend h to a continuous function $\xi : F_\alpha \to \mathbb{I}$. Moreover, since F_α is C^*-embedded in X, we can extend ξ to a continuous function $\eta : X \to \mathbb{I}$.

We claim that there is a cozero-subset W of X such that $F_\alpha \subseteq W$ and
$$g \restriction (W \cap V) = \eta \restriction (W \cap V).$$
Indeed, we write V as $\bigcup_{n<\omega} A_n$, where each A_n is closed in X. For all $n < \omega$ and $k \geq 1$, let
$$A_n^k = \{x \in A_n : |g(x) - \eta(x)| \geq 2^{-k}\}.$$
Clearly, A_n^k is closed in X and disjoint from F_α since $g \restriction (F_\alpha \cap V) = \eta \restriction (F_\alpha \cap V)$. As F_α is a nicely placed P-subset of X there is a cozero subset W of X such that
$$F_\alpha \subseteq W \subseteq X \setminus \bigcup_{n<\omega} \bigcup_{k\geq 1} A_n^k.$$
It is clear that W is as required. Now put $V_\alpha = V \cup W$ and $f_\alpha = g \cup (\eta \restriction W)$.

At the end of the recursion we let $\bar{f} = \bigcup_{\alpha<\omega_1} f_\alpha$; this is the desired continuous extension of f. □

Remark 2.2. The referee noticed, as did we, that in the proof of Theorem 2.1 we only need that X is covered by a family $\mathcal{F}$ such that $|\mathcal{F}| \leq \omega_1$ and each element F of $\mathcal{F}$ has the property that its closure, B_F, in βX is both a P-set in βX and an F-space. Indeed, by compactness, each B_F is nicely placed and C^*-embedded in βX. Hence $Y = \bigcup_{F\in\mathcal{F}} B_F$ is an F-space by Theorem 2.1. But then X is an F-space as well since it is clearly C^*-embedded in Y. When writing the paper, we decided not to formulate Theorem 2.1 in this form since the condition that each B_F is both a P-set and an F-space is not an 'internal' one: our theorem gives a condition under which building blocks that are F-spaces yield an F-space, whereas the other formulation would show when building blocks that need not be F-spaces combine into an F-space.

But it is potentially a weaker condition than the ones that we stated in Theorem 2.1 and so we believe that it should be studied more closely.

3. The first example

We shall describe an example of a locally compact space that is not an F-space yet it admits a clopen cover of size ω_2 consisting of compact zero-dimensional F-spaces. This shows that Theorem 2.1 is false for unions of families of size ω_2. Our example is a modification of the example in [1].

Our starting point is the compact space G obtained from the topological sum of $\omega^* \times (\omega_1 + 1)$ and $\beta\omega$ by identifying the points $\langle u, \omega_1 \rangle$ and u, for every point u of ω^*.

Observe that after this identification ω is an open F_σ-subset of G and that $\beta\omega$ is a P-set of character ω_1 in G. Moreover, the weight of G is equal to $\mathfrak{c}$ and G is zero-dimensional.

Our next step is to put $Y = \omega \times G$. Let $\pi : Y \to G$ denote the projection map and let $\pi^* : Y^* \to G$ be the restriction of the Stone extension of π. As π^* is closed the preimage $(\pi^*)^{\leftarrow}[\beta\omega]$ is not open since $\beta\omega$ is not open in G.

The space Y^* is a compact zero-dimensional F-space of weight $\mathfrak{c}$ and $(\pi^*)^{\leftarrow}[\beta\omega]$ is a P-set of character ω_1. The problem is that $(\pi^*)^{\leftarrow}[\omega]$ is not dense in $(\pi^*)^{\leftarrow}[\beta\omega]$. To remedy this let f be the restriction of π^* to $(\pi^*)^{\leftarrow}[\beta\omega]$. Now f maps the closed P-set $(\pi^*)^{\leftarrow}[\beta\omega]$ onto the compact F-space $\beta\omega$. Hence [6, Lemma 1.4.1] applies to show that the adjunction space $\Omega = Y^* \cup_f \beta\omega$ is a compact F-space of weight $\mathfrak{c}$. It is also easily seen to be zero-dimensional. Thus we have replaced $(\pi^*)^{\leftarrow}[\beta\omega]$ in Y^* by (a copy of) $\beta\omega$; in this way we get an open F_σ-subset C in Ω whose closure is a P-set of character ω_1: let $C = \omega$.

We can give an explicit increasing sequence $\langle V_\alpha : \alpha \in \omega_1 \rangle$ of clopen sets in Ω such that $\omega \setminus \operatorname{cl}_\Omega C$ is equal to $\bigcup_{\alpha\in\omega_1} V_\alpha$. Indeed, in G we have the clopen initial segments of $\omega^* \times (\omega_1 + 1)$: put $G_\alpha = \omega^* \times (\alpha + 1)$ for each α. These are transported into Y^*, and hence into Ω, by taking preimages: let $V_\alpha = (\pi^*)^{\leftarrow}[G_\alpha]$ for all α.

Now we perform the same construction as in [1] with ω^* replaced by Ω. Let X be $\omega_2 + 1$ endowed with G_δ-topology. We observe that $X \times \Omega$ is an F-space by [7], and that its weight is equal to $\omega_2 \cdot \mathfrak{c}$. This implies that $K = \beta(X \times \Omega)$ is an F-space as well and its weight is equal to $(\omega_2 \cdot \mathfrak{c})^\omega = \omega_2 \cdot \mathfrak{c}$.

Next let $L = \{\alpha \in \omega_2 + 1 : \operatorname{cf} \alpha \geq \omega_1\}$. We let T be the closure in K of $L \times C$; note that $T = \operatorname{cl}_K(L \times \operatorname{cl}_\Omega C)$ also. The complement U of T in K is our example.

That U is not an F-space is proven in exactly the same way as in [1].

To finish we show that U is the union of ω_2 many clopen subset of K. Each of these is trivially a nicely placed and C-embedded P-set, and an F-space because K is.

The first ω_1 many clopen sets are the closures $\operatorname{cl}_K(X \times V_\alpha)$, for $\alpha \in \omega_1$; these cover the points of U that do not belong to $\operatorname{cl}_K(X \times \operatorname{cl}_\Omega C)$, as we shall see presently.

The other ω_2 many clopen sets will appear in the course of the following argument. Let $u \in U$ and let W be a clopen neighbourhood of u in K that is disjoint from T. We let $A = \bigl\{\alpha \in X \setminus L : (\exists m \in \operatorname{cl}_\Omega C)(\langle \alpha, m\rangle \in W)\bigr\}$; note that, because W is clopen, it is even the case that $W \cap (\{\alpha\} \times C) \neq \emptyset$ whenever $\alpha \in A$.

Claim 1. A is countable.

Proof. If A is uncountable then, as a set or ordinals, it has an initial segment of order type ω_1; we simply assume that the order type of A itself is ω_1. Let $\beta = \sup A$. Then $\beta \in L$. Moreover, for every $\alpha \in A$, pick, by the above remark, an element $m_\alpha \in C$ such that $\langle \alpha, m_\alpha \rangle \in W$. Since C is countable there is an $m \in C$ such that $M = \{\alpha : m_\alpha = m\}$ has cardinality ω_1. This then implies that $\langle \beta, m \rangle \in W \cap (L \times C)$, a contradiction. $\square$

Claim 2. There exists $\alpha < \omega_1$ such that

$$W \cap (X \times \Omega) \subseteq (X \times V_\alpha) \cup (A \times \Omega).$$

Proof. To begin we observe that for every $\gamma \in X \setminus A$ there is an α such that

$$W \cap (\{\gamma\} \times \Omega) \subseteq \{\gamma\} \times V_\alpha.$$

This follows because $W \cap (\{\gamma\} \times \Omega)$ is compact and disjoint from $\{\gamma\} \times \mathrm{cl}_\Omega C$.

We claim that for each α the set $O_\alpha = \{\gamma \notin A : (\{\gamma\} \times \Omega) \cap W \subseteq \{\gamma\} \times V_\alpha\}$ is open in X. Indeed, $X \setminus O_\alpha = A \cup \pi_X[W \cap (X \times (\Omega \setminus V_\alpha))]$, and this set closed because A is closed and because the projection $\pi_X : X \times (\Omega \setminus V_\alpha) \to X$ is closed (by compactness of $\Omega \setminus V_\alpha$).

By repeated application of the pressing-down lemma one readily proves that $X \setminus A$ is Lindelöf, so that there is $\beta \in \omega_1$ such that $X \setminus A \subseteq \bigcup_{\alpha<\beta} O_\alpha$.

But this then implies that $W \cap (X \times \Omega) \subseteq (X \times V_\beta) \cup (A \times \Omega)$. $\square$

Since $(X \times V_\beta) \cup (A \times \Omega)$ is clopen in $X \times \Omega$ we see that $W \subseteq \mathrm{cl}_K(X \times V_\beta) \cup \mathrm{cl}_K(A \times \Omega)$.

From this we extract our second family of clopen sets: all sets of the form $\mathrm{cl}_K(A \times \Omega)$ for countable $A \subseteq X \setminus L$.

We finish by observing that $[X \setminus L]^\omega$ has a cofinal subfamily $\mathcal{A}$ of cardinality ω_2: for each $\alpha \in \omega_2$ the set $[\alpha \setminus L]^\omega$ has a cofinal subfamily $\mathcal{A}_\alpha$ of cardinality ω_1, obtained via an injection from α into ω_1. Then $\mathcal{A} = \bigcup_{\alpha<\omega_2} \mathcal{A}_\alpha$ is as required.

Hence the clopen families $\{\mathrm{cl}_K(X \times V_\alpha) : \alpha \in \omega_1\}$ and $\{\mathrm{cl}_K(A \times \Omega) : A \in \mathcal{A}\}$ is the required cover of U.

4. The second example

We shall describe an example of a space that admits a cover of size ω_1 consisting of C^*-embedded F-subspaces that are P-sets yet it is not an F-space. This shows that Theorem 2.1 is false for unions of P-sets that are not nicely placed. The space is Example 1.9 from [2].

Let $X = \omega_1 \cup \{p\}$, where neighborhoods of p are cocountable and ω_1 is discrete. Let $S = \omega_1 \times \omega^*$, where again ω_1 has the discrete topology. Let $C \subseteq \omega^*$ be a cozero subset whose closure is not a zero-set.

For $\alpha \in \omega_1$, let $C_\alpha = \{\alpha\} \times C$, and put

$$K = \bigcap_{\alpha \in \omega_1} \mathrm{cl}_{\beta S} \left(\bigcup_{\gamma > \alpha} C_\gamma \right).$$

Then $Y = \beta S \setminus K$ is a locally compact F-space, and $X \times Y$ is not an F-space [2].

The crucial property of Y is the following: if for each α one takes a zero subset Z_α of $\{\alpha\} \times \omega^*$ that contains C_α then

$$Y \cap \bigcap_{\alpha \in \omega_1} \mathrm{cl}_{\beta S} \left(\bigcup_{\gamma > \alpha} Z_\gamma \right) \neq \emptyset. \tag{†}$$

Lemma 4.1. *Let x be a P-point in a space D and let E be a locally compact space. Then $\{x\} \times E$ is a P-set in $D \times E$.*

Proof. Let F be an F_σ-subset of $D \times E$ which is disjoint from $\{x\} \times E$, we show that $\mathrm{cl}\, F$ is also disjoint from $\{x\} \times E$.

To this end let $y \in E$ and let C be a compact neighborhood of y in E. The projection map $\pi_D : D \times C \to D$ is closed, hence $H = \pi_D[F \cap (D \times C)]$ is an F_σ-subset of D that does not contain x. Hence $U = D \setminus \mathrm{cl}\, H$ is a neighborhood of x since x is a P-point in D. So the product of U and the interior of C is a neighborhood of $\langle x, y \rangle$ that is disjoint from F, so that $\langle x, y \rangle \notin \mathrm{cl}\, F$. □

From this Lemma we conclude that the collection

$$\big\{ \{x\} \times Y : x \in X \big\}$$

consists of P-subsets of $X \times Y$ that are themselves F-spaces and clearly C^*-embedded. Since $X \times Y$ is not an F-space, at least one of them cannot be nicely placed by Theorem 2.1. Since $\{q\} \times Y$ is clopen in $X \times Y$ for every $q \in X \setminus \{p\}$ the only candidate for such P-set is $E = \{p\} \times Y$. It is instructive to provide a direct argument that E is not nicely placed in $X \times Y$.

To this end put

$$A = \bigcup_{\alpha \in \omega_1} \{\alpha\} \times C_\alpha.$$

It was shown in the proof of Theorem 1.7 in [2] that A is a cozero subset of Y. Since A is disjoint from the P-set E there is a neighbourhood O of E that is disjoint from A. If E were nicely placed in $X \times Y$ then there would be a cozero-set V in $X \times Y$ such that $E \subseteq V \subseteq O$. Hence $Z = (X \times Y) \setminus V$ is a zero-set in $X \times Y$ that contains A but misses E.

For every $\alpha < \omega_1$, put $Z_\alpha = Z \cap (\{\alpha\} \times \omega^*)$, this is a zero-set in $\{\alpha\} \times \omega^*$ that contains C_α. By (†) the intersection

$$Y \cap \bigcap_{\alpha \in \omega_1} \mathrm{cl}_{\beta S} \left(\bigcup_{\gamma > \alpha} Z_\gamma \right)$$

is nonempty. This intersection is a subset of $Z \cap E$ which was assumed to be empty.

5. The third example

The only question left is whether the hypothesis of being C^*-embedded is essential for Theorem 2.1. Unfortunately, we are unable to answer this question. A simpler question is: Is it true that every P-subset which is nicely placed in an F-space is C^*-embedded in that space? If the answer is positive, the condition on C^*-embeddedness in Theorem 2.1 would be superfluous. We can show that the assumption $2^{\omega_1} = \omega_2$ implies the answer is negative.

The equality $2^{\omega_1} = \omega_2$ implies that there is a maximal almost disjoint family on ω_1 of cardinality 2^{ω_1}, that is, a collection $\mathcal{A}$ of subsets of ω_1 with the following properties:

(1) $\mathcal{A} \subseteq [\omega_1]^{\omega_1}$,
(2) if $A, B \in \mathcal{A}$ are distinct, then $|A \cap B| \leq \omega$,
(3) $\mathcal{A}$ is maximal with respct to the properties (1) and (2),
(4) $|\mathcal{A}| = 2^{\omega_1}$.

Let X be $\omega_1 \cup \mathcal{A}$ and topologize X in the standard way as follows: the points of ω_1 are isolated and a neighborhood of $A \in \mathcal{A}$ contains $\{A\}$ and all but countably many elements from A. Then X is a P-space, and by Jones' Lemma, the set $\mathcal{A}$ is not C^*-embedded in X. However, by maximality of $\mathcal{A}$, every neighborhood of $\mathcal{A}$ has a countable complement and is therefore clopen. So, every neighborhood of $\mathcal{A}$ is clopen, and therefore $\mathcal{A}$ is nicely placed in the P-space X for trivial reasons.

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(K. P. Hart, L. Luo and J. van Mill) Faculty of Electrical Engineering, Mathematics and Computer Science, TU Delft, Postbus 5031, 2600 GA Delft, The Netherlands
E-mail address: k.p.hart@tudelft.nl
E-mail address: l.luo@tudelft.nl
E-mail address: j.van.mill@vu.nl

(J. van Mill) Faculty of Sciences, VU University Amsterdam, De Boelelaan 1081A, 1081 HV Amsterdam, The Netherlands

(J. van Mill) Department of Mathematical Sciences, University of South Africa, P. O. Box 392, 0003 Unisa, South Africa