Houston Journal of Mathematics
© 2014 University of Houston
Volume 40, No. 2, 2014

# ON UNIQUELY HOMOGENEOUS SPACES, II

ALEXANDER ARHANGEL'SKII AND JAN VAN MILL

Abstract. It is shown that there is an example of a uniquely homogeneous separable metrizable space that is Abelian but not Boolean. It is also shown that such an example cannot be a Baire space. This answers several problems on (unique) homogeneity.

## 1. Introduction

*All spaces under discussion are Tychonoff. By a homeomorphism of $X$ we will always mean a homeomorphism of $X$ onto itself. For a function $f\colon X \to Y$ such that $f\colon X \to f(X)$ is a homeomorphism and $f(X) \neq Y$, we use the term embedding.*

A space $X$ is called *uniquely homogeneous* provided that for all $x, y \in X$ there is a unique homeomorphism of $X$ that takes $x$ onto $y$. This concept is due to Burgess [9] who asked in 1955 whether there exists a non-trivial uniquely homogeneous metrizable continuum. Ungar [19] showed in 1975 that there are no such finite-dimensional metrizable continua and a few years later, Barit and Renaud [5] showed that the assumption on finite-dimensionality is superfluous. A somewhat different argument was given by Keesling and Wilson [13]. A nontrivial uniquely homogeneous Baire space of countable weight was constructed by van Mill [16]. This example is a topological group. There are also uniquely homogeneous spaces that do not admit the structure of a topological group, [17]. It is unknown whether there is a non-trivial Polish uniquely homogeneous space.

In Arhangel′skii and van Mill [2], the authors identified two properties of topological spaces called *skew-2-flexibility* and *2-flexibility* respectively that are useful in studying unique homogeneity. It was shown among other things that every locally compact homogeneous metrizable space is both skew-2-flexible and 2-flexible

2000 *Mathematics Subject Classification.* 54C05, 54H11, 54H15, 54G20.

*Key words and phrases.* Homogeneity, unique homogeneity, semitopological group, quasitopological group, topological group.

and that there is an example of a homogeneous Polish space that is skew-2-flexible but not 2-flexible. In addition, in the presence of unique homogeneity, 2-flexibility for $X$ is equivalent to $X$ being *Abelian*, i.e., all homeomorphisms of $X$ commute. Moreover, in the presence of unique homogeneity, skew 2-flexibility for $X$ implies 2-flexibility and is equivalent to $X$ being *Boolean*, i.e., all homeomorphisms on $X$ are involutions. This left open the question whether in the class of uniquely homogeneous spaces, 2-flexibility and skew 2-flexibility are equivalent notions. The aim of this paper is to answer this question in the negative by constructing a uniquely homogeneous (separable metrizable) space $X$ that is Abelian but not Boolean. In fact, no homeomorphism on $X$ except for the identity is an involution.

Our example is not a Baire space. We will also prove that such an example cannot be a Baire space so that what we have seems to be optimal.

## 2. Preliminaries

**(A) Groups.** A *semitopological group* (respectively, *paratopological group*) is a group endowed with a topology for which the product is separately (respectively, jointly) continuous. See Bouziad [8], Arhangel'skii and Choban [1], and [4] for conditions guaranteeing that a semitopological group (respectively, paratopological group) is a topological group.

For an Abelian group $G$ and $A \subseteq G$ we let $\langle\!\langle A\rangle\!\rangle$ denote the subgroup of $G$ generated by $A$. Moreover, for a subgroup $A$ of $G$ we let $[\![A]\!]$ denote the subgroup

$$\{x \in G : (\exists\, n \in \mathbb{Z})(nx \in A)\}$$

of $G$. Observe that if $G$ is a torsion-free Abelian group and $A \subseteq G$ is a countable subgroup, then $[\![A]\!]$ is countable as well.

Let $G$ be a torsion-free Abelian group. A subset $A$ of $G \setminus \{0\}$ is *algebraically independent* if for all pairwise distinct $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n \in \mathbb{Z}$ such that $\sum_{i=1}^n m_i a_i = 0$ we have $m_1 = \cdots = m_n = 0$. Observe that every uncountable set $A$ in a torsion-free Abelian group $G$ contains an uncountable algebraically independent subset. For if $B \subseteq A$ is countable, then so are $C = \langle\!\langle B\rangle\!\rangle$ and $D = [\![C]\!]$. Hence no maximal algebraically independent subset of $A$ is countable.

If $X$ is uniquely homogeneous, for all $x, y \in X$ we let $f^x_y$ denote the unique homeomorphism of $X$ that sends $x$ to $y$. For a fixed $e \in X$, define a binary operation $X \times X \to X$ by $x{\cdot}y = f^e_x(y)$. This is a group operation on $X$ having the property that all left translations of $X$ are homeomorphisms of $X$. That is, $X$ is a left topological group. For details, see [2, Proposition 4.1]. This group operation is called the *standard group operation on $X$*.

**(B) Topology.** We will need van Douwen's [10, 4.2] generalization of a classical result due to Souslin ([15, p. 437]).

**Theorem 2.1.** *Let $X$ and $Y$ be Polish spaces, and let $\mathcal{F}$ be a countable family of continuous functions from $X$ to $Y$ such that:*

$$\text{for every countable } A \subseteq Y: \ \{f^{-1}(A) : f \in \mathcal{F}\} \text{ does not cover } X.$$

*Then there exists a Cantor set $K \subseteq X$ such that $f \restriction K$ is injective for every $f \in \mathcal{F}$.*

Let $X$ be a space. We say that a subset $A$ of $X$ is a *bi-Bernstein set* (abbreviated: BB-set) in $X$ if $A$ as well as $X \setminus A$ intersects every Cantor set in $X$. Observe that a BB-set in $X$ intersects every Cantor set in a set of size $\mathfrak{c}$, since we can split every Cantor set in a family consisting of $\mathfrak{c}$ pairwise disjoint Cantor sets.

We let $K$ denote the standard Cantor set in $\mathbb{I}$.

**(C) Measurable functions.** We let $\mathbb{I}$ denote the closed unit interval $[0,1]$. Let $X$ be a space. A function $f\colon \mathbb{I} \to X$ is said to be *measurable* if $f^{-1}(U)$ is a Borel subset of $\mathbb{I}$ for every open subset $U$ of $X$. We are particularly interested in countable spaces. Observe that if $X$ is countable, then $f\colon \mathbb{I} \to X$ is measurable if and only if $f^{-1}(x)$ is Borel for every $x \in X$. Measurable functions $f, g\colon \mathbb{I} \to X$ are said to be *equivalent* provided that

$$\lambda(\{t \in \mathbb{I} : f(t) \neq g(t)\}) = 0,$$

where $\lambda$ denotes Lebesgue measure on $\mathbb{R}$.

Let $\mathcal{F}$ denote the collection of all measurable functions from $\mathbb{I}$ to $\mathbb{R}$.

The sequence $(f_n)_n$ in $\mathcal{F}$ *converges to zero in measure* if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \lambda(\{t \in \mathbb{I} : |f_n(t)| \geq \varepsilon\}) = 0.$$

Let $(f_n)_n$ be a sequence in $\mathcal{F}$. Then $(f_n)_n$ converges *to zero almost everywhere* if there exists a set $E$ of measure zero such that for every $x$ not in $E$ and $\varepsilon > 0$ there exists $n_0$ such that $|f_n(x)| < \varepsilon$ for every $n \geq n_0$.

These concepts are known to be related as follows. For completeness sake, we provide a sketch of proof of it below.

**Lemma 2.2.** *A sequence of functions $(f_n)_n$ in $\mathcal{F}$ converges to zero in measure if and only if every subsequence of $(f_n)_n$ contains a subsequence which converges to zero almost everywhere.*

Proof. Indeed, first assume that $(f_n)_n$ converges to zero in measure. Every subsequence of $(f_n)_n$ converges to zero in measure, hence is fundamental in measure by [12, Theorem C on page 92]. Hence some subsequence of it is almost uniformly fundamental in measure by [12, Theorem D on page 93]. But then this subsequence converges to zero almost everywhere by [12, Theorem B on page 89].

Conversely, assume that every subsequence of $(f_n)_n$ contains a subsequence which converges to zero almost everywhere. If $(f_n)_n$ does not converge to zero in measure, then there exist $\varepsilon > 0$, a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and $\delta > 0$ such that for every $k$,

$$\lambda(\{t \in \mathbb{I} : |f_{n_k}(t)| \geq \varepsilon\}) \geq \delta.$$

But this subsequence clearly does not have a subsequence that converges to zero almost everywhere. $\square$

**(D) The spaces $FM_X$.**

By $M_X$ we denote the space consisting of all equivalence classes of measurable functions from $\mathbb{I}$ into $X$ endowed with the topology of convergence in measure. For a measurable function $f$ we let $[f]$ denote its equivalence class.

The topology on $M_X$ is induced by the metric

$$\hat{d}([f],[g]) = \sqrt{\int_0^1 d(f(t), g(t))^2 dt}, \tag{1}$$

where $d$ is any admissible bounded metric on $X$. The topology on the space $M_X$ is independent of the (bounded) metric that is chosen to induce its topology. For completeness sake and for later use, we repeat the argument in [6, p. 192].

**Lemma 2.3.** *For a sequence $([f]_n)_n$ in $M_X$ and an element $[f]$ in $M_X$, the following statements are equivalent:*

1. *$([f]_n)_n$ converges to $[f]$,*
2. *every subsequence of the sequence $(f_n)_n$ contains a subsequence that converges pointwise to $f$ almost everywhere.*

PROOF. Simply observe that $\lim_{n\to\infty} \hat{d}([f]_n, [f]) = 0$ if and only if the sequence of functions $\big(\xi_n : t \mapsto d(f_n(t), f(t))\big)_n$ converges to zero in measure if and only if every subsequence of $(\xi_n)_n$ contains a subsequence that converges to zero almost everywhere (Lemma 2.2). But this is equivalent to the statement that every subsequence of $(f_n)_n$ contains a subsequence that converges pointwise to $f$ almost everywhere. $\square$

Since the Lemma 2.3(2) does not mention metrics, we see that indeed the topology on $M_X$ is independent of the chosen (bounded) metric on $X$.

**Corollary 2.4.** *Let $\varphi\colon X \to Y$ be a homeomorphism. Then the function $\bar{\varphi}\colon M_X \to M_Y$ defined by $\bar{\varphi}([f]) = [\varphi \circ f]$ is a homeomorphism.*

Our main interest is in the subspace

$$FM_X = \{[f] \in M_X : (\exists\, g \in [f])(\text{range}(g) \text{ is finite})\}$$

of $M_X$.

The function $x \mapsto [f_x]$, where $f_x \colon \mathbb{I} \to X$ is the constant function with value $x$, maps $X$ isometrically onto a closed subset of $M_X$. For this we only need to prove that the set $\{[f_x] : x \in X\}$ is closed in $M_X$. But this is easy. For suppose that for $f \colon \mathbb{I} \to X$ we have that $[f]$ is not the equivalence class of a constant (function). Then there are two distinct elements $x, y \in X$ such that $\delta_x = \lambda(f^{-1}(x)) > 0$ and $\delta_y = \lambda(f^{-1}(y)) > 0$. Any measurable $g \colon \mathbb{I} \to X$ such that $\hat{d}([f],[g]) < \min\{\frac{1}{2}d(x,y)\cdot\delta_x, \frac{1}{2}d(x,y)\cdot\delta_y\}$ is not equivalent to a constant function, which does the job.

Bessaga and Pełczyński proved the following fundamental fact about these spaces.

**Theorem 2.5** (Bessaga and Pełczyński [6, Theorem 7.1])**.** *The space $M_X$ is homeomorphic to the separable Hilbert space $\ell^2$ if and only if $X$ is Polish and contains more than one point.*

## 3. The group

Let $G$ be the subgroup of $\mathbb{R}$ consisting of all rational numbers, i.e., $G = \mathbb{Q}$.

We endow $G$ with the Sorgenfrey topology. That is, we take the collection of all intervals of the form $[x, y)$, where $x, y \in G$ and $x < y$, as an open base. Observe that $G$ with this topology is an Abelian paratopological group, but that inversion is (badly) discontinuous. Moreover, the Sorgenfrey base is countable since $G$ is, hence $G$ is metrizable.

Since $G$ is obviously dense in itself, $G$ is homeomorphic to $\mathbb{Q}$, but the homeomorphism cannot be chosen to have really nice algebraic properties.

Let $d$ be a metric bounded by 1 generating the topology on $G$.

We now consider the space $M_G$. For $[f], [g] \in M_G$, define $[f+g] \in M_X$ by the rule

$$(f+g)(t) = f(t) + g(t) \qquad (t \in \mathbb{I}).$$

Clearly, $f+g$ is measurable, and $+$ is an Abelian group operation on $M_X$.

**Lemma 3.1.** *$M_G$ is a paratopological group.*

Proof. Fix $[f], [g] \in M_G$, and let $([f_n])_n$ and $([g_n])_n$ be sequences converging to $[f]$ respectively $[g]$ in $M_G$. We have to show that $[f_n + g_n] \to [f+g]$ in $M_G$. By Lemma 2.3, every subsequence of $(f_n)_n$ contains a subsequence that converges pointwise to $f$ almost everywhere. Similarly for $g$. But then since $G$ is a paratopological group, every subsequence of $(f_n + g_n)_n$ contains a subsequence that converges pointwise to $f + g$ almost everywhere. Hence we are done by Lemma 2.3. $\square$

The constant functions form a closed subgroup of $M_G$ which is isometric to $G$. We write $G^*$ for this closed subgroup of $M_G$. Hence inversion on $M_G$ is as badly discontinuous on $M_G$ as it is on $G$. Observe that $FM_G$ is a subgroup of $M_G$ that obviously contains $G^*$.

Since $M_G$ contains a closed copy of the rational numbers, it is not Polish. In fact, the closed copy of $\mathbb{Q}$ gives us that $M_G$ is not hereditarily Baire. It can be shown that $M_G$ is Borel, hence Čech-analytic. From this it follows from Bouzhiad [8] that $M_G$ is not a Baire space since $M_G$ is not a topological group. We do not present the details of this since the group we are after is a subgroup of $FM_G$, and for that space it is obvious that it is not a Baire space, as we will now show.

Write $G$ as $\bigcup_{n<\omega} G_n$, where each $G_n$ is finite and $G_n \subseteq G_{n+1}$ for every $n$. For every $n$, put

$$FM_n = \{[f] \in FM_G : (\exists\, g \in [f])(g(\mathbb{I}) \subseteq G_n)\}.$$

Clearly, $FM_n$ is naturally homeomorphic to $M_{G_n}$, $FM_n \subseteq FM_{n+1}$ for every $n$, and $FM_G = \bigcup_{n<\omega} FM_n$.

**Lemma 3.2.** *For every $n$, $FM_n$ is a nowhere dense closed subspace of $FM_G$ which is homeomorphic to $\ell^2$.*

Proof. That $FM_n$ is homeomorphic to $\ell^2$ follows from Theorem 2.5. To prove it is closed, take any $[f] \in FM_G \setminus FM_n$. Then there is an element $x \in G \setminus G_n$ such that $\lambda(f^{-1}(x)) > 0$. Assume that $([f_i])_i$ is a sequence in $FM_n$ converging to $[f]$. We assume without loss of generality that $f_i(\mathbb{I}) \subseteq G_n$ for every $i$. By Lemma 2.3 we may assume that $(f_i)_i$ converges to $f$ almost everywhere. So there exists $p \in f^{-1}(x)$ such that $f_i(p) \to f(p)$. Hence there exists $i$ such that $f_i(p) \notin G_n$, which is a contradiction. To prove it is nowhere dense, take any $[f] \in FM_G$. Pick $x \in G$ such that $\lambda(f^{-1}(x)) > 0$. Split $f^{-1}(x)$ into two Borel sets, each of positive measure, say $A$ and $B$. Define a function $g\colon \mathbb{I} \to G$ by $g{\restriction}(\mathbb{I} \setminus B) = f{\restriction}\mathbb{I} \setminus B$, and $g{\restriction}B$ is the constant function with value $a$, where $a$ is an element of $G \setminus G_n$ with very small distance towards $x$. Then $[f]$ and $[g]$ are very close, and $[g]$ does not belong to $FM_n$. $\square$

We conclude from the previous lemma, that $FM_G$ is *strongly $\sigma$-complete*, i.e., a countable union of closed Polish subspaces. Hence $FM_G$ is Borel, but not Baire.

Inversion on $FM_G$ is badly discontinuous, but not if we consider inversion on one of its building blocks $FM_n$.

**Lemma 3.3.** *For a fixed $n$ define $i\colon FM_n \to FM_G$ by $i([f]) = [-f]$. Then $i$ is an embedding.*

Proof. Put $H = \{-s : s \in G_n\}$. Since $G_n$ and $H$ are finite, the function $\varphi\colon G_n \to H$ defined by $\varphi(s) = -s$ is a homeomorphism. Hence we are done by Corollary 2.4 since the homeomorphism $\bar{\varphi}\colon M_{G_n} \to M_H$ defined there is identical to $i$. ☐

**Lemma 3.4.** *$M_G$ is torsion-free.*

Proof. Take $[f] \in FM_G$ and assume that $nf$ is the constant function with value 0 for some $n \in \mathbb{Z}\setminus\{0\}$. Let $x \in \operatorname{range}(f)$. For $p \in f^{-1}(x)$ we have that $nf(p) = 0$, hence $x = f(p) = 0$ since $G$ is torsion-free. We conclude that $f$ is the constant function with value 0. ☐

Hence we almost completed the proof of the following result.

**Theorem 3.5.** *There is a separable metrizable torsion-free Abelian paratopological group $H$ that can be written as $\bigcup_{n<\omega} H_n$ such that $H_n \subseteq H_{n+1}$ for every $n$, while moreover:*

1. *$H$ contains a countable subgroup $G$ on which inversion is discontinuous,*
2. *every $H_n$ is closed in $H$ and homeomorphic to Hilbert space $\ell^2$,*
3. *for every $n$ there exists $m \geq n$ such that $H_n + H_n$ and $H_n - H_n$ are both contained in $H_m$,*
4. *for every $n$ the function $i\colon H_n \to H$ defined by $i(x) = -x$ is an embedding.*

Proof. Of course, we set $H = FM_G$ and $H_n = FM_n$ for all $n$. By Theorem 2.5, and Lemmas 3.1, 3.2, 3.3 and 3.4, the only thing left to prove is (3). But this is trivial, since for given $n$, we may take $m$ so large that $G_m$ contains both $G_n + G_n$ and $G_n - G_n$, and then $m$ is as required. ☐

**Corollary 3.6.** *For given $n$ and $m \geq n$, let $f\colon S \to H_m$ be continuous, where $S \subseteq H_n$. Then the functions $\xi, \eta\colon S \to H$ defined by $\xi(x) = x + f(x)$ and $\eta(x) = x - f(x)$ are continuous. Moreover, there exists $k$ such that the ranges of both $\xi$ and $\eta$ are contained in $H_k$.*

Proof. Simply apply Theorem 3.5(4) and (3) and the fact that $H$ is a paratopological group. ☐

We finish this section by the following technical result, which we will need in the forthcoming §4.

**Lemma 3.7.** *Let $L \subseteq H$ be any Cantor set. Then $L$ contains an algebraically independent Cantor set.*

Proof. In this proof we will say that a subset $A$ of $H \setminus \{0\}$ is $k$-algebraically independent for some $k \geq 1$ if for all pairwise distinct $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n \in [-k, k]$ such that $\sum_{i=1}^n m_i a_i = 0$ we have $m_1 = \cdots = m_n = 0$.

**Claim 1.** Let the pairwise distinct $a_1, \ldots, a_m$ in $H_n$ be $k$-algebraically independent. Then there are pairwise disjoint neighborhoods $U_1, \ldots, U_m$ of $a_1, \ldots, a_m$ in $H_n$ such that for any choice $b_1 \in U_1, \ldots, b_m \in U_m$ we have that $b_1, \ldots, b_m$ is $k$-algebraically independent.

If this is not true, then there are sequences $(a_i^j)_j$ such that $a_i^j \to a_i$ for every $i \leq m$ and $a_1^j, \ldots, a_m^j$ is not $k$-algebraically independent for every $j$. This means that there is a linear combination $\sum_{i=1}^m k_i a_i^j$, where $k_1, \ldots, k_m \in [-k, k]$, which is 0 while yet some $k_{i_0} \neq 0$. But then infinitely often we have that the $k_1, \ldots, k_m$ and the $k_{i_0}$ are the same. So we may assume without loss of generality that they are always equal to say $k_1, \ldots, k_m$ and $k_{i_0}$. Observe that by Theorem 3.5(3) and (4) the function $H_n^m \to H$ defined by $(t_1, \ldots, t_m) \mapsto \sum_{i=1}^m k_i t_i$ is continuous. Hence $\sum_{i=1}^m k_i a_i = 0$, and hence $k_{i_0} \neq 0$ contradicts the $k$-algebraic independence of $a_1, \ldots, a_m$.

Since $L = \bigcup_{n<\omega} L \cap H_n$, the Baire Category Theorem implies that we may assume without loss of generality that $L \subseteq H_n$ for some $n$. Since $L$ is uncountable and $H$ is torsion-free, it contains an uncountable algebraically independent subset, say $E$. By the Cantor-Bendixson Theorem, [11, 1.7.11], we may assume that $E$ is dense-in-itself. By using disjoint balls about points of $E$, we may now construct a Cantor set in the standard manner. A little extra care made possible by Claim 1 will ensure that it will be $k$-algebraically independent for every $k$, i.e., it will be algebraically independent. □

## 4. Unique homogeneity

Let $H$ be the group from Theorem 3.5. We now closely follow the construction in van Mill [18], but considerable extra care is needed. As in §3, let $K$ be the Cantor set in $\mathbb{I}$. Consider the collection

$$\mathcal{K} = \{\langle f, g\rangle : f, g\colon K \to H \text{ are embeddings and the functions } f{+}g \text{ and } f{-}g \text{ are one-to-one}\}.$$

Observe that if $\langle f, g\rangle \in \mathcal{K}$, then also $\langle g, f\rangle \in \mathcal{K}$. For every $\langle f, g\rangle \in \mathcal{K}$ we would like to 'kill' the homeomorphism $g \circ f^{-1}\colon f(K) \to g(K)$ or, if this is not possible, the homeomorphism $f \circ g^{-1}\colon g(K) \to f(K)$. It is clear that $|\mathcal{K}| \leq \mathfrak{c}$, hence we may enumerate $\mathcal{K}$ as $\{\langle f_\alpha, g_\alpha\rangle : \alpha < \mathfrak{c}\}$ (repetitions permitted).

By transfinite induction on $\alpha < \mathfrak{c}$, we will pick a point $x_\alpha \in K$ and points $p_\alpha, q_\alpha \in H \setminus \{0\}$ such that

(1) $\{p_\alpha, q_\alpha\} = \{f_\alpha(x_\alpha), g_\alpha(x_\alpha)\}$,
(2) $\langle\!\langle \{p_\beta : \beta \leq \alpha\} \cup G\rangle\!\rangle \cap \{q_\beta : \beta \leq \alpha\} = \emptyset$.

Assume that we picked $x_\beta, p_\beta$ and $q_\beta$ for every $\beta < \alpha$, where $\alpha < \mathfrak{c}$ (possibly $\alpha = 0$). Put $A = \langle\!\langle \{p_\beta : \beta < \alpha\} \cup G\rangle\!\rangle$ and $V = \{q_\beta : \beta \leq \alpha\}$, respectively. Then $\max\{|A|, |V|\} \leq |\alpha|\cdot\omega < \mathfrak{c}$, and $A \cap V = \emptyset$. For convenience, put $f = f_\alpha$ respectively $g = g_\alpha$.

**Lemma 4.1.** $E_f = \{x \in K : \langle\!\langle \{f(x)\} \cup A\rangle\!\rangle \cap V \neq \emptyset\}$ *has cardinality less than* $\mathfrak{c}$.

Proof. For every $x \in E_f$ there exists $n_x \in \mathbb{Z}$ such that $n_x\cdot f(x) \in V - A$. Since $A \cap V = \emptyset$, $V - A \subseteq H \setminus \{0\}$, so always $n_x \neq 0$. Suppose that $|E_f| = \mathfrak{c}$. Then since $|V - A| < \mathfrak{c}$, there are distinct $x, y \in E_f$ and $n \in \mathbb{Z} \setminus \{0\}$ such that $n = n_x = n_y$ and $n\cdot f(x) = n\cdot f(y)$. But then $f(x) = f(y)$ since $H$ is torsion free, which violates $f$ being injective. □

By precisely the same argument, we obtain:

**Lemma 4.2.** $E_g = \{x \in K : \langle\!\langle \{g(x)\} \cup A\rangle\!\rangle \cap V \neq \emptyset\}$ *has cardinality less than* $\mathfrak{c}$.

We now come to the crucial step in our argumentation.

**Lemma 4.3.** *If* $F \subseteq K$ *has cardinality* $\mathfrak{c}$, *then there exists* $x \in F$ *such that* $f(x) \notin \langle\!\langle \{g(x)\} \cup A\rangle\!\rangle$ *or* $g(x) \notin \langle\!\langle \{f(x)\} \cup A\rangle\!\rangle$.

Proof. Let $F \subseteq K$ have size $\mathfrak{c}$, and assume that for every $x \in F$ we have that $f(x) \in \langle\!\langle \{g(x)\} \cup A\rangle\!\rangle$ and $g(x) \in \langle\!\langle \{f(x)\} \cup A\rangle\!\rangle$. Let $\kappa = |A|\cdot\omega$. Then $|\mathbb{Z} \times A| = \kappa < \mathfrak{c}$, so there are $n \in \mathbb{Z}$, $a \in A$ and $\hat{F} \subseteq F$ of cardinality greater than $\kappa$ such that for every $x \in \hat{F}$, $f(x) = n\cdot g(x) + a$. Since the functions $f + g$ and $f - g$ are both one-to-one and $|A| \leq \kappa$, we get $n \notin \{1, -1\}$. By the same argumentation, there exist a subset $\tilde{F}$ of $\hat{F}$ of size bigger than $\kappa$, $m \in \mathbb{Z} \setminus \{1, -1\}$ and $\bar{a} \in A$ such that for every $x \in \tilde{F}$, $g(x) = m\cdot f(x) + \bar{a}$. For $x \in \tilde{F}$ we consequently have

$$f(x) = n\cdot g(x) + a = nm\cdot f(x) + (n\cdot\bar{a}) + a,$$

hence

$$(nm - 1)\cdot f(x) = \tilde{a},$$

where $\tilde{a} = (n{\cdot}\bar{a}) + a$. Since $nm - 1 \neq 0$, $H$ is torsion free and $f$ is one-to-one, we reached a contradiction. $\square$

Now let $E_f$ and $E_g$ be as in Lemmas 4.1 and 4.2, and put $F = K \setminus (E_f \cup E_g)$. By Lemma 4.3 we may assume without loss of generality that there exists $x \in F$ such that $f(x) \notin \langle\langle \{g(x)\} \cup A \rangle\rangle$. Now put $x_\alpha = x$, $p_\alpha = g(x_\alpha)$ and $q_\alpha = f(x_\alpha)$. It is clear that our choices are as required. This completes the transfinite induction.

Put $X = \langle\langle \{p_\alpha : \alpha < \mathfrak{c}\} \cup G \rangle\rangle$. We claim that $X$ has no homeomorphisms other than translations. This will show that $X$ is uniquely homogeneous and Abelian, but not Boolean. If fact, no nontrivial translation is an involution since $X$ is torsion-free, hence $X$ does not have any involution other than the identity function.

**Lemma 4.4.** *For every $n < \omega$, $X \cap H_n$ is a BB-set in $H_n$.*

Proof. Let $L \subseteq H_n$ be a Cantor set. By Lemma 3.7 we may assume without loss of generality that $L$ is algebraically independent. Now let $L_0$ and $L_1$ be disjoint Cantor sets in $L$, and let $\alpha\colon K \to L_0$ and $\beta\colon K \to L_1$ be arbitrary homeomorphisms. Then $\langle \alpha, \beta \rangle \in \mathcal{K}$, hence $X$ intersects $L_0 \cup L_1 \subseteq L$ by construction. Similarly, $L$ intersects $H_n \setminus X$. $\square$

We now formulate and prove a curious property of $X$.

**Lemma 4.5.** *Let $f\colon X \to X$ be a homeomorphism. Then there is a countable subgroup $A$ of $X$ such that for every $x \in X$ there exists $a \in A$ such that $f(x) = x + a$ or $f(x) = -x + a$.*

Proof. Let $E \subseteq X$ be maximal with respect to the properties that the functions $\xi, \eta\colon E \to X$ defined by

$$\xi(x) = x + f(x), \qquad \eta(x) = x - f(x)$$

are one-to-one.

We will prove that $E$ is countable. Striving for a contradiction, assume that $E$ is uncountable. We have to do some thinning out first. Since $H = \bigcup_{n<\omega} H_n$, we may fix an integer $n$ such that $E_0 = E \cap H_n$ is uncountable. Since $f{\restriction}E_0$ is one-to-one, its range is uncountable. Hence there is $k \geq n$ such that the set $E_1 = \{x \in E_0 : f(x) \in H_k\}$ is uncountable. Pick $m \geq k$ such that $H_k + H_k$ and $H_k - H_k$ are both contained in $H_m$. Observe that the ranges of the functions $f{\restriction}E_1$, $\xi{\restriction}E_1$ and $\eta{\restriction}E_1$ are all contained in $H_m$. Since $H_n$ and $H_k$ are Polish, being homeomorphic to $\ell^2$, there are $G_\delta$-subsets $S$ of $H_n$ and $T$ of $H_k$ such that $E_1$ is dense in $S$ and $f{\restriction}E_1$ can be extended to a homeomorphism $\bar{f}\colon S \to T$

([11, Theorem 4.3.21]). Define $\bar{\xi}\colon S \to H_m$ by $\bar{\xi}(x) = x+\bar{f}(x)$, and, similarly, $\bar{\eta}\colon S \to H_m$ by $\bar{\eta}(x) = x-\bar{f}(x)$. Then $\bar{\xi}$ and $\bar{\eta}$ are continuous by Corollary 3.6. Since the functions $\bar{\xi}, \bar{\eta}$ and $\bar{f}$ are one-to-one on $E_1$, there is by Theorem 2.1 a Cantor set $L$ in $S$ such that $\bar{\xi}$, $\bar{\eta}$ and $\bar{f}$ are all one-to-one on $L$. Let $\alpha\colon K \to L$ be an arbitrary homeomorphism. Consider the pair of functions $\langle \alpha, \bar{f}{\restriction}L \circ \alpha\rangle \in \mathcal{K}$. By construction there exists $x \in L$ such that either $x \in X$ and $\bar{f}(x) \notin X$, or $\bar{f}(x) \in X$ and $x \notin X$. Suppose first that $x \in X$ and $\bar{f}(x) \notin X$. Then since $\bar{f}$ extends $f$, we get $\bar{f}(x) = f(x) \in X$ which is a contradiction. Suppose next that $\bar{f}(x) \in X$ and $x \notin X$. Since $E_1$ is dense in $S \supseteq L$, there is a sequence $(p_i)_i$ in $E_1$ which converges to $x$. Hence $\bar{f}(p_i) \to \bar{f}(x)$. But $\bar{f}$ extends $f$, hence $f(p_i) \to \bar{f}(x)$. There exists $p \in X$ such that $f(p) = \bar{f}(x)$ since $f$ is a homeomorphism. Hence $f(p_i) \to f(p)$ which means that $p_i \to p$. We conclude that $x = p \in X$, which is a contradiction. This completes the proof of our claim.

So we conclude that $E$ is indeed countable. Let $A = \langle\!\langle E \cup f(E)\rangle\!\rangle$. If $x \in E$, then $f(x)-x \in A$ so there is nothing to prove. Assume that $x \notin E$. By maximality of $E$, $\xi{\restriction}E \cup \{x\}$ or $\eta{\restriction}E \cup \{x\}$ is not one-to-one. Suppose first that there exists $e \in E$ such that $\xi(e) = \xi(x)$. Then $e+f(e) = x+f(x) \in A$ and hence we are done. If $\eta{\restriction}E \cup \{x\}$ is not one-to-one, then we can argue similarly. $\square$

We now come to our main result.

**Theorem 4.6.** *Every homeomorphism of $X$ is a translation.*

Proof. Let $f\colon X \to X$ be a homeomorphism, and let $A$ be the subgroup of $X$ we get from Lemma 4.5. For every $a \in A$ and $\varepsilon \in \{-1, 1\}$ we put

$$S_a^\varepsilon = \{x \in X : f(x) = \varepsilon{\cdot}x + a\}.$$

We claim that at most one element of the cover $\mathcal{S} = \{S_a^\varepsilon : \varepsilon \in \{-1, 1\}, a \in A\}$ of $X$ is uncountable. Striving for a contradiction, assume that there are at least two elements of $\mathcal{S}$ that are uncountable. Pick $n$ such that at least two elements of $\mathcal{S}$ intersect $X \cap H_n$ in an uncountable set, say $S_{a_0}^{\varepsilon_0}$ and $S_{a_1}^{\varepsilon_1}$.

For every $a \in A$ and $\varepsilon \in \{-1, 1\}$ put $S_{n,a}^\varepsilon = S_a^\varepsilon \cap H_n$.

**Claim 1.** For every $a \in A$ and $\varepsilon \in \{-1, 1\}$, $S_{n,a}^\varepsilon$ is closed in $X \cap H_n$.

This is easy. Let $(x_i)_i$ be a sequence in $S_{n,a}^\varepsilon$ converging to some element $x \in X \cap H_n$. Then $f(x_i) = \varepsilon{\cdot}x_i + a$ for every $i$. But $\varepsilon{\cdot}x_i + a \to \varepsilon{\cdot}x + a$ by Theorem 3.5(4) and (3) and the fact that $H$ is a paratopological group. Hence $f(x) = \varepsilon{\cdot}x + a$.

For every $a \in A$ and $\varepsilon \in \{-1, 1\}$, let $T_{n,a}^\varepsilon$ be the closure of $S_{n,a}^\varepsilon$ in $H_n$.

**Claim 2.** $T = H_n \setminus \bigcup\{T^\varepsilon_{n,a} : \varepsilon \in \{-1,1\}, a \in A\}$ is countable

This is clear since $T$ is a $G_\delta$-subset of $H_n$ that misses $X \cap H_n$. Hence it has to be countable since if it were uncountable, then it would contains a Cantor set which would intersect $X \cap H_n$ (Lemma 4.4).

**Claim 3.** If $a, a' \in A$ are distinct, then $T^\delta_{n,a} \cap T^\varepsilon_{n,a'}$ is countable for all $\delta, \varepsilon \in \{-1,1\}$.

Striving for a contradiction, assume that for certain distinct $a, a' \in A$ and $\delta, \varepsilon \in \{-1,1\}$ we have that $T^\delta_{n,a} \cap T^\varepsilon_{n,a'}$ is uncountable. Then $T^\delta_{n,a} \cap T^\varepsilon_{n,a'}$ contains a Cantor set which consequently intersects $X \cap H_n$ in a set of size $\mathfrak{c}$ (Lemma 4.4). So we may pick $x \in X \setminus [\![A]\!]$ such that $f(x) = \varepsilon{\cdot}x + a = \delta{\cdot}x + a'$. If $\varepsilon = \delta$, then $a = a'$. Hence $\varepsilon \neq \delta$ which implies that $2x \in A$, which is a contradiction.

Now consider the countable subset

$$S = T \cup \bigcup \big\{T^\delta_{n,a} \cap T^\varepsilon_{n,a'} : a, a' \in A, a \neq a', \delta, \varepsilon, \in \{-1,1\}\big\}$$

of $H_n$. Since $H_n \approx \ell^2$, $H_n \setminus S$ is path-connected ([6, Theorem 6.4 on Page 166]). Pick arbitrary $x \in S^{\varepsilon_0}_{a_0} \setminus S$ and $y \in S^{\varepsilon_1}_{a_1} \setminus S$. There is an arc $J$ in $H_n \setminus S$ which contains both $x$ and $y$. Then the collection

$$\big\{T^\varepsilon_{n,a} \cap J : a \in A, \varepsilon \in \{-1,1\}\big\}$$

is a partition of $J$ in at least two nonempty and at most countably many nonempty closed sets. But this violates the Sierpiński Theorem, [11, Theorem 6.1.27].

This means that at most one element of the closed cover $\mathcal{S}$ of $X$ is uncountable. But $X$ is locally of cardinality $\mathfrak{c}$ by Lemma 4.4. As a consequence, since $\mathcal{S}$ is countable, there is exactly one element of $\mathcal{S}$ that is nonempty, and hence is equal to $X$. There are two cases to consider. First assume that $S^{-1}_a = X$ for some $a \in A$. Then $f(x) = -x + a$ for every $x \in X$. But $H$ contains the countable group $G$, the paratopological group we started with. There is a sequence $(x_i)_i$ in $G$ such that $x_i \to 0$ but $(-x_i)_i$ is closed and discrete in $G$ and hence in $H$ and hence in $X$. Since the translation $x \mapsto x{+}a$ is a homeomorphism of $X$, this implies that the sequence $(-x_i + a)_i$ is closed and discrete in $X$. But this contradicts the continuity of $f$. Hence there is a unique $a \in A$ such that $f(x) = x + a$ for every $x \in X$, i.e., $f$ is a translation. $\square$

## 5. Baire uniquely homogeneous spaces

The example constructed in the previous section is not a Baire space. Here we prove that it cannot be a Baire space.

**Theorem 5.1.** *Let $X$ be a metrizable Baire space that is uniquely homogeneous. If $X$ is Abelian, then $X$ is an Abelian topological group.*

Proof. There are several ways we can arrive at the desired conclusion.

For example, by [2, Theorem 5.4], the standard group operation on $X$ is semitopological and Abelian. However, every metrizable semitopological group with the Baire property is a paratopological group, by Bouziad [7, Corollary 5]. But every symmetrizable paratopological group with the Baire property is a topological group, as was shown by Arhangel′skii and Reznichenko [3, Theorem 1.4].

Another route is to use Theorem 2 in Kenderov, Kortezov, and Moors [14]. $\square$

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Received February 18, 2011
Revised version received April 19, 2012

(Arhangel'skii) Moscow, 121165, Russia
*E-mail address*: `arhangel.alex@gmail.com`

(Van Mill) Faculty of Sciences, Department of Mathematics, VU University Amsterdam, 1081 HV Amsterdam, The Netherlands
*E-mail address*: `j.van.mill@vu.nl`