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Universal frames

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For a class of frames we define the notion of a universal element and prove that in the class of all frames of weight less than or equal to a fixed infinite cardinal number τ there are such elements.

0. Introduction

Universal objects play an important role in mathematics. In topology, there is a wealth of such objects that go back a long time. For example, the Cantor set is a universal space for the class of all zero-dimensional separable metrizable spaces, the n-dimensional Nöbeling spaces for the class of all n-dimensional separable metrizable spaces, the Hilbert space for the class of all separable metrizable spaces, etc. There are also many interesting results for isometries. For example, the Urysohn Universal Metric Space and the space $C[0,1]$ of all continuous functions on $[0,1]$ with the uniform convergence contain isometrically all separable metric spaces. For details and references on universality, see Iliadis [2]. For basic information about frames or pointfree topology, see Picado and Pultr [4] or Johnstone [3].

The aim of this note is to construct a universal frame in the class of all frames of weight at most a given infinite cardinal number τ. Some but not all frames have a topological space as underlying object, so our result is independent of that of the existence of universal objects in the class of all topological T_0-spaces

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of weight τ given in [2]. We employ the general technique of constructing universal objects as given by Iliadis [2]. In so doing, a novel method for constructing new frames involving very set theoretical methods is presented.

1. Universal frames

1.1. Definitions and notation. Recall that a *frame* is a complete lattice L in which

$$x \wedge \bigvee S = \bigvee \{x \wedge s\colon\ s \in S\}$$

for any $x \in L$ and any $S \subseteq L$. Our notations shall be fairly standard from Picado and Pultr [4]. For instance we denote the top element and the bottom element of L by 1_L and 0_L respectively. As usual, $\prec$ is the *rather below* relation whilst $\prec\!\!\prec$ denotes the *completely below* relation. For $x, y \in L$, $x \prec y$ iff $x^* \vee y = 1_L$ where $x^* = \bigvee\{t \in L\colon\ t \wedge x = 0\}$ whilst $x \prec\!\!\prec y$ iff there is a system $\{c_r \in L\colon\ r \in \mathbb{Q} \cap [0,1]\}$ such that $c_0 = x$, $c_1 = y$ and $c_r \prec c_s$ whenever $r < s$. A frame L is *completely regular* if for each $x \in L$, $x = \bigvee\{y \in L\colon\ y \prec\!\!\prec x\}$ and *regular* frames are those in which each element is the join of elements rather below it. If $x \vee x^* = 1_L$ then x is *complemented* in L. The frame L is *zero-dimensional* provided that each element of L is a join of complemented elements (see Banaschewski [1]). A frame *homomorphism* is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. We denote the class of all frames that are completely regular by **CRegFrm**, regular by **RegFrm**, and zero-dimensional by **0-DFrm**.

A subset B of a complete lattice (or frame) L is called *base* for L if each element of L is a supremum of a subset of B. The *weight* of a complete lattice (or frame) L is the minimal cardinal κ for which there exists a base B for L of cardinality κ.

An *ordinal number* is the set of smaller ordinal numbers, and a *cardinal number* is an initial ordinal number. By ω we denote the least infinite cardinal. By τ we denote a fixed infinite cardinal and by $\mathcal{F}$ the set of all non-empty finite subsets of τ.

Each mapping φ of τ onto a set X is called an *indexation* of X and will be denoted by

$$X = \{x_0, \ldots, x_\delta, \ldots\},$$

where $x_\delta = \varphi(\delta)$, $\delta \in \tau$.

1.2. Definition of a universal complete lattice (or frame). Let $\mathbb{L}$ be a class of (non-empty) complete lattices (or frames). We say that a complete lattice (or frame) T is *universal* in this class if (a) $\mathrm{T} \in \mathbb{L}$ and (b) for every $L \in \mathbb{L}$ there exists a homomorphism of T onto L.

1.3. Theorem. *In the class $\mathbb{L}$ of all frames of weight $\leqslant \tau$ there exist universal elements.*

Proof. Without loss of generality we can suppose that $\mathbb{L}$ is a set and that its elements are mutually disjoint.

For every $L \in \mathbb{L}$ we denote by

$$B^L = \{a_0^L, a_1^L, \ldots, a_\delta^L, \ldots\}$$

an indexed base of L of cardinality $\leqslant \tau$ such that $a_0^L = 0$ and $a_1^L = 1$. We assume without loss of generality that B^L is closed under taking finite infima. By θ_L we denote a fixed mapping of the set $\mathcal{F}$ into τ such that for every $t \in \mathcal{F}$ we have

$$a^L_{\theta_L(t)} = \bigwedge\{a^L_\delta\colon\ \delta \in t\},$$

with the additional condition that if $t = \{\delta\}$, then $\theta_L(t) = \delta$. We note that for distinct elements L and G in $\mathbb{L}$ the mappings θ_L and θ_G need not have any relation to each other.

Fix $s \in \mathcal{F}$ for a moment. By $\sim^s$ we denote the equivalence relation on the family $\mathbb{L}$ defined as follows: two elements $L, G \in \mathbb{L}$ are $\sim^s$-equivalent if and only if for all $t \subseteq s$ such that $t \neq \emptyset$ we have $\theta_L(t) = \theta_G(t)$.

It is easy to see that if $\emptyset \neq s' \subseteq s$, then $\sim^s \subseteq \sim^{s'}$. We set $\mathrm{R} = \{\sim^s\colon\ s \in \mathcal{F}\}$.

We denote by $\mathrm{C}(\sim^s)$ the set of all equivalence classes of the relation $\sim^s$, $s \in \mathcal{F}$, and put

$$\mathrm{C}(\mathrm{R}) = \bigcup \bigl\{\mathrm{C}\bigl(\sim^s\bigr)\colon\ s \in \mathcal{F}\bigr\}.$$

It is easy to see that the cardinality of the set $\mathrm{C}(\sim^s)$ is less than or equal to τ and, therefore, the cardinality of the set $\mathrm{C}(\mathrm{R})$ is also less than or equal to τ.

Now we consider the free union $\mathbf{U}$ of the elements of $\mathbb{L}$, i.e.,

$$\mathbf{U} = \bigcup \{L\colon\ L \in \mathbb{L}\}.$$

For every $s \in \mathcal{F}$, $\mathbf{H} \in \mathrm{C}(\sim^s)$, and $\delta \in s$ we put

$$A_\delta(\mathbf{H}) = \bigl\{a^L_\delta\colon\ L \in \mathbf{H}\bigr\} \cup \bigl\{a^L_0 = 0_L\colon\ L \in \mathbb{L} \setminus \mathbf{H}\bigr\} \subset \mathbf{U}.$$

Obviously, for every $L \in \mathbb{L}$ the intersection $L \cap A_\delta(\mathbf{H})$ consists of one element only: the element a^L_δ if $L \in \mathbf{H}$ and the element 0_L if $L \notin \mathbf{H}$. We put

$$\mathrm{B} = \bigl\{A_\delta(\mathbf{H})\colon\ \delta \in s, s \in \mathcal{F}, \mathbf{H} \in \mathrm{C}\bigl(\sim^s\bigr)\bigr\}.$$

Obviously, the cardinality of B is $\leqslant \tau$.

For every subset

$$M = \bigl\{A_{\delta_j}(\mathbf{H}_j)\colon\ j \in J\bigr\} \subseteq \mathrm{B}$$

we shall define a subset of $\mathbf{U}$, denoted by $\tilde{M}$. For the definition of this set we first fix an element $L \in \mathbb{L}$ and denote by M_L the set of elements a^L_δ of L for which there exists $j \in J$ such that

$$\bigl\{a^L_\delta\bigr\} = L \cap A_{\delta_j}(\mathbf{H}_j).$$

Therefore, a^L_δ coincides either with $a^L_{\delta_j}$ or with 0_L. Then, we put

$$\tilde{M} = \{\sup M_L\colon\ L \in \mathbb{L}\}.$$

Observe that the element $\sup M_L \in \tilde{M}$ in general is not an element of M_L.

Now we consider the set

$$\mathrm{T} = \{\tilde{M}\colon\ M \subseteq \mathrm{B}\}.$$

The subset $M \subseteq \mathrm{B}$ will be called a *generator* of $\tilde{M} \in \mathrm{T}$. The generators of the elements of T are not uniquely determined. It is easy to see that the union of any number of generators of an element of T is also a generator of this element. Therefore, the union of all generators of a certain element of T is the *maximal generator* of that element.

We note that each element $A_\delta(\mathbf{H}) \in \mathrm{B}$ is an element of T, that is $\mathrm{B} \subseteq \mathrm{T}$. Indeed, for the set $N = \{A_\delta(\mathbf{H})\} \subseteq \mathrm{B}$, it is easy to see that $\tilde{N} = A_\delta(\mathbf{H})$.

In T we define an order $\leqslant$ by putting $\tilde{M} \leqslant \tilde{N}$ for $\tilde{M}, \tilde{N} \in \mathrm{T}$ if and only if for every $L \in \mathbb{L}$ we have $\sup M_L \leqslant \sup N_L$. Obviously, the pair $(\mathrm{T}, \leqslant)$ is a poset. □

Lemma 1. The poset $(\mathrm{T}, \leqslant)$ is a complete lattice of weight $\leqslant \tau$.

Proof. To prove the lemma it suffices to prove the existence of suprema of arbitrary subsets (Picado and Pultr [4, 4.3.1]), and that the weight of T is at most τ. This last fact will follow once we show that B is a base for T.

The existence of suprema. Let

$$G = \{\tilde{M}^\lambda\colon\ \lambda \in \Lambda\} \subseteq \mathrm{T}.$$

We must prove that the supremum of G exists. Denote by M the union of all generators M^λ of all elements of G, i.e.,

$$M = \bigcup \{M^\lambda\colon\ \lambda \in \Lambda\}.$$

Then, the set

$$\tilde{M} = \{\sup M_L\colon\ L \in \mathbb{L}\}$$

is an element of T. We shall prove that $\tilde{M} = \sup G$.

Indeed, consider an element $L \in \mathbb{L}$. Then, it is easy to see that

$$M_L = \bigcup \{M_L^\lambda\colon\ \lambda \in \Lambda\}$$

and, therefore,

$$\sup M_L = \sup\Bigl(\bigcup \{M_L^\lambda\colon\ \lambda \in \Lambda\}\Bigr) = \sup\{\sup M_L^\lambda\colon\ \lambda \in \Lambda\}. \tag{1}$$

This relation shows $\sup M_L^\lambda \leqslant \sup M_L$, that is $\tilde{M}^\lambda \leqslant \tilde{M}$ for every $\lambda \in \Lambda$.

Now, suppose that N is an element of T such that $M^\lambda \leqslant N$ for every $\lambda \in \Lambda$, that is

$$\sup M_L^\lambda \leqslant \sup N_L$$

for every $L \in \mathbb{L}$ and for every $\lambda \in \Lambda$. Then,

$$\sup\{\sup M_L^\lambda\colon\ \lambda \in \Lambda\} \leqslant \sup N_L$$

and by relation (1) we have

$$\sup M_L \leqslant \sup N_L,$$

that is $\tilde{M} \leqslant \tilde{N}$ proving that $\tilde{M} = \sup G$.

The set B is a base for T. Consider an arbitrary element $\tilde{M}$ of T and let

$$M = \{A_{\delta_\lambda}(\mathbf{H}_\lambda)\colon\ \lambda \in \Lambda\}.$$

We put $M^\lambda = \{A_{\delta_\lambda}(\mathbf{H}_\lambda)\}$. Then,

$$\tilde{M}^\lambda = A_{\delta_\lambda}(\mathbf{H}_\lambda) \in \mathrm{B} \subseteq \mathrm{T} \quad \text{and} \quad M = \bigcup \{M^\lambda\colon\ \lambda \in \Lambda\}.$$

Consider the set

$$G = \{\tilde{M}^{\lambda}\colon\ \lambda \in \Lambda\} \subseteq \mathrm{B}.$$

As in the above we can prove that

$$\tilde{M} = \sup G = \sup\{\tilde{M}^{\lambda}\colon\ \lambda \in \Lambda\}.$$

This relation shows that B is a base for T and, therefore, the weight of T is at most τ. The proof of the lemma is complete. □

It follows from Lemma 1 that T has a bottom and a top. For later use, it will be convenient to have an explicit description of these two elements.

The existence of the bottom. Let $s = \{0\} \in \mathcal{F}$. Consider the set

$$M = \{A_0(\mathbf{H})\colon\ \mathbf{H} \in \mathrm{C}(\sim^s)\}.$$

By the definition of the indexation of B^L, for every $\mathbf{H} \in \mathrm{C}(\sim^s)$ we have

$$A_0(\mathbf{H}) = \{0_L\colon\ L \in \mathbf{H}\} \cup \{0_L\colon\ L \in \mathbb{L} \setminus \mathbf{H}\}.$$

Then, for every $L \in \mathbb{L}$, $M_L = \{0_L\}$ and, therefore, $\sup M_L = \sup\{0_L\} = 0_L$. Thus,

$$\tilde{M} = \{0_L\colon\ L \in \mathbb{L}\}.$$

Obviously, by the definition of the order in T for every $\tilde{N} \in \mathrm{T}$ we have $\tilde{M} \leqslant \tilde{N}$. Therefore, the constructed element $\tilde{M} \in \mathrm{T}$ is the least element of the poset $(\mathrm{T}, \leqslant)$ and will be denoted by 0_{T}.

The existence of the top. Let $s = \{1\} \in \mathcal{F}$. Consider the set

$$M = \{A_1(\mathbf{H})\colon\ \mathbf{H} \in \mathrm{C}(\sim^s)\}.$$

By the definition of the indexation of B^L,

$$A_1(\mathbf{H}) = \{1_L\colon\ L \in \mathbf{H}\} \cup \{0_L\colon\ L \in \mathbb{L} \setminus \mathbf{H}\}, \quad \mathbf{H} \in \mathrm{C}(\sim^s).$$

Since the distinct classes of $\sim^s$ are disjoint and the union of all classes is $\mathbb{L}$, for every $L \in \mathbb{L}$ we have $M_L = \{1_L\}$ and, therefore, $\sup M_L = \sup\{1_L\} = 1_L$. Then,

$$\tilde{M} = \{1_L\colon\ L \in \mathbb{L}\}.$$

Obviously, by the definition of the order in T for every $\tilde{N} \in \mathrm{T}$ we have $\tilde{N} \leqslant \tilde{M}$. Therefore, the constructed element $\tilde{M} \in \mathrm{T}$ is the greatest element of $(\mathrm{T}, \leqslant)$ and will be denoted by 1_{T}.

Lemma 2. For every $L \in \mathbb{L}$ there exists a homomorphism h_L of T onto L.

Proof. Let L be a fixed element of $\mathbb{L}$.

The definition of h_L. We define the mapping h_L setting for every element $\tilde{M} \in \mathrm{T}$:

$$h_L(\tilde{M}) \text{ is the unique element in } L \cap \tilde{M}.$$

Obviously,

$$h_L(\tilde{M}) = \sup M_L.$$

We note that $h_L(\tilde{M})$ is independent of the generator M of the element $\tilde{M}$.

Since for any elements $\tilde{N}$ and $\tilde{M}$ of T the relation $\tilde{N} \leqslant \tilde{M}$ means that $\sup N_L \leqslant \sup M_L$, the mapping h_L is order preserving.

By the definition of h_L it follows immediately that $h_L(0_{\mathrm{T}}) = 0_L$ and $h_L(1_{\mathrm{T}}) = 1_L$.

The mapping h_L is onto. Let $a \in L$. Since B^L is a base for L, there exists a subset $\kappa \subseteq \tau$ such that for

$$A = \{a^L_\delta\colon\ \delta \in \kappa\} \subseteq B^L \subseteq L$$

we have $a = \sup A$. For every $s \in \mathcal{F}$ denote by $\mathbf{H}^L_s$ the equivalence class of $\mathrm{C}(\sim^s)$ containing L. For every $\delta \in s \cap \kappa$ consider the set $A_\delta(\mathbf{H}^L_s)$ and put

$$M = \{A_\delta(\mathbf{H}^L_s)\colon\ s \in \mathcal{F}, \delta \in s \cap \kappa\}.$$

Then,

$$M_L = \bigcup\{\{a^L_\delta\} = L \cap A_\delta(\mathbf{H}^L_s)\colon\ s \in \mathcal{F}, \delta \in s \cap \kappa\}.$$

The above equality means that $A = M_L$. Hence, $\sup M_L = a$, so $h_L(\tilde{M}) = a$ showing that h_L is indeed onto.

The mapping h_L preserves suprema. Let

$$\tilde{M} = \sup\{\tilde{M}^\lambda\colon\ \lambda \in \Lambda\}.$$

Then,

$$\begin{aligned} h_L(\tilde{M}) &= \sup M_L = \sup\Big(\bigcup\{M^\lambda_L\colon\ \lambda \in \Lambda\}\Big) \\ &= \sup\{\sup M^\lambda_L\colon\ \lambda \in \Lambda\} = \sup\{h_L(\tilde{M}^\lambda)\colon\ \lambda \in \Lambda\}, \end{aligned}$$

that is, h_L preserve suprema.

The mapping h_L preserves finite infima. Let $A_{\delta_j}(\mathbf{H}_j)$ and $A_{\eta_i}(\mathbf{H}_i)$ be two fixed elements of B such that $\delta_j \in s_j$, $\mathbf{H}_j \in \mathrm{R}(\sim^{s_j})$, $\eta_i \in s_i$, and $\mathbf{H}_i \in \mathrm{R}(\sim^{s_i})$. We shall define a subset of B, denoted by

$$M_{(j,i)} = A_{\delta_j}(\mathbf{H}_j) \Delta A_{\eta_i}(\mathbf{H}_i)$$

as follows. Let $s = s_j \cup s_i$. Then, for every $L, G \in \mathbf{H} \in \mathrm{C}(\sim^s)$ we have $\theta_L(t) = \theta_G(t)$ where $t = \{\delta_j, \eta_i\}$. In this case we put $\theta_{\mathbf{H}}(t) = \theta_L(t)$. By the properties of the elements of the family R it follows that if $\mathbf{H}$ is an element of $\mathrm{C}(\sim^s)$, then either $\mathbf{H} \subseteq \mathbf{H}_j \cap \mathbf{H}_i$ or $\mathbf{H} \cap (\mathbf{H}_j \cup \mathbf{H}_i) = \emptyset$. If the set $\mathbf{H}_j \cap \mathbf{H}_i$ is empty, then we put

$$M_{(j,i)} = \{A_0(\mathbf{H})\colon\ \mathbf{H} \in \mathrm{C}(\sim^s)\}.$$

If $\mathbf{H}_j \cap \mathbf{H}_i \neq \emptyset$, then we put

$$M_{(j,i)} = \{A_\delta(\mathbf{H})\colon\ \mathbf{H} \in \mathrm{C}(\sim^s),\ \mathbf{H} \subseteq \mathbf{H}_j \cap \mathbf{H}_i,\ \delta = \theta_{\mathbf{H}}(t)\}.$$

We note that if $s_j = s_i$, $\mathbf{H}_j = \mathbf{H}_i$, and $\delta_j = \eta_i$, then $\theta_L(t) = \delta_j$ and $s = s_j$. Therefore, in this case

$$M_{(j,i)} = \{A_{\delta_j}(\mathbf{H}_j)\} = \{A_{\eta_i}(\mathbf{H}_i)\}.$$

It is easy to see that for every $A_\delta(\mathbf{H}) \in M_{(j,i)}$ we have

$$A_\delta(\mathbf{H}) \leqslant A_{\delta_j}(\mathbf{H}_j) \quad \text{and} \quad A_\delta(\mathbf{H}) \leqslant A_{\eta_i}(\mathbf{H}_i). \tag{2}$$

Now, let $\tilde{N}$ and $\tilde{K}$ be two elements of T. Suppose that

$$N = \{A_{\delta_j}(\mathbf{H}_j)\colon\ j \in J,\ \mathbf{H}_j \in \mathrm{C}(\sim^{s_j}),\ \delta_j \in s_j \in \mathcal{F}\}$$

and

$$K = \{A_{\eta_i}(\mathbf{H}_i)\colon\ i \in I,\ \mathbf{H}_i \in \mathrm{C}(\sim^{s_i}),\ \eta_i \in s_i \in \mathcal{F}\}.$$

Without loss of generality we can suppose that N and K are the maximal generators of $\tilde{N}$ and $\tilde{K}$, respectively.

Consider the set

$$M = \bigcup\{M_{(j,i)} = A_{\delta_j}(\mathbf{H}_j)\Delta A_{\eta_i}(\mathbf{H}_i)\colon\ (j,i) \in J \times I\}.$$

First, we shall prove that

$$\tilde{M} = \tilde{N} \wedge \tilde{K}. \tag{3}$$

Indeed, let $L \in \mathbb{L}$. Obviously, for every $(j,i) \in J \times I$ we have

$$A_{\delta_j}(\mathbf{H}_j) \leqslant \tilde{N} \quad \text{and} \quad A_{\eta_i}(\mathbf{H}_i) \leqslant \tilde{K}. \tag{4}$$

Then, since N and K are maximal generators by the definition of the set M and the relations (2) and (4) for every $(j,i) \in J \times I$ we have

$$M_{(j,i)} \subseteq N \quad \text{and} \quad M_{(j,i)} \subseteq K$$

and, therefore,

$$\bigcup\{M_{(j,i)}\colon\ (j,i) \in J \times I\} \subseteq N \quad \text{and} \quad \bigcup\{M_{(j,i)}\colon\ (j,i) \in J \times I\} \subseteq K.$$

This means that

$$\sup M_L = \sup\Bigl(\bigcup\{(M_{(j,i)})_L\colon\ (j,i) \in J \times I\}\Bigr) \leqslant \sup N_L \tag{5}$$

and similarly

$$\sup M_L \leqslant \sup K_L. \tag{6}$$

The relations (5) and (6) imply that

$$\tilde{M} \leqslant \tilde{N} \quad \text{and} \quad \tilde{M} \leqslant \tilde{K}.$$

Suppose now that for an element $\tilde{P} \in \mathrm{T}$ we have

$$\tilde{P} \leqslant \tilde{N} \quad \text{and} \quad \tilde{P} \leqslant \tilde{K}.$$

Since N and K are maximal generators we have

$$P \subseteq N \quad \text{and} \quad P \subseteq K.$$

Let $A_\delta(\mathbf{H})$ be an element of P. Then, $A_\delta(\mathbf{H}) \in N$ and $A_\delta(\mathbf{H}) \in K$. Setting $A_{\delta_j}(\mathbf{H}_j) = A_{\eta_i}(\mathbf{H}_i) = A_\delta(\mathbf{H})$, by the above, we have $M_{(j,i)} = \{A_\delta(\mathbf{H})\}$. Therefore, $A_\delta(\mathbf{H}) \in M$, that is $P \subseteq M$ which means that $\tilde{P} \leqslant \tilde{M}$. Thus, relation (3) is proved.

We shall now prove that

$$\sup M_L = \sup N_L \wedge \sup K_L. \tag{7}$$

First, we shall prove that if $a^L_{\delta_j} \in N_L$ and $a^L_{\eta_i} \in K_L$, then

$$a^L_{\delta_j} \wedge a^L_{\eta_i} \in M_L. \tag{8}$$

Indeed, let $a^L_{\delta_j} \in N_L$ and $a^L_{\eta_i} \in K_L$. Then there exist elements $\mathbf{H}_j \in \mathrm{C}(\sim^{s_j})$ and $\mathbf{H}_i \in \mathrm{C}(\sim^{s_i})$ for some $s_j, s_i \in \mathcal{F}$ such that $L \in \mathbf{H}_j \cap \mathbf{H}_i$, $A_{\delta_j}(\mathbf{H}_j) \in N$ and $A_{\eta_i}(\mathbf{H}_i) \in K$. Let $t = \{\delta_j, \eta_i\}$, $s = s_j \cup s_i$, and $\mathbf{H}$ be the element of $\mathrm{C}(\sim^s)$ containing L. Then, $A_{\theta_{\mathbf{H}}(t)}(\mathbf{H}) \in M_{(j,i)} \subseteq M$. Since $a^L_{\theta_{\mathbf{H}}(t)} = a^L_{\theta_L(t)} = a^L_{\delta_j} \wedge a^L_{\eta_i}$ we have $a^L_{\delta_j} \wedge a^L_{\eta_i} \in M_L$.

Now, let $a \in M_L$. Then, there exist

$$\delta_j \in s_j \in \mathcal{F}, \qquad \mathbf{H}_j \in \mathrm{C}\big(\sim^{s_j}\big), \qquad \eta_i \in s_i \in \mathcal{F} \quad \text{and} \quad \mathbf{H}_i \in \mathrm{C}\big(\sim^{s_i}\big)$$

such that $a \in (M_{(j,i)})_L$, where

$$M_{(j,i)} = A_{\delta_j}(\mathbf{H}_j) \Delta A_{\eta_i}(\mathbf{H}_i).$$

Therefore, we have

$$a \in A_{\theta_{\mathbf{H}}(t)}(\mathbf{H}) \in M_{(j,i)},$$

where $t = \{\delta_j, \eta_i\}$, $s = s_j \cup s_i$, and $\mathbf{H}$ is the element of $\mathrm{C}(\sim^s)$ containing L. This means that

$$a = a^L_{\delta_j} \wedge a^L_{\eta_i}. \tag{9}$$

The relations (8) and (9) show that

$$M_L = \big\{a^L_{\delta_j} \wedge a^L_{\eta_i}\colon\ a^L_{\delta_j} \in N_L, a^L_{\eta_i} \in K_L\big\}$$

and, therefore,

$$\sup M_L = \sup\big\{a^L_{\delta_j} \wedge a^L_{\eta_i}\colon\ a^L_{\delta_j} \in N_L, a^L_{\eta_i} \in K_L\big\}.$$

Since L is a frame we have

$$\sup M_L = \sup\{a_{\delta_j}^L\colon\ a_{\delta_j}^L \in N_L\} \wedge \sup\{a_{\eta_i}^L\colon\ \eta_i \in K_L\} = \sup N_L \wedge \sup K_L$$

proving relation (7). Relations (3) and (7) prove that the mapping h_L preserves finite infima. Hence we are done. □

Lemma 3. The complete lattice $(\mathrm{T}, \leqslant)$ is a frame.

Proof. We must prove that

$$\tilde{N} \wedge \sup\{\tilde{K}^\lambda\colon\ \lambda \in \Lambda\} = \sup\{\tilde{N} \wedge \tilde{K}^\lambda\colon\ \lambda \in \Lambda\} \tag{10}$$

for all $\tilde{N}, \tilde{K}^\lambda \in \mathrm{T}$. Without loss of generality we can suppose that N and K^λ, $\lambda \in \Lambda$, are the maximal generators of $\tilde{N}$ and $\tilde{K}^\lambda$, respectively.

Suppose that

$$N = \{A_{\delta_j}(\mathbf{H}_j)\colon\ j \in J, \delta_j \in s_j \in \mathcal{F}, \mathbf{H}_j \in \mathrm{C}(\sim^{s_j})\}$$

and

$$K^\lambda = \{A_{\eta_{i(\lambda)}}(\mathbf{H}_{i(\lambda)})\colon\ i(\lambda) \in I(\lambda), \eta_{i(\lambda)} \in s_{i(\lambda)} \in \mathcal{F}, \mathbf{H}_{i(\lambda)} \in \mathrm{C}(\sim^{s_{i(\lambda)}})\}.$$

Then, the set

$$\begin{aligned} K &= \bigcup\{K^\lambda\colon\ \lambda \in \Lambda\} \\ &= \{A_{\eta_{i(\lambda)}}(\mathbf{H}_{i(\lambda)})\colon\ i(\lambda) \in I(\lambda), \eta_{i(\lambda)} \in s_{i(\lambda)} \in \mathcal{F}, \mathbf{H}_{i(\lambda)} \in \mathrm{C}(\sim^{s_{i(\lambda)}}), \lambda \in \Lambda\} \end{aligned}$$

is a generator of the element $\sup\{\tilde{K}^\lambda\colon\ \lambda \in \Lambda\} \in \mathrm{T}$ and, therefore,

$$\tilde{K} = \sup\{\tilde{K}^\lambda\colon\ \lambda \in \Lambda\}.$$

By the relation (4) for every $\lambda \in \Lambda$ the set

$$\begin{aligned} M^\lambda = \bigcup\{A_{\delta_j}(\mathbf{H}_j)\,\Delta\, A_{\eta_{i(\lambda)}}(\mathbf{H}_{i(\lambda)})\colon\ & j \in J, \delta_j \in s_j \in \mathcal{F}, \mathbf{H}_j \in \mathrm{C}(\sim^{s_j}), i(\lambda) \in I(\lambda), \\ & \eta_{i(\lambda)} \in s_{i(\lambda)} \in \mathcal{F}, \mathbf{H}_{i(\lambda)} \in \mathrm{C}(\sim^{s_{i(\lambda)}})\} \end{aligned}$$

is a generator of the element $\tilde{N} \wedge \tilde{K}^\lambda \in \mathrm{T}$. Hence, the set

$$\begin{aligned} M &= \bigcup\{M^\lambda\colon\ \lambda \in \Lambda\} \\ &= \{A_{\delta_j}(\mathbf{H}_j)\Delta A_{\eta_{i(\lambda)}}(\mathbf{H}_{i(\lambda)})\colon\ j \in J, \delta_j \in s_j \in \mathcal{F}, \mathbf{H}_j \in \mathrm{C}(\sim^{s_j}), i(\lambda) \in I(\lambda), \\ &\qquad \eta_{i(\lambda)} \in s_{i(\lambda)} \in \mathcal{F}, \mathbf{H}_{i(\lambda)} \in \mathrm{C}(\sim^{s_{i(\lambda)}}), \lambda \in \Lambda\} \end{aligned}$$

is a generator of the element $\sup\{\tilde{N} \wedge \tilde{K}^\lambda\colon\ \lambda \in \Lambda\} \in \mathrm{T}$. Therefore,

$$\tilde{M} = \sup\{\tilde{N} \wedge \tilde{K}^\lambda\colon\ \lambda \in \Lambda\}.$$

Thus, relation (10) takes the form

$$\tilde{N} \wedge \tilde{K} = \tilde{M}.$$

Suppose that $\tilde{K}' = \tilde{K}$, where

$$K' = \{A_{\delta_{j'}}(\mathbf{H}_{j'}) \colon\ j' \in J', \delta_{j'} \in s_{j'} \in \mathcal{F}, \mathbf{H}_{j'} \in \mathrm{C}(\sim^{s_{j'}})\}$$

is the maximal generator of the element $\tilde{K} \in \mathrm{T}$ (therefore, containing K).

The set

$$P = \{A_{\delta_j}(\mathbf{H}_j)\Delta A_{\delta_{j'}}(\mathbf{H}_{j'}) \colon\ j \in J, \delta_j \in s_j \in \mathcal{F}, \mathbf{H}_j \in \mathrm{C}(\sim^{s_j}), j' \in J', \\ \delta_{j'} \in s_{j'} \in \mathcal{F}, \mathbf{H}_{j'} \in \mathrm{C}(\sim^{s_{j'}})\}$$

is a generator of the element $\tilde{N} \wedge \tilde{K} \in \mathrm{T}$ and, therefore,

$$\tilde{P} = \tilde{N} \wedge \tilde{K} = \tilde{N} \wedge \tilde{K}'.$$

Since $K \subseteq K'$ we have $M \subseteq P$. Therefore,

$$\tilde{M} \leqslant \tilde{N} \wedge \tilde{K} = \tilde{N} \wedge \tilde{K}'.$$

Thus, to prove the lemma it suffices to prove that

$$\tilde{N} \wedge \tilde{K}' \leqslant \tilde{M}$$

or equivalently

$$\sup P_L \leqslant \sup M_L$$

for every $L \in \mathbb{L}$. For this purpose it suffices to prove that if $L \in \mathbb{L}$ and $a \in P_L$, then $a \leqslant \sup M_L$. Let $a \in P_L$. Then, there exist

$$A_{\delta_j}(\mathbf{H}_j) \in N \quad \text{and} \quad A_{\delta_{j'}}(\mathbf{H}_{j'}) \in K'$$

such that $a \in L \cap A_{\theta_{\mathbf{H}}(t)}(\mathbf{H})$, where $\mathbf{H}$ is the element of $\mathrm{C}(\sim^s)$ containing L, $s = s_j \cup s_{j'}$, and $t = \{\delta_j, \delta_{j'}\}$. Therefore,

$$a = a^L_{\theta_L(t)} = a^L_{\delta_j} \wedge a^L_{\delta_{j'}}.$$

If $a^L_{\delta_{j'}} = a^L_{\eta_{i(\lambda)}}$ for some $\lambda \in \Lambda$ and $i(\lambda) \in I(\lambda)$, then $a \in M_L$. Thus,

$$\sup\{a^L_{\delta_j} \wedge a^L_{\delta_{j'}} \colon\ j \in J, j' \in J'\} = \sup\{a^L_{\delta_j} \colon\ j \in J\} \wedge \sup\{a^L_{\delta_{j'}} \colon\ j' \in J'\} \\ = \sup\{a^L_{\delta_j} \colon\ j \in J\} \wedge \sup\{a^L_{\eta_{i(\lambda)}} \colon\ \lambda \in \Lambda, i(\lambda) \in I(\lambda)\}$$

proving the lemma. □

Lemmas 1–3 prove the theorem. □

1.4. Some problems. Consider the class of all frames having a given property **P**. It is interesting for what properties **P** this class contains universal elements. For example:

1. Are there universal elements in the class **CRegFrm**?
2. Are there universal elements in the class **RegFrm**?
3. Are there universal elements in the class **0-DFrm**?

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