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Homeomorphism groups of homogeneous compacta need not be minimal

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ABSTRACT

It is shown that the homeomorphism group of the n-dimensional Menger universal continuum is not minimal. This answers a question by Stojanov from about 1984.

1. Introduction

All spaces under discussion are Hausdorff.

A topological group G is called *minimal* if its topology cannot be properly weakened to another group topology. It is known that a minimal Abelian topological group is precompact (Prodanov and Stojanov [23]), and that for non-Abelian groups this need not hold (Gaughan [17]). For information on minimal groups, see e.g., Dikranjan, Prodanov and Stojanov [10], Dikranjan and Megrelishvili [9] and Lukács [19].

It was asked by Stojanov (see Arhangel'skii [3, VI.7] or Comfort, Hofmann and Remus [7, 3.3.3(a)]), whether the homeomorphism group $\mathscr{H}(X)$ of a homogeneous compactum is minimal. As usual, $\mathscr{H}(X)$ is endowed with the compact-open topology. It is known that this is the case for X the Cantor set (Gamarnik [16]; see also Uspenskiy [24]), but it is not known for X the Hilbert cube (this is a question of Uspenskiy [25]). The aim of this note is to answer Stojanov's question in the negative.

A topological group is *non-archimedean* if it has a local base at the identity consisting of open subgroups. A non-archimedean topological group is clearly zero-dimensional. The group of rational numbers with its usual topology is an example of a zero-dimensional group which is not non-archimedean.

The aim of this note is to prove the following result.

Theorem 1.1. *For $n \geqslant 1$, let X be an n-dimensional compact space such that for every nonempty open subset U of X there is a compact subset A of U that homotopically dominates the n-sphere. Then $\mathscr{H}(X)$ admits a weaker non-archimedean group topology whose weight does not exceed the weight of X.*

For the proof of Theorem 1.1 we make good use of the proof of Theorem 5 in Oversteegen and Tymchatyn [22]. Similar arguments were also used by Anderson [1] (for details, see [6, Theorem 1.3]).

What we will describe is actually a (simple) method for constructing potentially interesting non-archimedean group topologies on homeomorphism groups $\mathscr{H}(X)$ for compact spaces X. This method may have the potential of applications way beyond the scope of this note.

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For $n \geqslant 1$, let μ^n denote the n-dimensional universal Menger continuum (Menger [20]). These spaces are obtained from finite-dimensional cubes by drilling holes in them in a way similar to the creation of the Cantor ternary set by repeatedly deleting the open middle thirds of a set of line segments. See [14, §1.11] for details. From the definition of μ^n it is clear that every nonempty open subset of it contains a copy of $\mathbb{S}^n$. Hence μ^n satisfies the conditions mentioned in Theorem 1.1.

Bestvina [5] provided elegant characterizations of these spaces and proved their homogeneity (for $n = 1$ this was done earlier by Anderson [2]). We denote the group of homeomorphisms of μ^n by $\mathscr{H}^n$. It was shown in Oversteegen and Tymchatyn [22, Theorem 5] that $\dim \mathscr{H}^n \leqslant 1$. Dijkstra [8, Theorem 7] established that $\mathscr{H}^n$ contains a copy of the famed Erdős space $\mathscr{E}$ from [15] which is 1-dimensional. The surprising and highly counterintuitive conclusion of these results is that $\dim \mathscr{H}^n = 1$.

By Theorem 1.1, $\mathscr{H}^n$ admits a weaker (separable metrizable) non-archimedean group topology. This topology is strictly weaker than the 1-dimensional compact-open topology on $\mathscr{H}^n$ and so μ^n solves Stojanov's problem in the negative.

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2. Preliminaries

For $n \in \mathbb{N}$, let $\mathbb{S}^n$ denote the euclidean sphere $\{x \in \mathbb{R}^{n+1}\colon \|x\| = 1\}$. As usual, by $f \simeq g$ we mean that f and g are homotopic functions. It is well known, and easy to prove, that if $f, g : X \to \mathbb{S}^n$ are such that for each $x \in X$, $f(x)$ and $g(x)$ are not antipodal, then $f \simeq g$ (Dugundji [11, XV.1.2(1)]). In particular, if $\|f(x) - g(x)\| < 1$ for every $x \in X$, then $f \simeq g$.

Let X and Y be spaces. We say that X *homotopically dominates* Y if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity function on Y.

Let $f, f' : X \to Y$ and $g, g' : Y \to Z$. If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$. This elementary fact about the homotopy relation will be used without explicit reference from now on.

If X and Y are topological spaces, then $C(X, Y)$ denotes the set of all continuous functions from X to Y endowed with the compact-open topology. Moreover, let $\mathscr{H}(X, Y)$ denote $\{h \in C(X, Y)\colon h \text{ is a homeomorphism}\}$. If $X = Y$, then $\mathscr{H}(X)$ abbreviates $\mathscr{H}(X, X)$. Hence $\mathscr{H}(X)$ is the *group of homeomorphisms of X with the compact-open topology*. It is not necessarily a topological group with function composition as the group operation. But for a compact space X, $\mathscr{H}(X)$ *is* a topological group with the relative topology from $C(X, X)$ and function composition as the group operation.

Let G be a group, and let $\mathscr{G}$ be a collection of subsets of G with the following properties:

(G1) $G^{-1} = G$ for every $G \in \mathscr{G}$,
(G2) for every $G \in \mathscr{G}$, there exists $H \in \mathscr{G}$ such that $H^2 \subseteq G$,
(G3) for every $G \in \mathscr{G}$ and $x \in G$, there is $H \in \mathscr{G}$ such that $x^{-1}Hx \subseteq G$.

Let $\mathscr{H}$ denote the family of all finite intersections of members of $\mathscr{G}$. Then

$$\tau = \big\{O \subseteq G\colon (\forall x \in O)\, (\exists H \in \mathscr{H})\, (Hx \subseteq O)\big\}$$

is a group topology on G. If $\mathscr{G}$ moreover satisfies

(G4) $\{e\} = \bigcap \mathscr{G}$,

then τ is Hausdorff. For details, see [19, Proposition 1.12] (or [18, II.4.5], [4, 1.3.12]). Observe that a T_1-topological group is Tychonoff, see e.g. [18, II.8.4] or [4, 3.3.11].

The identity function on a set X is denoted by id_M or 1_M.

3. Proof of Theorem 1.1

Let X be a compact space satisfying the hypotheses stated in Theorem 1.1. In addition, let U be a dense subset of $C(X, \mathbb{S}^n)$ with extra conditions to be specified later. For every $u \in U$ we put

$$C_u = \big\{h \in \mathscr{H}(X)\colon u \circ h \simeq u\big\}.$$

Lemma 3.1. *For $u \in U$, C_u is a clopen subgroup of $\mathscr{H}(X)$.*

Proof. Let $f, g \in C_u$. Then $f \circ g \in C_u$ since $u \circ (f \circ g) = (u \circ f) \circ g \simeq u \circ g \simeq u$. Moreover, $f^{-1} \in C_u$ since $u = u \circ f \circ f^{-1} \simeq u \circ f^{-1}$. Hence C_u is a subgroup. To prove it is clopen, take an arbitrary $h \in \mathscr{H}(X)$. There exists a neighborhood N of h in $\mathscr{H}(X)$ such that for every $g \in N$, $\|(u \circ h) - (u \circ g)\| < 1$. Hence if $g \in N$, then $u \circ g \simeq u \circ h$, and so $g \in C_u$ if and only if $h \in C_u$. This clearly implies that C_u is open and that $\mathscr{H}(X) \setminus C_u$ is open. □

Lemma 3.2. *For every $u \in U$ and $g \in \mathscr{H}(X)$ there exists $v \in U$ such that $gC_vg^{-1} \subseteq C_u$.*

Proof. Pick $v \in U$ such that $\|v - (u \circ g)\| < 1$. Now if $h \in C_v$, then $v \circ h \simeq v$, and hence

$$u \circ g \circ h \simeq v \circ h \simeq v \simeq u \circ g.$$

So $u \circ g \circ h \circ g^{-1} \simeq u \circ g \circ g^{-1} = u$, as required. □

Hence $\mathscr{C} = \{C_u\colon\ u \in U\}$ satisfies the conditions (G1) through (G3) in Section 2.

Remark 3.3. The collection $\mathscr{C}$ determines a group topology τ on $\mathscr{H}(X)$. Observe that besides compactness, the conditions on X were not used so far. In addition, with respect to homotopies the only thing we used is that 'close' maps into $\mathbb{S}^n$ are homotopic. So we can replace $\mathbb{S}^n$ by any ANR. The problem with this topology is of course that it may not be Hausdorff. Consider for example the case that $\mathscr{H}(X)$ is connected. Below we use the conditions on the space X in Theorem 1.1 to prove Hausdorffness. Different ANR's and different arguments may yield Hausdorffness in different situations.

Now we impose extra conditions on U. Assume that U is a dense subset of $C(X, \mathbb{S}^n)$ whose cardinality does not exceed the weight of X [13, Theorem 3.4.16].

Lemma 3.4. *Let $g \in \mathscr{H}(X)$ not be the identity. Then there exists $u \in U$ such that $g \notin C_u$.*

Proof. Since g is not the identity, there is a nonempty open subset V of X such that $V \cap g(V) = \emptyset$. Let A be a compact subset of V which homotopically dominates $\mathbb{S}^n$. Let $\xi : \mathbb{S}^n \to A$ and $\eta : A \to \mathbb{S}^n$ be continuous functions such that $\eta \circ \xi$ is homotopic to the identity function on $\mathbb{S}^n$. Define $\alpha : A \cup g(A) \to \mathbb{S}^n$ as follows:

$$\alpha(x) = \begin{cases} \eta(x) & (x \in A), \\ (1, 0, \ldots, 0) & (x \in g(A)). \end{cases}$$

Since $\dim X = n$, α can be extended to a continuous function $\bar{\alpha} : X \to \mathbb{S}^n$ ([14, 3.2.10]). Pick $u \in U$ such that $\|\bar{\alpha} - u\| < 1$. We claim that $g \notin C_u$. Striving for a contradiction, assume that $u \circ g \simeq u$. Since $\bar{\alpha} \simeq u$, we have

$$\bar{\alpha} \circ g \simeq u \circ g \simeq u \simeq \bar{\alpha},$$

hence $\bar{\alpha} \circ g \circ \xi \simeq \bar{\alpha} \circ \xi$. But $\bar{\alpha} \circ g \circ \xi$ is the constant function with value $(1, 0, 0, \ldots)$, and $\bar{\alpha} \circ \xi = \eta \circ \xi$ is homotopic to the identity function on $\mathbb{S}^n$. This violates the Brouwer Fixed-Point Theorem. □

Hence $\mathscr{C}$ satisfies condition (G4) in Section 2. Since $\mathscr{C}$ consists of clopen subgroups of $\mathscr{H}(X)$, we consequently conclude that there is a Hausdorff group topology τ on $\mathscr{H}(X)$ such that $\mathscr{C}$ is a neighborhood subbase at e in $(\mathscr{H}(X), \tau)$. Hence τ is contained in the original topology on $\mathscr{H}(X)$, and the elements of $\mathscr{C}$ are clopen in $(\mathscr{H}(X), \tau)$. As a consequence, $(\mathscr{H}(X), \tau)$ is non-archimedean.

Lemma 3.5. *The weight of $(\mathscr{H}(X), \tau)$ does not exceed the weight of X.*

Proof. Let $\kappa \geqslant \omega$ be the weight of X. As we observed in Section 2, the weight and hence the Lindelöf number of $\mathscr{H}(X)$ does not exceed κ. This implies that the Lindelöf number of $(\mathscr{H}(X), \tau)$ does not exceed κ. But $|\mathscr{C}| \leqslant \kappa$, hence the neutral element of $(\mathscr{H}(X), \tau)$ has a neighborhood base of size at most κ. This clearly implies that the weight of $(\mathscr{H}(X), \tau)$ is at most $\kappa \cdot \kappa = \kappa$. □

It is natural to ask whether τ is a 'nice' topology in the sense that the natural action

$$\big(\mathscr{H}(X), \tau\big) \times X \to X$$

is continuous. We will show that for the spaces μ^n, this is not the case.

Proposition 3.6. *Let C be a clopen subgroup of $\mathscr{H}(X)$, where X is a homogeneous compact space. Then for every $x \in X$ we have that Cx is clopen in X.*

Proof. By the Effros Theorem from [12] (see also [21]), Cx is open in X for every $x \in X$. Now pick an arbitrary $x \in X$, and take $y \in \overline{Cx}$. Then $Cy \cap Cx \neq \emptyset$ since Cy is open. Pick $\alpha, \beta \in C$ such that $\alpha x = \beta y$. Then $(\beta^{-1}\alpha)x = y$, i.e., $y \in Cx$ since $\beta^{-1}\alpha \in C$. □

Hence if the space X in Proposition 3.6 is a nontrivial continuum, then for every clopen subgroup C of $\mathscr{H}(X)$ and every $x \in X$ we have that $Cx = X$. This evidently implies that for a weaker non-archimedean topology $\mathscr{T}$ on $\mathscr{H}(X)$, the natural action $X \times (\mathscr{H}(X), \mathscr{T}) \to X$ is badly discontinuous. Simply observe that if V is any proper nonempty open subset of X, then the preimage of V under the natural action is not open.

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