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A compact F-space with noncoinciding dimensions

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ABSTRACT

We prove that there exists a compact F-space of weight $\mathfrak{c}^+$ with noncoinciding dimensions.

1. Introduction

All spaces under discussion here are Tychonoff.

It is well known that the three fundamental dimension functions take on the same values for all separable metrizable spaces. This does not hold for various other classes of spaces with very nice properties, see e.g., Fedorchuk [10], Kozlov [15] and Charalambous [2] for results and references. The first examples of compact spaces with noncoinciding dimensions were constructed by Lunc and Lokucievskiĭ [16] in 1949. Lokucievskiĭ's now classical example is a compact space X of weight ω_1 such that $\dim X = \operatorname{ind} X = 1$ but $\operatorname{Ind} X = 2$. In Hart and van Mill [14, Theorem 2.1] it was proved under the Continuum Hypothesis that for every compact F-space X of weight $\mathfrak{c}$ we have $\operatorname{ind} X = \operatorname{Ind} X = \dim X$. It is not known whether this is a theorem of ZFC. The aim of this paper is to show that this cannot be generalized to compact F-spaces of larger weight. We will use the example by Lokucievskiĭ [16] for the construction of the following example:

Example 1.1. There is a compact F-space $\mathbb{X}$ of weight $\mathfrak{c}^+$ with $\operatorname{Ind} \mathbb{X} = 2$ and closed subspaces X_0 and X_1 such that $\mathbb{X} = X_0 \cup X_1$ and $\operatorname{Ind} X_0 = \operatorname{Ind} X_1 = 1$.

This implies that $\dim \mathbb{X} = 1$. From the construction it will be clear that $\operatorname{ind} \mathbb{X} = 2$.

2. Preliminaries

2.1. Notation and terminology

For a subset A of a space X its *boundary* $\operatorname{Fr} A$ is the set $\overline{A} \setminus \operatorname{Int} A$. Hence if U is open, then $\operatorname{Fr} U = \overline{U} \setminus U$. A subset of a space X is *clopen* if it is both open and closed. A continuous surjection $f : X \to Y$ is called *irreducible* provided that there does not exist a proper closed subset A of X such that $f(A) = Y$. It is not difficult to show that if $f : X \to Y$ is a continuous

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surjection with compact fibers, then there is a closed subspace A of X such that $f \restriction A : A \to Y$ is irreducible (and hence, onto) [8, Exercise 3.1.C].

Let A and B be disjoint closed subsets of a space X. We say that a closed subset K of X is a *partition* between A and B provided that $X \setminus K$ can be written as $U \cup V$, where U and V are disjoint open subsets of X such that $A \subseteq U$ and $B \subseteq V$.

A subset A of a space X is called a *P-set* provided that the intersection of countably many neighborhoods of A is again a neighborhood of A.

If X is a space then βX denotes its Čech–Stone compactification, and $X^* = \beta X \setminus X$. If X is normal, and A is closed in X, then we can and will identify βA and the closure of A in βX [8, 3.6.8]. Observe that if A and B are closed subsets of the metrizable space X, and $A^* = B^*$, then the symmetric difference $A \triangle B$ of A and B has compact closure in X. To prove this, assume that e.g., $A \setminus B$ does not have compact closure in X. By [8, 4.1.17], there is a countably infinite discrete set D in $A \setminus B$ which is closed in X. Since $D \cap B = \emptyset$, we have $\overline{D} \cap \overline{B} = \emptyset$ (here 'closure' means closure in βX). Since D is infinite, there is a point $p \in D^*$. Then $p \in A^* \setminus B^*$, which is a contradiction.

If Y is compact and $f : X \to Y$ is continuous, then f can be extended to a continuous function $\beta f : \beta X \to Y$. This (unique) function is called the *Stone extension* of f. We put $\bar{f} = \beta f \restriction X^* : X^* \to Y$.

Let $\{x_n : n < \omega\}$ be a sequence of points in the compact space X. Then for every ultrafilter ξ on ω, let $\lim_\xi \{x_n : n < \omega\}$ be the unique point in the intersection $\bigcap_{P \in \xi} \overline{\{x_n : n \in P\}}$. This point is called the *ξ-limit* of the sequence $\{x_n : n \in \omega\}$. Let f denote the obvious function $\omega \to \{x_n : n < \omega\}$. It is easy to see that $\lim_\xi \{x_n : n < \omega\}$ is equal to $\beta f(\xi)$.

If $U \subseteq X$ is open, then $\operatorname{Ex} U = \beta X \setminus (\overline{X \setminus U})$ is open in βX. Clearly, $\operatorname{Ex} U$ is the largest open subset of βX whose intersection with X equals U. Let X be normal. If $F \subseteq X$ is closed, and $U \subseteq X$ is open, and $F \subseteq U$, then $\overline{F} \subseteq \operatorname{Ex}(U)$. Simply observe that $F \cap (X \setminus U) = \emptyset$ and hence $\overline{F} \cap \overline{X \setminus U} = \emptyset$. This fact will be used frequently in the forthcoming, and without explicit reference.

Cardinals are initial (von Neumann) ordinals, and get the discrete topology; $\mathfrak{c}$ is the cardinality of the continuum. If X is a set and κ is a cardinal number, then $[X]^\kappa$, $[X]^{<\kappa}$ and $[X]^{\leqslant\kappa}$ denote $\{A \subseteq X : |A| = \kappa\}$, $\{A \subseteq X : |A| < \kappa\}$ and $\{A \subseteq X : |A| \leqslant \kappa\}$, respectively.

2.2. Dimension theory

A space is called *zero-dimensional* if it has a base consisting entirely of clopen sets. In this paper we are only interested in the dimension theory of compact spaces. Our basic dimension function is the *covering dimension* $\dim X$ of a space X. So if we say that a compact space X is n-dimensional, this always refers to the covering dimension. Besides the covering dimension, there are the so-called *small* and *large* inductive dimension functions ind and Ind, respectively. For more information on dimension theory and definitions, see [9]. For us, the following well-known results will be important:

Theorem 2.1. *If X is a compact space, then $\dim X \leqslant \operatorname{ind} X \leqslant \operatorname{Ind} X$. Moreover,*

$$\dim X = 0 \quad \Leftrightarrow \quad \operatorname{ind} X = 0 \quad \Leftrightarrow \quad \operatorname{Ind} X = 0.$$

Proof. The inequality $\operatorname{ind} X \leqslant \operatorname{Ind} X$ holds for all normal spaces X [9, 1.6.3]. Moreover, the inequality $\dim X \leqslant \operatorname{ind} X$ holds for all strongly paracompact spaces X [9, 3.1.29]. The second part of the theorem is a direct consequence of [9, 3.1.30]. □

So for a compact space there is only one notion of zero-dimensionality. In fact, a compact space is zero-dimensional if and only if it does not contain any nontrivial continuum [9, 1.4.5].

None of these inequalities is sharp, even for spaces with very nice properties, see e.g., Fedorchuk [10], Kozlov [15] and Charalambous [2] for results and references.

Proposition 2.2. *Let X and Y be compact spaces and $f : X \to Y$ a continuous surjection. Put $A = \overline{\bigcup\{f^{-1}(y) : (y \in Y) \,\&\, (|f^{-1}(y)| > 1)\}}$ and $B = f(A)$, respectively. Assume moreover that $\operatorname{Ind} X \leqslant 1$. Then the following hold*:

(1) *If* $\operatorname{Ind} B \leqslant 0$, *then* $\operatorname{Ind} Y \leqslant 1$.
(2) *If* $\operatorname{Ind} A \leqslant 0$, *and f is* $(\leqslant)2$*-to-one, then* $\operatorname{Ind} Y \leqslant 1$.

Proof. For (1), let E and F be arbitrary disjoint closed subsets of Y. There are relatively clopen disjoint sets C and D in B such that $C \cup D = B$, and $(E \cup C) \cap (F \cup D) = \emptyset$. Let R be a partition between $f^{-1}(E \cup C)$ and $f^{-1}(F \cup D)$ in X such that $\operatorname{Ind} R \leqslant 0$. Write $X \setminus R$ as $U \cup V$, where U and V are disjoint open subsets of X such that $f^{-1}(E \cup C) \subseteq U$ and $f^{-1}(F \cup D) \subseteq V$. Let $p \in U \cup R$ and $q \in V \cup R$ be distinct points such that $f(p) = f(q)$. We may assume without loss of generality that $f(p) = f(q) \in C$. But then $q \in U$, which is a contradiction. Hence such points p and q do not exist, from which it follows that $f(U \cup R) \cap f(V \cup R) = f(R)$. Hence $f(R)$ is by compactness a partition between E and F in X and is homeomorphic to R, which is as desired.

For (2), let E and F be disjoint closed subsets of Y, and let C be a clopen subset of A such that $f^{-1}(E) \cap A \subseteq C \subseteq A \setminus (f^{-1}(F) \cap A) = A \setminus f^{-1}(F)$. Let D be a partition between $f^{-1}(E) \cup C$ and $f^{-1}(F) \cup (A \setminus C)$ in X such that $\operatorname{Ind} D \leqslant 0$.

Write $X \setminus D$ as $U \cup V$, where U and V are nonempty open subsets of X such that $f^{-1}(E) \cup C \subseteq U$ and $f^{-1}(F) \cup (A \setminus C) \subseteq V$. Let $\hat{U} = U \cup D$ and $\hat{V} = V \cup D$, respectively. Observe that both $\hat{U}$ and $\hat{V}$ are closed, that $\hat{U} \cup \hat{V} = X$, and that $\hat{U} \cap \hat{V} = D$ misses A. Moreover, $\hat{U} \cap f^{-1}(F) = \emptyset$ and $\hat{V} \cap f^{-1}(E) = \emptyset$. From this it clearly follows that

$$\hat{D} = f(\hat{U}) \cap f(\hat{V})$$

is a partition between E and F. We claim that $\dim \hat{D} \leqslant 0$ (and hence $\operatorname{Ind} \hat{D} \leqslant 0$ by Theorem 2.1). To prove this, pick an arbitrary $y \in \hat{D}$. There are $u \in \hat{U}$ and $v \in \hat{V}$ such that $f(u) = f(v) = y$.

Case 1. $u \notin A$ and $v \in A$.

Then $f^{-1}(y)$ contains two distinct points and hence must be contained in A. Since $u \notin A$, this is impossible.

Case 2. $u \notin A$ and $v \notin A$.

Observe that $f \restriction (X \setminus A)$ is one-to-one. Hence $u = v \in D$.

Case 3. $u \in A$ and $v \in A$.

Then $u \in C$ and $v \in C \setminus A$.

The conclusion is that $\hat{D}$ is contained in the disjoint union of the compact sets $f(D)$ and $f(C')$, where $C' = \{u \in C\colon (\exists v \in C \setminus A)\ (f(u) = f(v))\}$, and that f is one-to-one on both D and C' (here we use that f is ($\leqslant$)2-to-one). Observe that by compactness, both $f \restriction D$ and $f \restriction C'$ are homeomorphisms. Since $\operatorname{Ind} A \leqslant 0$ and $\operatorname{Ind} D \leqslant 0$, this means that the compact space $\hat{D}$ is contained in the union of two zero-dimensional compact spaces and hence is zero-dimensional itself [9, 3.1.8]. □

2.3. *F-spaces*

An *F-space* is a space in which every cozero-set is C^*-embedded, see [12]. It is easy to see that a compact space X is an F-space if the following holds: if F and G are F_σ-subsets of X with $\overline{F} \cap G = \emptyset = F \cap \overline{G}$, then $\overline{F} \cap \overline{G} = \emptyset$ (van Douwen [4, p. 239]).

Let X be a compact F-space, and let D in X be F_σ (for example, D is countable). We claim that D is C^*-embedded in X. For this it suffices to prove that $\overline{D} = \beta D$. Indeed, if A and B are relatively closed disjoint subsets of D, then A and B are F_σ-subsets of X such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, and so by van Douwen's result, $\overline{A} \cap \overline{B} = \emptyset$. Since every infinite space contains an infinite discrete space, this shows that every infinite compact F-space contains a copy of $\beta\omega$ and hence has weight at least $\mathfrak{c}$. See Comfort, Hindman and Negrepontis [3] and Woods [18] for stronger results about C^*-embedded subspaces of compact F-spaces.

It is clear that in a compact space X van Douwen's condition is equivalent to the following statement: every two disjoint open F_σ-subsets of X have disjoint closures.

A closed subspace of a compact F-space is again a compact F-space, as well as the topological sum of finitely many compact F-spaces. These facts follow easily from van Douwen's criterion and will be used without explicit reference in the forthcoming.

The basic examples of compact F-spaces are the spaces of the form X^*, where X is any locally compact, σ-compact space [12, 14.27]. Such a remainder also has the property that every nonempty G_δ of it has nonempty interior, see Fine and Gillman [11, p. 377].

2.4. *Continua in βX*

For each n, let X_n be a nontrivial continuum. We assume that the sequence $\{X_n\colon n < \omega\}$ is pairwise disjoint. We let X denote the topological sum of the X_n's. Finally, let $\pi : X \to \omega$ be the 'projection' defined by $f(x) = n$ iff $x \in X_n$.

The collection of components of X^* coincides with the collection

$$\{\bar{\pi}^{-1}(p)\colon\ p \in \omega^*\}.$$

A moment's reflection consequently shows that a closed subset C of X^* is a component of X^* if and only if there exists $p \in \omega^*$ such that

$$C = \bigcap_{P \in P} \overline{\bigcup_{n \in P} X_n}.$$

Observe that all these components are nontrivial. For details, see e.g. Hart [13].

2.5. Adjunction spaces

Let X and Y be two disjoint compact spaces, let $A \subseteq X$ be closed, and let $f : A \to Y$ be continuous. The decomposition

$$\mathcal{Q} = \{f^{-1}(y) \cup \{y\}\colon\ y \in f(A)\} \cup \{\{x\}\colon\ x \in (X \setminus A) \cup (Y \setminus f(A))\}$$

of the topological sum $X+Y$ of X and Y is upper semi-continuous, and the quotient space $(X+Y)/\mathcal{Q}$ is denoted by $X \cup_f Y$. It is called the *adjunction space* obtained from X and Y by f. Let $p_f : X+Y \to X \cup_f Y$ be the natural quotient map.

That X and Y are disjoint is not essential of course. A similar construction can also be performed when X and Y intersect. We simply replace X by $X \times \{0\}$ and Y by $Y \times \{1\}$, and proceed as before. So when we talk about adjunction spaces in the sequel, we will always implicitly assume that the spaces under consideration are disjoint.

Observe that we can think of $X \cup_f Y$ as created in two steps. In the first step we replace A in X by $f(A)$ thus obtaining the space X'. In X' and the space Y there are two copies of $f(A)$ that are being identified in the second step. That brings us to $X \cup_f Y$. We concentrate on compact spaces here. For noncompact spaces one can perform similar constructions, see [7, p. 127] for details.

In case $f : A \to Y$ is surjective, there is no need to consider the topological sum $X+Y$ for the construction of $X \cup_f Y$. For then, $X \cup_f Y$ is simply $X/\mathcal{Q}$, where $\mathcal{Q}$ is the upper semi-continuous decomposition

$$\mathcal{Q} = \{f^{-1}(y)\colon\ y \in Y\} \cup \{\{x\}\colon\ x \in X \setminus A\}$$

of X.

The following result is implicit in Balcar, Frankiewicz and Mills [1] (see also [17, 1.4.1]). For completeness sake, we will include the simple proof.

Lemma 2.3. *Let X and Y be two compact F-spaces. If A is a closed P-set in X and $f : A \to Y$ is continuous, then $X \cup_f Y$ is a compact F-space.*

Proof. Compactness is clear. To prove that $X \cup_f Y$ is an F-space, let U and V be disjoint open F_σ-subsets of $X \cup_f Y$. We have to show that $\overline{U} \cap \overline{V} = \emptyset$, Section 2.3. It will be convenient to identify Y and $p_f(Y)$, and, similarly, $X \setminus A$ and $p_f(X \setminus A)$. Since Y is a compact F-space, $\overline{U \cap Y} \cap \overline{V \cap Y} = \emptyset$. Let E and F be disjoint closed G_δ-subsets of $X \cup_f Y$ that are neighborhoods of $\overline{U \cap Y}$ respectively $\overline{V \cap Y}$. Then $U \setminus E$ and $V \setminus F$ are disjoint open F_σ-subsets of $X \cup_f Y$ which both do not meet A.

Claim 1. $\overline{U \setminus E} \cap \overline{V} = \emptyset$ and $\overline{V \setminus F} \cap \overline{U} = \emptyset$.

Striving for a contradiction, assume that there exists e.g. an element $p \in \overline{U \setminus E} \cap \overline{V}$. Observe that $U \setminus E$ is an open F_σ-subset of X that misses A. Hence $\overline{U \setminus E} \cap A = \emptyset$ since A is a P-set. Let K be an open F_σ-subset of X such that $\overline{U \setminus E} \subseteq K \subseteq \overline{K} \subseteq X \setminus A$. Then, clearly, $p \in K \cap \overline{V} \subseteq \overline{K \cap V}$. Hence $U \setminus E$ and $K \cap V$ are disjoint open F_σ's of X such that $p \in \overline{U \setminus E} \cap \overline{K \cap V}$, which contradicts X being an F-space.

Since $\overline{U} = \overline{U \setminus E} \cup \overline{U \cap E}$, $\overline{V} = \overline{V \setminus F} \cup \overline{V \cap F}$, and, clearly, $\overline{U \cap E} \cap \overline{V \cap F} = \emptyset$, we get by Claim 1 that $\overline{U} \cap \overline{V} = \emptyset$, as required. □

3. Reflections on $\beta[0,1)$

Let D denote an arbitrary countable dense subset of $(0,1)$. In $\mathbb{I}$ we split each point $d \in D$ in two points, d^- and d^+. The points in $\mathbb{I} \setminus D$ will not be split. Order the set

$$\triangle = (\mathbb{I} \setminus D) \cup \{d^-, d^+\colon\ d \in D\}$$

in the natural way, where d^- always precedes d^+. Endow $\triangle$ with the order topology derived from this order. It is clear that topologically, $\triangle$ is nothing but the ordinary Cantor middle-third set in $\mathbb{I}$. Let $f : \triangle \to \mathbb{I}$ be the unique order preserving function that maps for each $d \in D$ the points d^- and d^+ to d. Clearly, f is a continuous surjection, and for $y \in \mathbb{I}$, $|f^{-1}(y)| = 2$ if $y \in D$ and $|f^{-1}(y)| = 1$ otherwise. That such a map exists is well known of course and goes back to Alexandroff and Hausdorff (see [9, 1.3.D]). Observe that f is irreducible.

A set of the form $[d_0^+, d_1^-]$, where $d_0, d_1 \in D$ and $d_0 < d_1$, is called a *clopen segment* of $\triangle$.

Let $\varepsilon : \mathbb{I} \to \mathbb{I}$ be a homeomorphism such that $\varepsilon(0) = 0$, $\varepsilon(1) = 1$ and $\varepsilon(D) \cap D = \emptyset$. It will be convenient to denote $\varepsilon(D)$ by E.

Put $\mathbb{K} = \triangle \setminus \{1\}$, and let $g_0 = f \restriction \mathbb{K} : \mathbb{K} \to [0,1)$ and $g_1 = (\varepsilon \circ g_0) \restriction \mathbb{K} : \mathbb{K} \to [0,1)$, respectively. Then g_0 and g_1 are both perfect since $f^{-1}(1) = \{1\}$ and $\varepsilon^{-1}(1) = \{1\}$. Observe that both g_0 and g_1 are irreducible.

Observe that $\mathbb{K}$ is a σ-compact zero-dimensional space. It consequently follows that $\operatorname{Ind} \beta\mathbb{K} = \operatorname{Ind} \mathbb{K}^* = 0$ [9, 2.2.10]. Moreover, $\mathbb{K}$ has weight ω, hence the weight of $\beta\mathbb{K}$ is easily seen to be $\mathfrak{c}$ (prove that $\mathbb{K}$ has $\mathfrak{c}$ many clopen subsets).

Consider for $i = 0, 1$ the Stone extensions

$$\beta g_i : \beta\mathbb{K} \to \beta[0, 1),$$

and let

$$\bar{g}_i = \beta g_i \restriction \mathbb{K}^* : \mathbb{K}^* \to [0, 1)^*.$$

We claim that the $\bar{g}_i$ are ($\leqslant$)2-to-one continuous surjections. For this, first observe that the $\bar{g}_i$ are continuous surjections since the g_i are perfect surjections. That they are ($\leqslant$)2-to-one is a consequence of van Douwen [5, Lemma 4.3].

Lemma 3.1. *Both $\bar{g}_0$ and $\bar{g}_1$ are irreducible.*

Proof. It is clear that it suffices to prove this for $\bar{g}_0$. So let A be a proper closed subset of $\mathbb{K}^*$. There is a noncompact clopen subset V of $\mathbb{K}$ such that $V^* \subseteq \mathbb{K}^* \setminus A$. Write V as $\bigcup_{n<\omega} V_n$, where for each n, $V_n = [d_n^+, e_n^-]$ is a clopen segment and $V_n \cap V_m = \emptyset$ if $n \neq m$. For every n, pick a point $p_n \in (d_n, e_n)$. Observe that $P = \{p_n\colon n < \omega\}$ is closed in $[0, 1)$ but not compact. Hence we may pick a point $p \in P^*$. Put $W = \mathbb{K} \setminus V$. Then W is clopen and $A \subseteq \overline{W}$. Moreover, $g_0(W)$ is a closed subset of $[0, 1)$ that misses P, i.e., $p \notin \overline{g_0(W)} \supseteq \bar{g}_0(A)$. □

Proposition 3.2. *Let $U \subseteq [0, 1)^*$ be nonempty. If U is not dense, then $\overline{\bar{g}_0^{-1}(U)}$ is not clopen or $\overline{\bar{g}_1^{-1}(U)}$ is not clopen.*

Proof. Striving for a contradiction, assume that $\overline{\bar{g}_0^{-1}(U)}$ and $\overline{\bar{g}_1^{-1}(U)}$ are both clopen. Since U is not dense in $[0, 1)^*$, $\overline{\bar{g}_0^{-1}(U)}$ and $\overline{\bar{g}_1^{-1}(U)}$ are proper subsets of $\mathbb{K}^*$.

Claim 1. If A is a clopen subset of $\mathbb{K}$ such that $A^* = \overline{\bar{g}_0^{-1}(U)}$, then $\operatorname{Fr} g_0(A)$ is nonempty and is contained in D.

Proof. Since $\overline{\bar{g}_0^{-1}(U)} \neq \mathbb{K}^*$, it follows that A is a proper closed subset of $\mathbb{K}$ and A is nonempty since U is nonempty. Hence since g_0 is perfect and irreducible, $g_0(A)$ is a proper nonempty closed subset of $[0, 1)$.

By connectivity of $[0, 1)$, the boundary of $g_0(A)$ is nonempty. Take an arbitrary $p \in \operatorname{Fr} g_0(A)$. We will prove that $p \in D$. If $|g_0^{-1}(p)| = 1$, then $q = g_0^{-1}(p)$ belongs to A since $p \in g_0(A)$. Since A is open and g_0 is perfect, there is a neighborhood V of p such that $g_0^{-1}(V) \subseteq A$. But this means that p is in the interior of $g_0(A)$, which is a contradiction. Hence $|g_0^{-1}(p)| = 2$, i.e., $p \in D$. □

Since ε is a homeomorphism, the following result has an identical proof.

Claim 2. If B is a clopen subset of $\mathbb{K}$ such that $B^* = \overline{\bar{g}_1^{-1}(U)}$, then $\operatorname{Fr} g_0(B)$ is nonempty and is contained in E.

Now let A and B be arbitrary clopen sets such as in Claims 1 and 2.

Claim 3. $g_0(A)^* = g_1(B)^* = \overline{U}$.

Proof. It suffices to prove that $g_0(A)^* = \overline{U}$. First observe that $\beta g_0(\overline{A}) = \overline{g_0(A)}$. This holds since $g_0(A)$ is dense in $\beta g_0(\overline{A})$ as well as $\overline{g_0(A)}$. Now $\overline{A} = A \cup A^*$, hence

$$\beta g_0(\overline{A}) = g_0(A) \cup \bar{g}_0\big(A^*\big) = g_0(A) \cup \bar{g}_0\big(\overline{\bar{g}_0^{-1}(U)}\big) = g_0(A) \cup \overline{U}.$$

Since

$$\overline{g_0(A)} = g_0(A) \cup g_0(A)^*,$$

we consequently get what we want. □

Claim 4. The open set $[0, 1) \setminus (g_0(A) \cup g_1(B))$ does not have compact closure in $[0, 1)$.

Proof. By Claim 3 we get that $(g_0(A) \cup g_1(B))^* = g_0(A)^* \cup g_1(B)^* = \overline{U}$ is a proper subset of $[0, 1)^*$, from which the desired result follows immediately. □

So from Claim 3 and the remarks in Section 2.3 we have that $g_0(A) \vartriangle g_1(B)$ has compact closure in $[0, 1)$. Hence there exists $t \in [0, 1)$ such that $g_0(A) \cap [t, 1) = g_1(B) \cap [t, 1)$. By Claim 4 we may assume without loss of generality that $t \in [0, 1) \setminus (g_0(A) \cup g_1(B) \cup D \cup E)$. Now put

$$\tilde{A} = A \cap \left[g_0^{-1}(t), 1\right), \qquad \tilde{B} = B \cap \left[g_1^{-1}(t), 1\right).$$

Then $\tilde{A}$ and $\tilde{B}$ are clopen subsets of $\mathbb{K}$ such that $\tilde{A}^* = A^*$ and $\tilde{B}^* = B^*$. Moreover,

$$g_0(\tilde{A}) = g_0(A) \cap [t, 1), \qquad g_1(\tilde{B}) = g_1(B) \cap [t, 1).$$

Hence $g_0(\tilde{A}) = g_1(\tilde{B})$. By Claims 1 and 2, $\emptyset \neq \operatorname{Fr} g_0(\tilde{A}) \subseteq D$ and $\emptyset \neq \operatorname{Fr} g_1(\tilde{B}) \subseteq E$. But this is a contradiction since $D \cap E = \emptyset$. $\square$

4. The example

Our example will be based on Lokucievskiĭ's paper [16] in which he constructs a simple compact space with noncoinciding dimensions. Since we aim at an F-space, we have to adjust the construction.

4.1. Step 1

Let $L = [\mathbf{u}, \mathbf{v}]$ be a compact connected ordered space of weight $\mathfrak{c}^+$ in which $\mathbf{v}$ is a $P_{\mathfrak{c}^+}$-point. (Such a space is easily found. For example, let L denote the long-segment of length $\mathfrak{c}^+$. That is, L is the one-point compactification of the product $\mathfrak{c}^+ \times [0, 1)$ endowed with the lexicographical order.)

Put $L_0 = L \setminus \{\mathbf{v}\}$, $S = L \times \mathbb{K}^*$, and $P = \{\mathbf{v}\} \times \mathbb{K}^*$, respectively. It is easy to prove that $\dim S = \operatorname{ind} S = \operatorname{Ind} S = 1$. Finally, observe that the weight of S is $\mathfrak{c}^+$ since the weight of $\mathbb{K}^*$ is $\mathfrak{c}$.

4.2. Step 2

Put $T = S \times \omega$. It will be convenient to represent a point from T as (x, p, n), where $x \in L$, $p \in \mathbb{K}^*$, and $n \in \omega$. Let $\pi : T \to S$ denote the projection $(x, p, n) \mapsto (x, p)$.

It is clear that $\operatorname{Ind} T = \operatorname{Ind} S = 1$, and it consequently follows that $\operatorname{Ind} \beta T = 1$ [9, 2.2.10]. Since T^* contains nontrivial continua by Section 2.4, we get $\operatorname{Ind} T^* = 1$. The weight of βT is easily seen to be equal to $(\mathfrak{c}^+)^\omega = \mathfrak{c}^+$.

The following lemma is a consequence of van Douwen and van Mill [6, Lemma 3]. For the convenience of the reader we will repeat its simple proof.

Lemma 4.1. *$\beta\pi^{-1}(P) = \overline{P \times \omega}$, hence $\bar{\pi}^{-1}(P) = (P \times \omega)^*$.*

Proof. That $\beta\pi^{-1}(P) \supseteq \overline{P \times \omega}$ is clear. Now take an arbitrary element $z \in \beta\pi^{-1}(P)$, and assume that $z \notin \overline{P \times \omega}$. There is a closed neighborhood C of z in βT which misses $P \times \omega$. Hence $C \cap T$ is a σ-compact subset of T which has z in its closure. But $\pi(C \cap T) \cap P = \emptyset$, hence $\overline{\pi(C \cap T)} \cap P = \emptyset$ since $P = \{\mathbf{v}\} \times \mathbb{K}^*$ is a P-set of S. Since $z \in \overline{C \cap T}$ and hence $\beta\pi(z) \in \overline{\pi(C \cap T)} \cap P$, this is a contradiction. $\square$

Fix arbitrary $\mu \in [\mathbf{u}, \mathbf{v})$ and $t \in \mathbb{K}^*$. Since $[\mu, \mathbf{v}]$ is a nontrivial continuum, every component of $([\mu, \mathbf{v}] \times \{t\} \times \omega)^*$ is by Section 2.4 of the form

$$\mathbb{I}(\mu, t, \xi) = \bigcap_{A \in \xi} \bigcup_{n \in A} \overline{[\mu, \mathbf{v}] \times \{t\} \times \{n\}} \tag{†}$$

for some free ultrafilter ξ on ω. Observe that

$$\mathbb{I}(\mu, t, \xi) \cap (P \times \omega)^* = \lim_{\xi} \left\{(\mathbf{v}, t, n) \colon n < \omega\right\} \tag{††}$$

is a single point.

4.3. Step 3

Let $\varphi = \bar{\pi} \restriction (P \times \omega)^* : (P \times \omega)^* \to P$, and consider the adjunction space

$$Y = T^* \cup_\varphi P.$$

Observe that Y is T^* with the $P_{\mathfrak{c}^+}$-set $(P \times \omega)^*$ of T^* replaced by P (in a natural way). Since $P = \{\mathbf{v}\} \times \mathbb{K}^* \approx \mathbb{K}^*$ is an F-space (Section 2.3), we get that Y is an F-space as well (Lemma 2.3). Since $\operatorname{Ind} P = 0$, we get $\operatorname{Ind} Y \leqslant 1$ by Proposition 2.2(1). Since Y contains a nontrivial continuum, $\operatorname{Ind} Y = 1$. Also, the weight of Y is easily seen to be equal to $\mathfrak{c}^+$.

Let $F : T^* \to Y$ denote the standard quotient map. We think of $T^* \setminus (P \times \omega)^*$ and $F(T^* \setminus (P \times \omega)^*)$ as the same spaces, 'identifying' p and $F(p)$ for every $p \in T^* \setminus (P \times \omega)^*$. Hence F restricts to the identity on $T^* \setminus (P \times \omega)^*$. It will also be convenient to think of P and $F((P \times \omega)^*)$ as the same spaces.

Observe that for $\xi \in \omega^*$ and $t \in \mathbb{K}^*$ we have that

$$(\dagger\dagger\dagger) \qquad F\Big(\lim_{\xi}\big\{(\mathbf{v}, t, n)\colon n < \omega\big\}\Big) = (\mathbf{v}, t).$$

Also observe that by (††) we have that the restriction of F to every continuum of the form $\mathbb{I}(\mu, t, \xi)$ is one-to-one. We will denote $F(\mathbb{I}(\mu, t, \xi))$ by $\hat{\mathbb{I}}(\mu, t, \xi)$, hence by our identifications,

$$\hat{\mathbb{I}}(\mu, t, \xi) = \Big(\mathbb{I}(\mu, t, \xi) \setminus \Big\{\lim_{\xi}\big\{(\mathbf{v}, t, n)\colon n < \omega\big\}\Big\}\Big) \cup \big\{(\mathbf{v}, t)\big\}.$$

There clearly is a map $G\colon Y \to S$ such that $\bar{\pi} = G \circ F$. The map G has the property that $G^{-1}(p) = \{p\}$ for every $p \in P$.

Our identifications are sometimes confusing. For example, P is a subset of S as well as Y, and a point in Y that belongs to P is just as in S represented by a pair $(\mathbf{v}, t)$, where $t \in \mathbb{K}^*$. However, without these identifications, the notation becomes cumbersome.

Lemma 4.2. *Let U be an open subset of $\mathbb{K}^*$. Moreover, let $t \in \overline{U}$ be such that every G_δ-subset of $\mathbb{K}^*$ that contains t meets U. Finally, let V be an open subset of Y such that $V \cap P = \{\mathbf{v}\} \times U$. Then* Fr V *contains a nontrivial continuum that misses P or $\overline{V}$ contains a closed G_δ-subset of P that contains $(\mathbf{v}, t)$.*

Proof. Put $K = Y \setminus V$, and $W = S \setminus G(K)$. Observe that since $G^{-1}(p) = \{p\}$ for every $p \in P$, W is an open subset of S such that $W \cap P = \{\mathbf{v}\} \times U$ and $G^{-1}(W) \subseteq V$.

Let E be a subset of U of size $\mathfrak{c}$ such that E meets every nonempty G_δ-subset of U. Since U has weight $\mathfrak{c}$, such a set is easily found. Since W is open, there is for every $e \in E$ an element $\kappa_e \in L_0$ such that $[\kappa_e, \mathbf{v}] \times \{e\} \subseteq W$. Since $\mathbf{v}$ is a $P_{\mathfrak{c}^+}$-point in L, there exists $\kappa \in L_0$ such that $\bigcup_{e \in E}[\kappa, \mathbf{v}] \times \{e\} \subseteq W$. Hence $\bigcup_{e \in E} G^{-1}([\kappa, \mathbf{v}] \times \{e\}) \subseteq V$ and $[\kappa, \mathbf{v}] \times \{t\} \subseteq \overline{W}$.

Claim 1. $F(([\kappa, \mathbf{v}] \times \{t\} \times \omega)^*) \subseteq \overline{V}$.

Proof. First observe that $F(([\kappa, \mathbf{v}] \times \{t\} \times \omega)^*) \cap P = \{(\mathbf{v}, t)\} \subseteq \overline{V}$. Now take an arbitrary $p \in ([\kappa, \mathbf{v}] \times \{t\} \times \omega)^* \setminus (P \times \omega)^*$ and an arbitrary open neighborhood Z of p in $T^* \setminus (P \times \omega)^* = Y \setminus P$. We will show that $Z \cap V \neq \emptyset$. We may assume without loss of generality that Z is of the form $\mathrm{Ex}(\bigcup_{n \in N} Z_n)$, where N is an infinite subset of ω, and for each $n \in N$, $Z_n \subseteq S \times \{n\}$ is open and intersects $[\kappa, \mathbf{v}) \times \{t\} \times \{n\}$, say in the point (x_n, t, n). For each $n \in N$, pick open neighborhoods A_n and B_n of x_n in L and t in $\mathbb{K}^*$ such that $A_n \times B_n \subseteq Z_n$. Then $t \in \bigcap_{n \in N} B_n$ and hence, by assumption, there exists an element $e \in \bigcap_{n \in N} B_n \cap E$. So $(x_n, e, n) \in A_n \times B_n$ for every $n \in N$. Take an arbitrary limit point q of the sequence $\{(x_n, e, n)\colon n \in N\}$. Then $q \in Z$, and

$$\begin{aligned} q &\in \big([\kappa, \mathbf{v}] \times \{e\} \times \omega\big)^* \setminus (P \times \omega)^* \\ &\subseteq \overline{\big([\kappa, \mathbf{v}] \times \{e\} \times \omega\big)} \\ &\subseteq G^{-1}\big([\kappa, \mathbf{v}] \times \{e\}\big) \\ &\subseteq V, \end{aligned}$$

as required. □

Assume that for some $\mu \in [\kappa, \mathbf{v})$ we have that $\hat{\mathbb{I}}(\mu, t, \xi)$, the image under F of the component of $([\mu, \mathbf{v}] \times \{t\} \times \omega)^*$ corresponding to $\xi \in \omega^*$, is contained in Fr V. Then we are done since $\hat{\mathbb{I}}(\mu, t, \xi)$ meets P in exactly one point, so a proper subcontinuum of it not containing that point is what we are after. Hence assume the contrary. Fix an arbitrary free ultrafilter ξ on ω. For every $\mu \in [\kappa, \mathbf{v})$ we have $\hat{\mathbb{I}}(\mu, t, \xi) \subseteq \overline{V}$ by Claim 1 and $\hat{\mathbb{I}}(\mu, t, \xi) \cap V \neq \emptyset$ by assumption. Observe again that $\hat{\mathbb{I}}(\mu, t, \xi) \cap P$ is a single point and hence, since $\hat{\mathbb{I}}(\mu, t, \xi)$ is a nontrivial continuum, $\hat{\mathbb{I}}(\mu, t, \xi) \cap (V \setminus P) \neq \emptyset$.

Let $\mathscr{H}$ be a clopen neighborhood base of t in $\mathbb{K}^*$ such that $|\mathscr{H}| \leqslant \mathfrak{c}$. In addition, let $\{\mathbf{v}_\delta\}_{\delta < \mathfrak{c}^+} \nearrow \mathbf{v}$ be a strictly increasing cofinal sequence in L_0. For every $\delta < \mathfrak{c}^+$ we will construct

(1) $\kappa_\delta \in [\kappa, \mathbf{v})$,
(2) $A_\delta \in \xi$,
(3) $f_\delta\colon A_\delta \to [\kappa_\delta, \mathbf{v})$,
(4) $g_\delta\colon A_\delta \to \mathscr{H}$,

such that

(5) if $\delta < \varepsilon < \mathfrak{c}^+$, then $\kappa_\varepsilon > \sup\{f_\delta(n)\colon n \in A_\delta\} \geqslant \max\{\kappa_\delta, \mathbf{v}_\delta\}$,
(6) $\{\{f_\delta(n)\} \times g_\delta(n) \times \{n\}\colon n \in A_\delta\} \subseteq V \setminus P$.

The construction is simple. At stage δ, let κ_δ be an arbitrary point from $(\mathbf{v}_\delta, \mathbf{v})$ greater than the supremum of the set

$$\{f_{\delta'}(n)\colon \delta' < \delta,\ n \in A_{\delta'}\}.$$

By assumption, $\hat{\mathbb{I}}(\kappa_\delta, t, \xi)$ meets $V \setminus P$, say in the point y. Since $V \setminus P$ is open, it contains a neighborhood of y of the form $\operatorname{Ex}(O)$, where O is open in T. It is clear that $A_\delta = \{n < \omega\colon ([\kappa_\delta, \mathbf{v}) \times \{t\} \times \{n\}) \cap O \neq \emptyset\} \in \xi$. For every $n \in A_\delta$ pick $y_n \in [\kappa_\delta, \mathbf{v})$ such that $(y_n, t, n) \in O$. Let $f_\delta : A_\delta \to [\kappa_\delta, \mathbf{v})$ be defined by $f_\delta(n) = (y_n, t, n)$. For every $n \in A_\delta$ there is an element $H_n \in \mathscr{H}$ such that $\{y_n\} \times H_n \times \{n\} \subseteq O$. Let $g_\delta : A_\delta \to \mathscr{H}$ be defined by $g_\delta(n) = H_n$ for every $n \in A_\delta$. These choices clearly satisfy the inductive requirements.

The function $\delta \mapsto (A_\delta, \bigcap_{n\in A_\delta} g_\delta(n))$ maps $\mathfrak{c}^+$ into a set of size $\mathfrak{c}$. Hence since $\mathfrak{c}^+$ is regular, there are $A \in \xi$ and a closed G_δ-subset H of $\mathbb{K}^*$ containing t such that the set $M = \{\kappa_\delta \in [\kappa, \mathbf{v})\colon A_\delta = A,\ \bigcap_{n\in A_\delta} g_\delta(n) = H\}$ is cofinal in L_0.

Claim 2. $\{\mathbf{v}\} \times H \subseteq \overline{V}$.

Proof. Pick an arbitrary $t' \in H$, and let Z be an arbitrary neighborhood of $(\mathbf{v}, t')$ in Y. We will prove that Z intersects V. Since $G^{-1}(\mathbf{v}, t') = \{(\mathbf{v}, t')\}$, there is a closed neighborhood Z' of $(\mathbf{v}, t')$ in S such that $G^{-1}(Z') \subseteq Z$. Pick $\delta < \mathfrak{c}^+$ such that $\kappa_\delta \in M$ and $[\kappa_\delta, \mathbf{v}] \times \{t'\} \subseteq Z'$. For every $n \in A_\delta$ we have $\pi(x_n^\delta, t', n) \in [\kappa_\delta, \mathbf{v}] \times \{t'\} \subseteq Z'$. Let $q = \lim_{\xi\restriction A_\delta}\{(x_n^\delta, t', n)\colon n \in A_\delta\}$. Then $q \in V \setminus P$ by (1), and clearly $\bar{\pi}(q) \in Z'$. From this we conclude that $q \in Z \cap V$, as required. $\square$

So we are done. $\square$

4.4. Step 4

Consider the ($\leqslant$)2-to-one functions $\bar{g}_i : \mathbb{K}^* \to [0,1)^*$, $i = 0, 1$, that we defined in Section 3. Define $h_i : P \to [0,1)^*$ in the obvious way by

$$h_i(\mathbf{v}, p) = \bar{g}_i(p) \quad (i = 0, 1).$$

Consider the adjunction spaces

$$X_i = Y \cup_{h_i} P \quad (i = 0, 1).$$

Hence the X_i are just Y with P replaced by (a copy of) $[0,1)^*$. First observe that the X_i are compact F-spaces by Lemma 2.3. Next, $\operatorname{Ind} X_i \leqslant 1$, $i = 0, 1$, by Proposition 2.2(2). Hence $\operatorname{Ind} X_0 = \operatorname{Ind} X_1 = 1$ since X_0 and X_1 contain nontrivial continua. Moreover, X_0 and X_1 have weight $\mathfrak{c}^+$.

Let $q_i : Y \to X_i$ denote the natural quotient maps, $i = 0, 1$. It will be convenient to identify $Y \setminus P$ and $X_i \setminus [0,1)^*$. Observe that it is not clear that X_0 and X_1 are homeomorphic, probably they are not. The point $p \in [0,1)^*$ 'corresponds' in X_0 to $\bar{g}_0^{-1}(p)$, while in X_1 it 'corresponds' to $\bar{g}_1^{-1}(p) = \bar{g}_0^{-1}(\bar{\varepsilon}^{-1}(p))$.

Proposition 4.3. *Let U be a proper, nonempty open set in $[0,1)^*$, and fix $i \in \{0, 1\}$. Assume that $\overline{\bar{g}_i^{-1}(U)}$ is not clopen. Then for every open subset V of X_i such that $V \cap [0,1)^* = U$, the boundary* $\operatorname{Fr} V$ *of V contains a nontrivial continuum.*

Proof. We assume without loss of generality that $i = 1$.

Let $t \in \overline{h_1^{-1}(U)}$ witness the fact that $\overline{h_1^{-1}(U)}$ is not clopen. That is, every neighborhood of t in P meets $W = P \setminus \overline{h_1^{-1}(U)}$.

Assume first that there is a closed G_δ-subset A of P such that $t \in A \subseteq \overline{h_1^{-1}(U)}$. Write $A = \bigcap_{n<\omega} U_n$, where each U_n is clopen in P and $U_{n+1} \subseteq U_n$ for every n. By recursion on n we will construct a nonempty clopen subset K_n of P such that $K_n \subseteq (U_n \cap W) \setminus \bigcup_{i<n} K_i$. If $n = 0$, then we get what we want from $U_0 \cap W \neq \emptyset$. Suppose that we constructed $K_0, \ldots, K_{n-1}$. Observe that $U_n \setminus \bigcup_{i<n} K_i$ is an open neighborhood of t in P, and hence intersects W. This means that $(U_n \cap W) \setminus \bigcup_{i<n} K_i \neq \emptyset$ and hence it is a triviality to pick K_n.

Clearly, $\overline{\bigcup_{n<\omega} K_n} \setminus \bigcup_{n<\omega} K_n \subseteq \bigcap_{n<\omega} U_n = A$, and $\overline{\bigcup_{n<\omega} K_n} \subseteq \overline{W}$. By Lemma 3.1 we may pick for every n a nonempty open subset L_n of $[0,1)^*$ such that $h_1^{-1}(L_n) \subseteq K_n$. Since $[0,1)^*$ is a continuum, there is for every n a nontrivial continuum $H_n \subseteq L_n$. Since $[0,1)^*$ is an F-space, every σ-compact subset of it is C^*-embedded, Section 2.3. Hence by Section 2.4 we may pick a nontrivial continuum

$$H \subseteq \overline{\bigcup_{n<\omega} H_n} \setminus \bigcup_{n<\omega} H_n.$$

Clearly, $H \subseteq h_1(\overline{\bigcup_{n<\omega} K_n} \setminus \bigcup_{n<\omega} K_n) \subseteq h_1(A) \subseteq h_1(\overline{h_1^{-1}(U)}) = \overline{U}$. Moreover, $H \subseteq h_1(\overline{W})$ and $h_1(\overline{W}) \cap U = \emptyset$. From this we conclude that $H \subseteq \overline{U} \setminus U \subseteq \operatorname{Fr} V$, as required.

Hence we may assume without loss of generality that every G_δ-subset of P that contains t meets W.

If $\mathrm{Fr}\, q_1^{-1}(V)$ contains a nontrivial continuum that misses P then we are clearly done. Hence by Lemma 4.2, we may assume without loss of generality that $\overline{q_1^{-1}(V)}$ contains a closed G_δ-subset C of P that contains t. By assumption, C meets W, hence $C \cap W$ has nonempty interior in P by Section 2.3. As above, there is a nontrivial continuum L of $[0,1)^*$ such that $h_1^{-1}(L) \subseteq C \cap W$. Then $L \subseteq \overline{V}$ but misses U. This evidently implies that $L \subseteq \overline{V} \setminus V = \mathrm{Fr}\, V$, as required. □

4.5. Step 5

It will be convenient to think of X_0 and X_1 as disjoint spaces. Consider the identity function $\mathrm{id} : [0,1)^* \to [0,1)^*$, and the adjunction space

$$\mathbb{X} = X_0 \cup_{\mathrm{id}} X_1.$$

We claim that $\mathbb{X}$ is the required example. First observe that $\mathbb{X}$ is a compact F-space by Lemma 2.3. Pick two distinct elements $p, q \in [0,1)^*$, and let V be an open neighborhood of p in $\mathbb{X}$ whose closure misses q. By Proposition 3.2, $\bar{g}_0^{-1}(V \cap [0,1)^*)$ is not clopen, or $\bar{g}_1^{-1}(V \cap [0,1)^*)$ is not clopen. Hence by Proposition 4.3, $\mathrm{Fr}(V \cap X_0)$ or $\mathrm{Fr}(V \cap X_1)$ contains a nontrivial continuum. That continuum is clearly also a subcontinuum of $\mathrm{Fr}\, V$. From this we conclude that $\mathrm{Ind}\, \mathbb{X} \geqslant \mathrm{ind}\, \mathbb{X} \geqslant 2$. Now, $\dim X_0 = \dim X_1 \leqslant 1$ since $\mathrm{Ind}\, X_0 = \mathrm{Ind}\, X_1 = 1$ (Theorem 2.1). As a consequence, $\dim \mathbb{X} = 1$ by the Countable Closed Sum Theorem [9, 3.1.8]. Clearly, the weight of $\mathbb{X}$ is $\mathfrak{c}^+$.

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