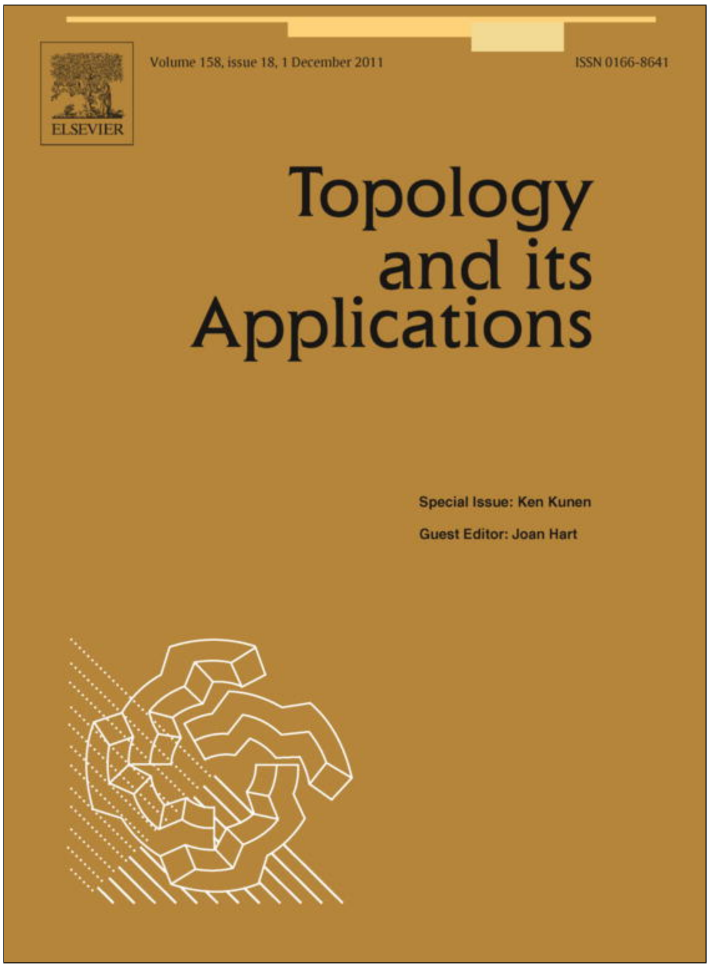
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Covering dimension and finite-to-one maps

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To Ken Kunen on the occasion of his retirement from teaching

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ABSTRACT

Hurewicz characterized the dimension of separable metrizable spaces by means of finite-to-one maps. We investigate whether this characterization also holds in the class of compact F-spaces of weight $\mathfrak{c}$. Our main result is that, assuming the Continuum Hypothesis, an n-dimensional compact F-space of weight $\mathfrak{c}$ is the continuous image of a zero-dimensional compact Hausdorff space by an at most 2^n-to-1 map.

0. Introduction

The starting point for this note is a theorem of Hurewicz from [5], which characterizes the dimension of separable metrizable spaces in terms of maps.

Theorem. *Let X be a separable and metrizable space and let n be a natural number. Then $\dim X \leqslant n$ iff there are a zero-dimensional separable metrizable space Y and a continuous and closed surjection $f : Y \to X$ such that $|f^{-1}(x)| \leqslant n+1$ for all x.*

Our aim is to generalize this result to a wider class of spaces.

One half of Hurewicz's theorem is a special case of the theorem on dimension raising maps. This special case can be generalized to the class of normal spaces (the hint to Problem 1.7.F(c) in [3] provides a proof):

Theorem. *Let $f : Y \to X$ be a closed continuous surjection between normal spaces and n a natural number. If Y is strongly zero-dimensional and if f is such that $|f^{-1}(x)| \leqslant n+1$ for all x then $\operatorname{Ind} X \leqslant n$.*

Hurewicz' proof of the other half was based on the interplay between the large inductive dimension and the covering dimension, using finite collections of closed sets of order $n+1$ to construct the preimage. Also Kuratowski's quantitative proof in [7] used covering dimension to show that in the case where X has no isolated points the set of surjections with at most $n+1$-point fibers is residual in the space of all surjections from the Cantor set onto X.

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This all suggests that we should look for classes of compact Hausdorff spaces where the covering dimension and the large inductive dimension coincide. We shall show that the compact F-spaces of weight $\mathfrak{c}$ form such a class, assuming the Continuum Hypothesis. Under that assumption these spaces are one step up from compact metrizable spaces: the weight is $\aleph_1$ and the F-space property lets one do countably many things at a time, which is quite helpful in inductions and recursions of length ω_1.

In Section 1 we give a variation of Hurewicz' proof of the second half his theorem; we do this with an eye to the proof of our main result and to make Hurewicz' argument better known. Next we establish, in Section 2, that the three major dimension functions coincide on the class of compact F-spaces of weight $\mathfrak{c}$. In Sections 3 and 4 we show that every n-dimensional compact F-space is an at most 2^n-to-1 continuous image of a zero-dimensional compact space. This leaves open an obvious question:

Question *(*CH*).* Is every n-dimensional compact F-space of weight $\mathfrak{c}$ a continuous image of some zero-dimensional compact space by a map whose fibers have size at most $n+1$?

0.1. Some preliminaries

We follow Engelking's book [3], regarding dimension theory except that by the order of a family $\mathcal{T}$ at a point x we simply mean the cardinality of $\{T \in \mathcal{T}\colon x \in T\}$. Thus a (normal) space has covering dimension at most n iff every finite open cover has an open refinement of order at most $n+1$.

A completely regular space X is an *F-space* if disjoint cozero sets are completely separated, i.e., if A and B are disjoint cozero sets then there is a continuous real-valued function f on X such that $f[A]=\{0\}$ and $f[B]=\{1\}$; for normal spaces this is equivalent to: disjoint cozero sets have disjoint closures. Whenever X is a σ-compact, locally compact space that is not compact then the remainder $\beta X \setminus X$ in its Čech–Stone compactification is an F-space. Gillman and Jerison's book [4], is still a good source of basic information on F-spaces.

A set is *regular open* if it is the interior of its closure. The family $\mathrm{RO}(X)$ of regular open sets in the space X forms a Boolean algebra under the following operations: $U \vee V = \operatorname{int}\operatorname{cl}(U \cup V)$, $U \wedge V = U \cap V$ and $U' = X \setminus \operatorname{cl} U$. In the proof of our main result we obtain our zero-dimensional preimage as the Stone space of a subalgebra of $\mathrm{RO}(X)$; we refer to Koppelberg's book [6], for a comprehensive treatment of this construction.

1. Making a zero-dimensional preimage

As announced we present in this section a variation of Hurewicz' construction of a zero-dimensional preimage of an n-dimensional compact metrizable space by a map whose fibers have cardinality at most $n+1$.

The key notion is that of a *tiling* of a space, which we define to be a finite pairwise disjoint family of regular open sets whose closures form a cover of the space. Given a tiling $\mathcal{T}$ and a point x we put $\mathcal{T}_x = \{T \in \mathcal{T}\colon x \in \operatorname{cl} T\}$.

In the proof we will construct ever finer tilings of the space; the following lemma will help us keep the cardinalities of the families $\mathcal{T}_x$ under control.

Lemma 1.1. *Let $\{B_i\colon i<k\}$ be a family of regular open sets in a space X and let T be a regular open set that is covered by this family. For each i put $C_i = T \cap (B_i \setminus \bigcup_{j<i} \operatorname{cl} B_j)$ and for $x \in \operatorname{cl} T$ put $F_x = \{i\colon x \in \operatorname{cl} C_i\}$. Then $x \in \operatorname{Fr} B_i$ whenever $i \in F_x$ is not maximal.*

Proof. This is clear: if $j<i$ then $B_j \cap \operatorname{cl} C_i = \emptyset$. Furthermore, if $j<i$ in F_x then $x \in \operatorname{cl} C_j \cap \operatorname{cl} C_i \subseteq \operatorname{cl} B_j \cap \operatorname{cl} C_i$; so $x \in \operatorname{cl} B_j \setminus B_j = \operatorname{Fr} B_j$. $\square$

The previous lemma implies that $\{i\colon x \in \operatorname{cl} C_i\}$ has at most one element more than $\{i\colon x \in \operatorname{Fr} B_i\}$. This behavior persists when we refine tilings.

Lemma 1.2. *Let $\mathcal{T}$ be a tiling of X that is a subset of the Boolean algebra generated by a finite family $\mathcal{B}$ of regular open sets and assume that for every x the family $\{B \in \mathcal{B}\colon x \in \operatorname{Fr} B\}$ has cardinality at least $|\mathcal{T}_x|-1$. Fix one $T \in \mathcal{T}$ and a finite family $\mathcal{C} = \{C_i\colon i<k\}$ of regular open sets such that $T \subseteq \bigcup \mathcal{C}$ and $\mathcal{B} \cap \mathcal{C} = \emptyset$; for $i<k$ put $R_i = T \cap (C_i \setminus \bigcup_{j<i} \operatorname{cl} C_j)$.*

Then the tiling $\mathcal{S} = \mathcal{T} \setminus \{T\} \cup \{R_i\colon i<k\}$ has the same property as $\mathcal{T}$: for every x the family $\{B \in \mathcal{B} \cup \mathcal{C}\colon x \in \operatorname{Fr} B\}$ has cardinality at least $|\mathcal{S}_x|-1$.

Proof. Let $x \in X$. If $x \notin \operatorname{cl} T$ then $\mathcal{S}_x = \mathcal{T}_x$ and we are done by the assumption.

Consider the case $x \in \operatorname{cl} T$. Then $\mathcal{S}_x$ consists of $\mathcal{T}_x \setminus \{T\}$ and $\{R_i\colon i \in G_x\}$, where $G_x = \{i\colon x \in \operatorname{cl} R_i\}$; this implies that $|\mathcal{S}_x| = |\mathcal{T}_x| - 1 + |G_x|$.

We can apply Lemma 1.1 to see that $x \in \operatorname{Fr} C_i$ if $x \in \operatorname{cl} R_i$ and $i \neq \max G_x$. This implies that $\{C \in \mathcal{C}\colon x \in \operatorname{Fr} C\}$ has cardinality at least $|G_x|-1$. By assumption the family $\{B \in \mathcal{B}\colon x \in \operatorname{Fr} B\}$ has cardinality at least $|\mathcal{T}_x|-1$.

Because $\mathcal{B}$ and $\mathcal{B}'$ are disjoint the family $\{B \in \mathcal{B} \cup \mathcal{C}\colon x \in \operatorname{Fr} B\}$ has cardinality at least $(|\mathcal{T}_x|-1)+(|G_x|-1)$, which is equal to $|\mathcal{S}_x|-1$. $\square$

Now we are ready to reprove the second half of Hurewicz' theorem.

Theorem 1.3. *Let X be a metrizable compact space with $\dim X \leqslant n$. Then there are a zero-dimensional compact metrizable space Y and a continuous surjection $f : Y \to X$ such that $|f^{-1}(x)| \leqslant n+1$ for all x.*

Proof. Let $\{B_i\colon i<\omega\}$ be a base for X consisting of regular open sets and such that $\bigcap_{i\in F} \operatorname{Fr} B_i = \emptyset$ whenever $F \in [\omega]^{n+1}$; see, e.g., [1, Corollary 6.12]. For technical reasons we assume that for every isolated point x in X the set $\{i\colon B_i = \{x\}\}$ is infinite; because $\operatorname{Fr}\{x\} = \emptyset$ whenever x is isolated this does not interfere with the intersection property of the family $\{\operatorname{Fr} B_i\colon i<\omega\}$.

By recursion we construct a sequence $\langle \mathcal{T}_k\colon k<\omega\rangle$ of tilings such that for all k the family $\mathcal{T}_{k+1}$ refines $\mathcal{T}_k$, such that $\operatorname{ord}\{\operatorname{cl} T\colon T\in\mathcal{T}_k\} \leqslant n+1$ for all k, and such that $\lim_k \max\{\operatorname{diam} T\colon T\in\mathcal{T}_k\} = 0$.

Each $\mathcal{T}_k$ is given the discrete topology and our compact zero-dimensional space Y is the subspace of the product $\prod_{k<\omega}\mathcal{T}_k$ defined by

$$\langle T_k\colon k<\omega\rangle \in Y \quad \text{iff} \quad (\forall k)(T_{k+1} \subseteq T_k).$$

Clearly Y is closed in the product, hence compact, metrizable and zero-dimensional. If $\langle T_k\colon k<\omega\rangle \in Y$ then $\bigcap_{k<\omega} \operatorname{cl} T_k$ consists of exactly one point (by compactness and the diameter condition); this defines a map f from Y to X.

The map is (uniformly) continuous: if $\varepsilon>0$ there is a k such that $\operatorname{diam} T<\varepsilon$ for all $T\in\mathcal{T}_k$. If $T=\langle T_k\colon k<\omega\rangle$ and $S=\langle S_k\colon k<\omega\rangle$ in Y are such that $T_k=S_k$ then $f(T), f(S)\in \operatorname{cl} T_k$ and hence $d(f(T),f(S))<\varepsilon$.

The map f is onto: if $x\in X$ then it is easy to find $T\in Y$ such that $x\in T_k$ for all k; then $x=f(T)$.

The map f is at most $n+1$-to-one. Indeed, let $x\in X$ and for each k let $\mathcal{T}_{k,x}=\{T\in\mathcal{T}_k\colon x\in\operatorname{cl} T\}$. Then $|\mathcal{T}_{k,x}|\leqslant|\mathcal{T}_{k+1,x}|\leqslant n+1$ for all k; this means that there is a k_0 such that $|\mathcal{T}_{k,x}|=|\mathcal{T}_{k_0,x}|$ for $k\geqslant k_0$. And this implies that $T\mapsto T_{k_0}$ is a bijective map from $f^{-1}(x)$ to $\mathcal{T}_{k_0,x}$ and thus: $|f^{-1}(x)|\leqslant n+1$.

It remains to construct the $\mathcal{T}_k$. We set $\mathcal{T}_0=\{X\}$. We assume we have found $\mathcal{T}_k$ as a subset of the Boolean algebra generated by $\{B_i\colon i<l\}$ for some l and that the assumptions of Lemma 1.2 are satisfied: for every $x\in X$ there are at least $|\mathcal{T}_{k,x}|-1$ indices $i<l$ such that $x\in\operatorname{Fr} B_i$.

Because for each isolated point x the set $\{x\}$ occurs as B_i infinitely often we know that for every m the family $\{B_i\colon i\geqslant m\}$ is a base for X. We can therefore find pairwise disjoint finite subsets F_T of (l,ω) for $T\in\mathcal{T}_k$ such that $\operatorname{cl} T\subseteq\bigcup_{i\in F_T} B_i$ and $\operatorname{diam} B_i<2^{-k}$ for all i. We can use these, as in Lemma 1.2 to define tilings of each $T\in\mathcal{T}_k$: for $i\in F_T$ put $C_{T,i}=T\cap(B_i\setminus\bigcup_{j\in F_T,\,j<i}\operatorname{cl} B_j)$. Repeated application of Lemma 1.2 shows that $\mathcal{T}_{k+1}=\{C_{T,i}\colon i\in F_T, T\in\mathcal{T}_k\}$ has the same property as $\mathcal{T}_k$: for each x there are at least $|\mathcal{T}_{k+1,x}|-1$ indices i in $l\cup\bigcup_{T\in\mathcal{T}_k} F_T$ such that $x\in\operatorname{Fr} B_i$.

To see that $|\mathcal{T}_{k,x}|\leqslant n+1$ for all k and all x we combine two inequalities: first, by construction we have $|\{i\colon x\in\operatorname{Fr} B_i\}|\geqslant|\mathcal{T}_{k,x}|-1$; second, by assumption on our base we have $n\geqslant|\{i\colon x\in\operatorname{Fr} B_i\}|$. This implies $|\mathcal{T}_{k,x}|-1\leqslant n$. $\square$

As mentioned in the introduction, in [7] Kuratowski gave a quantitative version of this half of Hurewicz' theorem: if X is compact, metrizable, n-dimensional and without isolated points then the set of maps all of whose fibers have size $n+1$ or less is residual in the space of all surjections from the Cantor set to X. The covering dimension is invoked to show that, given a natural number k, the set of maps with a fiber of size at least $n+2$ in which the points are at least distance 2^{-k} apart is nowhere dense (it is also readily seen to be closed).

2. Equality of dimensions

It is well known that the three fundamental dimension functions take on the same values for all separable metrizable spaces. We prove that this also holds in the class of compact F-spaces of weight $\mathfrak{c}$, provided the Continuum Hypothesis holds.

In the proof we use Hemmingsen's characterization of covering dimension [3, Theorem 1.6.9]: $\dim X\leqslant n$ iff every $n+2$-element open cover has an open shrinking with empty intersection.

Theorem 2.1 (CH)*. Let X be a compact F-space of weight $\mathfrak{c}$. Then $\dim X=\operatorname{ind} X=\operatorname{Ind} X$.*

Proof. The inequalities $\dim X\leqslant\operatorname{ind} X\leqslant\operatorname{Ind} X$ are well known. We show $\operatorname{Ind} X\leqslant\dim X$ by showing that $\dim X\leqslant n$ implies that between any two disjoint closed sets F and G one can find a partition L that satisfies $\dim L\leqslant n-1$. This is known to be true in case $n=0$, so we assume $n\geqslant 1$ from now on.

Fix a base $\mathcal{B}$ for X that consists of cozero sets, has cardinality $\aleph_1$ (by the CH) and is closed under countable unions and finite intersections.

Let $\langle\mathcal{B}_\alpha\colon\alpha<\omega_1\rangle$ enumerate the family of all $n+1$-element subfamilies of $\mathcal{B}$ with cofinal repetitions. We write $\mathcal{B}_\alpha=\{B_{\alpha,i}\colon i\leqslant n\}$.

We construct, by recursion, two sequences $\langle U_\alpha\colon\alpha<\omega_1\rangle$ and $\langle V_\alpha\colon\alpha<\omega_1\rangle$ in $\mathcal{B}$ such that

(1) $F\subseteq U_0$ and $G\subseteq V_0$;

(2) $\operatorname{cl} U_\alpha \subseteq U_\beta$ and $\operatorname{cl} V_\alpha \subseteq V_\beta$ whenever $\alpha < \beta$;
(3) $\operatorname{cl} U_\alpha \cap \operatorname{cl} V_\alpha = \emptyset$ for all α;
(4) if $U_\alpha \cup V_\alpha \cup \bigcup \mathcal{B}_\alpha = X$ then there is a subfamily $\mathcal{B}'_\alpha = \{B'_{\alpha,i}\colon i \leqslant n\}$ of $\mathcal{B}$ that refines $\mathcal{B}_\alpha$ and is such that $U_{\alpha+1} \cup V_{\alpha+1} \cup \bigcup \mathcal{B}'_\alpha = X$ and $\bigcap \mathcal{B}'_\alpha \subseteq U_{\alpha+1} \cup V_{\alpha+1}$.

Then $L = X \setminus \bigcup_\alpha (U_\alpha \cup V_\alpha)$ is a partition between F and G and $\dim L \leqslant n-1$. That L is a partition between F and G follows from (1), (2) and (3). To see that $\dim L \leqslant n-1$ let $\mathcal{O}$ be an $n+1$-element basic open cover of L. There is an α such that $\mathcal{O} = \{L \cap B_{\alpha,i}\colon i \leqslant n\}$ and such that $X \setminus (U_\alpha \cup V_\alpha) \subseteq \bigcup \mathcal{B}_\alpha$. By construction $\mathcal{O}' = \{L \cap B'_{\alpha,i}\colon i \leqslant n\}$ is a refinement of $\mathcal{O}$ such that $\bigcap \mathcal{O}' = \emptyset$.

It remains to perform the construction. We obtain U_0 and V_0 using compactness and the fact that $\mathcal{B}$ is closed under finite unions. If α is a limit we let $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$ and $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. Then $U_\alpha, V_\alpha \in \mathcal{B}$ by the assumption that $\mathcal{B}$ is closed under countable unions and $\operatorname{cl} U_\alpha \cap \operatorname{cl} V_\alpha = \emptyset$ because X is an F-space.

To deal with the successor case we first take elements C and D of $\mathcal{B}$ such that $\operatorname{cl} U_\alpha \subseteq C$, $\operatorname{cl} V_\alpha \subseteq D$ and $\operatorname{cl} C \cap \operatorname{cl} D = \emptyset$.

If $U_\alpha \cup V_\alpha \cup \bigcup \mathcal{B}_\alpha = X$ we put $E_i = B_{\alpha,i} \setminus (\operatorname{cl} U_\alpha \cup \operatorname{cl} V_\alpha)$ for $i \leqslant n$. Then we apply the inequality $\dim X \leqslant n$ to find a shrinking $\{B'_{\alpha,i}\colon i \leqslant n\} \cup \{O\}$ of $\{E_i\colon i \leqslant n\} \cup \{C \cup D\}$ such that $\operatorname{cl} O \cap \bigcap_i \operatorname{cl} B'_{\alpha,i} = \emptyset$. Let $O_1 = O \cap C$ and $O_2 = O \cap D$; also let $K = \bigcap_i \operatorname{cl} B'_{\alpha,i}$. Note that $\operatorname{cl} U_\alpha \subseteq O_1$ and $\operatorname{cl} V_\alpha \subseteq O_2$ and that the family $\{K, \operatorname{cl} O_1, \operatorname{cl} O_2\}$ is pairwise disjoint. We choose $U_{\alpha+1}$ and $V_{\alpha+1}$ in $\mathcal{B}$ with disjoint closures such that $K \cup \operatorname{cl} O_1 \subseteq U_{\alpha+1}$ and $\operatorname{cl} O_1 \subseteq U_{\alpha+1}$. Then the conclusion of (4) is satisfied.

If $U_\alpha \cup V_\alpha \cup \bigcup \mathcal{B}_\alpha \neq X$ we let $U_{\alpha+1} = C$ and $V_{\alpha+1} = D$. □

Remark 2.2. This proof is similar to the one given in [3] of the analogous result for compact metrizable spaces. That proof used a metric to guide the countably many steps toward a partition of covering dimension at most $n-1$. Of course in an infinite compact F-space there is no metric available; in our proof the role of the metric is taken over (in the background) by the unique uniformity that generates the topology of the compact F-space.

Remark 2.3. The second-named author has constructed an example of a compact F-space of weight $\mathfrak{c}^+$ with non-coinciding dimensions [8].

3. Special bases

In Section 1 we used the fact that a metrizable compact space X with $\dim X \leqslant n$ has a base $\{B_i\colon i < \omega\}$ with the property that $\bigcap_{i\in F} \operatorname{Fr} B_i = \emptyset$ whenever $|F| = n+1$. This is a special case of a stronger structural statement: every metrizable compact space has a base $\{B_i\colon i < \omega\}$ with the property that $\dim \bigcap_{i\in F} \operatorname{Fr} B_i \leqslant \dim \operatorname{Fr} B_{i_0} - |F| + 1$, where $i_0 = \min F$.

Our goal is to prove a similar statement for compact F-spaces of weight $\mathfrak{c}$, assuming the Continuum Hypothesis.

In general, if X is a compact space of weight $\aleph_1$ we shall assume it is embedded in the Tychonoff cube $[0,1]^{\omega_1}$ and for $\alpha < \omega_1$ we write $X_\alpha = \{x \restriction \alpha\colon x \in X\}$; this is the projection of X onto the first α coordinates. We denote this projection map by p_α, we reserve π_α for the projection onto the αth coordinate.

Lemma 3.1. *There is a closed and unbounded subset C of ω_1 such that $\dim X_\alpha = \dim X$ for $\alpha \in C$.*

Proof. The cube $[0,1]^{\omega_1}$ has a nice subbase, which consists of the strips $\pi_\alpha^{-1}[[0,q)]$ and $\pi_\alpha^{-1}[(q,1]]$, where $\alpha < \omega_1$ and $q \in \mathbb{Q} \cap (0,1)$. We close this subbase under finite unions and intersections to obtain a base $\mathcal{B}$ for the cube.

First we assume $\dim X = n < \infty$. In this case if $\mathcal{B}'$ is a finite subfamily of $\mathcal{B}$ that covers X then there is another finite subfamily $\mathcal{B}''$ of $\mathcal{B}$ that also covers X, refines $\mathcal{B}'$ and is such that $|\{B \in \mathcal{B}''\colon x \in B\}| \leqslant n+1$ for all $x \in X$.

Observe that each finite subfamily $\mathcal{C}$ of $\mathcal{B}$ is supported by a finite subset $F_{\mathcal{C}}$ of ω_1 (the coordinates of the strips used to make its elements). Next note that, given $\alpha < \omega_1$, there are only countably many finite subfamilies of $\mathcal{B}$ whose support lies below α. Thus we obtain a function $f : \omega_1 \to \omega_1$, defined by

> $f(\alpha)$ is the first countable ordinal β such that whenever $\mathcal{B}'$ is a finite subfamily of $\mathcal{B}$ that covers X and whose support lies below α then it has a refinement of order at most $n+1$ whose support lies below β.

The set $C = \{\delta < \omega_1\colon (\forall\alpha)(\alpha < \delta \to f(\alpha) < \delta)\}$ is closed and unbounded and it should be clear that $\dim X_\delta \leqslant n$ whenever $\delta \in C$. To get equality we note that there is also a finite cover $\mathcal{C}$ of X by members of $\mathcal{B}$ for which *every* open refinement has order $n+1$. Upon deleting an initial segment of C we can assume that $\mathcal{C}$ is supported below $\min C$; then $\mathcal{C}$ witnesses that $\dim X_\delta \geqslant n$ for all $\delta \in C$.

In case X is infinite-dimensional we have for each n a finite cover $\mathcal{C}_n$ such that every open refinement has order at least n. For any α above the supports of these covers we have $\dim X_\alpha = \infty$. □

The following proposition is instrumental in the construction of the type of base alluded to above. It also provides another proof of Theorem 2.1. In it we use the notion of a P-set: a subset of a space is a P-set if whenever it is disjoint from an F_σ-subset it is also disjoint the closure of that F_σ-set; in terms of neighborhoods: the intersection of countably many neighborhoods of the set is again a neighborhood of the set. The members of our base will have nowhere dense closed P-sets for boundaries.

Proposition 3.2 (CH). *Let X be a compact F-space of weight $\mathfrak{c}$. Let F and G be disjoint closed subsets of X and let $\mathcal{Q}$ be a family of no more than $\aleph_1$ many nowhere dense closed P-sets in X. There are disjoint regular open sets U and V such that*

(1) $F \subseteq U$ *and* $G \subseteq V$,
(2) $P = X \setminus (U \cup V)$ *is a nowhere dense P-set,*
(3) *if* $\dim X < \infty$ *then* $\dim P \leqslant \dim X - 1$,
(4) *if* $Q \in \mathcal{Q}$ *then* $P \cap Q$ *is nowhere dense in* Q *and if* $\dim Q < \infty$ *then* $\dim(P \cap Q) \leqslant \dim Q - 1$.

Proof. We may as well assume that $\dim X = n < \infty$ and that $\mathcal{Q}$ has cardinality $\aleph_1$. The proof is easily modified in case either of these is not the case.

Choose a closed and unbounded set C such that $\dim X_\delta = \dim X$ for $\delta \in C$ and assume without loss of generality (and by compactness) that $p_\delta[F] \cap p_\delta[G] = \emptyset$, where $\delta = \min C$.

Enumerate $\mathcal{Q}$ as $\{Q_\alpha \colon \alpha < \omega_1\}$ and choose for each α a closed and unbounded subset C_α of C such that $\dim p_\delta[Q_\alpha] = \dim Q_\alpha$ whenever $\delta \in C_\alpha$. Because the intersection of countably many closed and unbounded sets is again closed and unbounded we may as well assume that $C_\beta \subseteq C_\alpha$ whenever $\alpha < \beta$.

In case $\delta \in C_\alpha$ we can choose a zero-dimensional F_σ-set $Z_{\alpha,\delta}$ in X_δ such that $Z_{\alpha,\delta}$ is dense in X_δ, the intersection $Z_{\alpha,\delta} \cap p_\delta[Q_\alpha]$ is dense in $p_\delta[Q_\alpha]$, and such that

$$\dim(X_\delta \setminus Z_{\alpha,\delta}) \leqslant n - 1 \quad \text{and} \quad \dim\bigl(p_\delta[Q_\alpha] \setminus Z_{\alpha,\delta}\bigr) \leqslant \dim Q_\alpha - 1. \tag{3.1}$$

We start our recursive construction of P. Along the way we construct a sequence $\langle \delta_\alpha \colon \alpha < \omega_1 \rangle$ of ordinals.

Let $\delta_0 = \min C_0$ and choose a partition L in X_{δ_0} between $p_{\delta_0}[F]$ and $p_{\delta_0}[G]$ that is disjoint from Z_{0,δ_0}. Thus we obtain automatically that

- $\dim L \leqslant n - 1$,
- $\dim L \cap p_{\delta_0}[Q_0] \leqslant \dim Q_0 - 1$, and
- L is nowhere dense in X_{δ_0} and $L \cap p_{\delta_0}[Q_0]$ is nowhere dense in $p_{\delta_0}[Q_0]$.

Write $X_{\delta_0} \setminus L = U \cup V$, where U and V are open and disjoint sets around $p_{\delta_0}[F]$ and $p_{\delta_0}[G]$ respectively. Let $V_0 = X_\delta \setminus \operatorname{cl} U$ and $U_0 = X_\delta \setminus \operatorname{cl} V_0$; then

- U_0 and V_0 are regular open,
- $P_0 = X_\delta \setminus (U_0 \cup V_0)$ is a subset of L and a partition between $p_{\delta_0}[F]$ and $p_{\delta_0}[G]$ with $\dim P_0 \leqslant \dim L \leqslant n - 1$, and
- $\dim P_0 \cap p_{\delta_0}[Q_0] \leqslant \dim L \cap p_{\delta_0}[Q_0] \leqslant \dim p_{\delta_0}[Q_0] - 1 = \dim Q_0 - 1$.

Observe that $\operatorname{cl} U_0 = U_0 \cup P_0$ and $\operatorname{cl} V_0 = V_0 \cup P_0$.

To find δ_1 observe first that $p_{\delta_0}^{-1}[U_0]$ and $p_{\delta_0}^{-1}[V_0]$ are disjoint open F_σ-sets of X and hence have disjoint closures as X is an F-space. As with F and G we can find an ordinal η such that $p_\eta[\operatorname{cl} p_{\delta_0}^{-1}[U_0]]$ and $p_\eta[\operatorname{cl} p_{\delta_0}^{-1}[V_0]]$ are disjoint. Pick $\delta_1 \in C_1$ above η.

In X_{δ_1} we can find a partition L between $p_{\delta_1}[\operatorname{cl} p_{\delta_0}^{-1}[U_0]]$ and $p_{\delta_1}[\operatorname{cl} p_{\delta_0}^{-1}[V_0]]$ that is disjoint from $Z_{0,\delta_1} \cup Z_{1,\delta_1}$ — this is possible because $Z_{0,\delta_1} \cup Z_{1,\delta_1}$ is zero-dimensional by the countable closed sum theorem. We now obtain, by (3.1):

- $\dim L \leqslant n - 1$,
- $\dim L \cap p_{\delta_1}[Q_0] \leqslant \dim Q_0 - 1$, and
- $\dim L \cap p_{\delta_1}[Q_1] \leqslant \dim Q_1 - 1$.

Because of the density conditions on Z_{0,δ_1} and Z_{1,δ_1} we know that L is nowhere dense in X_{δ_1}, that $L \cap p_{\delta_1}[Q_0]$ is nowhere dense in $p_{\delta_1}[Q_0]$ and that $L \cap p_{\delta_1}[Q_1]$ is nowhere dense in $p_{\delta_1}[Q_1]$.

As above we find disjoint regular open sets U_1 and V_1 around $p_{\delta_1}[\operatorname{cl} p_{\delta_0}^{-1}[U_0]]$ and $p_{\delta_1}[\operatorname{cl} p_{\delta_0}^{-1}[V_0]]$ respectively such that $P_1 = X_{\delta_1} \setminus (U_1 \cup V_1) \subseteq L$. Note also that $p_{\delta_0}^{\delta_1}[P_1] \subseteq P_0$.

At stage α we consider the disjoint open F_σ-sets $A = \bigcup_{\beta<\alpha} p_{\delta_\beta}^{-1}[U_\beta]$ and $B = \bigcup_{\beta<\alpha} p_{\delta_\beta}^{-1}[V_\beta]$. There is a $\delta_\alpha \in C_\alpha$ above $\{\delta_\beta \colon \beta < \alpha\}$ such that $p_{\delta_\alpha}[\operatorname{cl} A]$ and $p_{\delta_\alpha}[\operatorname{cl} B]$ are disjoint.

The union $Z = \bigcup_{\beta \leqslant \alpha} Z_{\beta,\delta_\alpha}$ is zero-dimensional by the countable closed sum theorem so we can find a partition L in X_{δ_α} between $p_{\delta_\alpha}[\operatorname{cl} A]$ and $p_{\delta_\alpha}[\operatorname{cl} B]$ that is disjoint from Z. As before we find

- $\dim L \leqslant n-1$,
- $\dim L \cap p_{\delta_\alpha}[Q_\beta] \leqslant \dim Q_\beta - 1$ for $\beta \leqslant \alpha$, and
- L and $L \cap p_{\delta_\alpha}[Q_\beta]$ are nowhere dense in X_{δ_α} and $p_{\delta_\alpha}[Q_\beta]$ respectively.

As before we find disjoint regular open sets U_α and V_α around $p_{\delta_\alpha}[\operatorname{cl} A]$ and $p_{\delta_\alpha}[\operatorname{cl} B]$ respectively such that $P_\alpha = X_{\delta_\alpha} \setminus (U_\alpha \cup V_\alpha)$ is a subset of L.

At the end let $U = \bigcup_\alpha p_{\delta_\alpha}^{-1}[U_\alpha]$ and $V = \bigcup_\alpha p_{\delta_\alpha}^{-1}[V_\alpha]$. Then U and V are disjoint open sets around F and G respectively, so that $P = X \setminus (U \cup V)$ is a partition between F and G.

To see that P is a P-set observe that $\operatorname{cl} p_{\delta_\beta}^{-1}[U_\beta] \subseteq p_{\delta_\alpha}^{-1}[U_\alpha]$ whenever $\beta < \alpha$; by compactness this implies that $\operatorname{cl} K \subseteq U$ whenever K is an F_σ-subset contained in U. The same applies for V, so that $P \cap \operatorname{cl} K = \emptyset$, whenever K is an F_σ-set that is disjoint from P.

To see that $\dim P \leqslant n-1$ observe that $p_{\delta_\alpha}[P] \subseteq P_\alpha$ for all α. Any finite basic open cover of P is supported below δ_α for some α; because $\dim P_\alpha \leqslant n-1$ this cover has an open refinement of order at most n that is also supported below δ_α.

To see that P is nowhere dense let B be any basic open set in $[0,1]^{\omega_1}$ that meets X and choose α such that B is supported below δ_α. As P_α is nowhere dense there is a basic open set $B' \subseteq B$, also supported below δ_α, that meets X_{δ_α} but is disjoint from P_α. Reinterpreted in X this means that $B' \subseteq B$, that B' meets X and that $B' \cap P = \emptyset$.

To see that $\dim P \cap Q_\alpha < \dim Q_\alpha$ and $P \cap Q_\alpha$ is nowhere dense in Q_α apply the previous two paragraphs inside the space Q_α, both times taking suitable δ's inside C_α.

In case $\mathcal{Q}$ is countable one needs only one closed and unbounded set: the intersection of the countably many associated to X and the members of $\mathcal{Q}$.

In case $\dim X = \infty$ one chooses the dense zero-dimensional subsets $Z_{\alpha,\delta}$ as above, this to make all intersections $P \cap Q_\alpha$ nowhere dense, but one only worries about the value of $\dim(p_\delta[Q_\alpha] \setminus Z_{\alpha,\delta})$ in case $\dim Q_\alpha < \infty$. □

In the following theorem we adopt the convention that $\infty - n = \infty$ whenever n is a natural number — in this way the statement will be valid both for finite- and infinite-dimensional spaces.

Theorem 3.3 (CH). *Let X be a compact F-space of weight $\mathfrak{c}$. Then X has a base $\mathcal{B} = \{B_\alpha\colon \alpha < \omega_1\}$ such that $\dim \operatorname{Fr} B_\alpha \leqslant \dim X - 1$ for all α and $\dim \bigcap_{\alpha\in F} \operatorname{Fr} B_\alpha \leqslant \dim \operatorname{Fr} B_{\min F} - |F| + 1$ whenever F is a finite subset of ω_1.*

Proof. Let $\mathcal{C}$ be a base for X of cardinality $\aleph_1$ and let $\{\langle C_\alpha, D_\alpha\rangle\colon \alpha < \omega_1\}$ enumerate the set of pairs $\langle C, D\rangle \in \mathcal{C}^2$ that satisfy $\operatorname{cl} C \subseteq D$.

Apply Proposition 3.2 repeatedly to find, for each α, disjoint regular open sets U_α and V_α around $\operatorname{cl} C_\alpha$ and $X \setminus D_\alpha$ respectively such that $P_\alpha = X \setminus (U_\alpha \cup V_\alpha)$ is a nowhere dense P-set that satisfies

- $P_\alpha = \operatorname{Fr} U_\alpha$,
- $\dim P_\alpha \leqslant \dim X - 1$,
- for every finite subset F of α one has $\dim(P_\alpha \cap \bigcap_{\beta\in F} P_\beta) \leqslant \dim \bigcap_{\beta\in F} P_\beta - 1$.

Then $\{U_\alpha\colon \alpha < \omega_1\}$ is the base that we seek. □

A special case of this theorem is the one that we shall use in the next section.

Theorem 3.4 (CH). *Let X be a compact F-space of weight $\mathfrak{c}$ and of finite dimension n. Then X has a base $\mathcal{B} = \{B_\alpha\colon \alpha < \omega_1\}$, consisting of regular open sets, such that $\dim \bigcap_{\alpha\in F} \operatorname{Fr} B_\alpha = \emptyset$ whenever F is an $n+1$-element subset of ω_1.*

Proof. Let $\{B_\alpha\colon \alpha < \omega_1\}$ be a base as in Theorem 3.3. Then $\dim \operatorname{Fr} B_\alpha \leqslant n-1$ for all α, so if $|F| = n+1$ then $\dim \bigcap_{\alpha\in F} \operatorname{Fr} B_\alpha \leqslant n - 1 - (n+1) + 1 = -1$, which means that $\bigcap_{\alpha\in F} \operatorname{Fr} B_\alpha = \emptyset$. □

4. Finite-to-one maps

The purpose of this section is to show that, assuming the Continuum Hypothesis, every finite-dimensional compact F-space of weight $\mathfrak{c}$ is a finite-to-one continuous image of a compact zero-dimensional space of weight $\mathfrak{c}$.

Theorem 4.1 (CH). *Let X be a compact F-space of weight $\mathfrak{c}$ of finite dimension n. Then X is the at most 2^n-to-1 continuous image of a compact zero-dimensional space of weight $\mathfrak{c}$.*

Proof. Let $\mathcal{B} = \{B_\alpha\colon \alpha < \omega_1\}$ be a base for X as in Theorem 3.4. Let $\mathbb{B}$ be the Boolean subalgebra of $\operatorname{RO}(X)$ generated by this base and let Y be the Stone space of $\mathbb{B}$. If $y \in Y$ then $\bigcap\{\operatorname{cl} C\colon C \in y\}$ consists of exactly one point, which we denote by $f(y)$. Let $x \in X$ and put $F = \{\alpha\colon x \in \operatorname{Fr} B_\alpha\}$. If $f(y) = x$ then y determines a function $p_y : F \to 2$ by $p_y(\alpha) = 1$ iff $B_\alpha \in y$;

in addition if $\alpha \notin F$ then $x \in B_\alpha$ or $x \notin \operatorname{cl} B_\alpha$. It follows that if $f(y) = f(z) = x$ then $B_\alpha \in y$ iff $B_\alpha \in z$ for $\alpha \notin F$, so if $y \neq z$ then $p_y \neq p_z$. This implies that $|f^{-1}(x)| \leqslant 2^{|F|} \leqslant 2^n$. $\square$

Corollary 4.2 *(CH). If X is a one-dimensional compact F-space of weight $\mathfrak{c}$ then X is an at most 2-to-1 continuous image of a compact zero-dimensional space of weight $\mathfrak{c}$.*

Thus, for compact one-dimensional F-spaces we have a direct generalization of Hurewicz' theorem, as $1 + 1 = 2$.

One can give a proof of Theorem 4.1 along the lines of the proof in Section 1. We take a base as in Theorem 3.4 but enumerate it in such a way that every singleton open set is counted cofinally often.

Again one constructs tilings $\mathcal{T}_\alpha$ of order $n + 1$ but one can only ensure that $\mathcal{T}_{\alpha+1}$ refines $\mathcal{T}_\alpha$ for every α. The reason becomes apparent at stage ω: the common refinement of the tilings $\mathcal{T}_m$ will be infinite and not usable as a factor in a compact product. What one can do is start a fresh ω-sequence of tilings at each limit ordinal λ. The tilings $\mathcal{T}_{\lambda+m}$ will be constructed from the family $\{B_\alpha\colon \alpha \geqslant \delta\}$ for some δ (depending on λ). The zero-dimensional space Y will consist of the points $\langle T_\alpha\colon \alpha < \omega_1\rangle \in \prod_\alpha \mathcal{T}_\alpha$ with the following properties:

- $T_{\alpha+1} \subseteq T_\alpha$ for all α;
- $\{T_\alpha\colon \alpha < \omega_1\}$ has the finite intersection property.

For each x there will be at most n limit ordinals λ such that x is on the boundary of a tile in one of the $\mathcal{T}_{\lambda+k}$ (and hence in $\mathcal{T}_{\lambda+l}$ for $l \geqslant k$). Let $\langle \lambda_i\colon i < p\rangle$ enumerate these limit ordinals and for each i let m_i be the maximum of $\{|\mathcal{T}_{\lambda_i+k,x}|\colon k \in \omega\}$. The fiber of x under the obvious map from Y onto X has cardinality $\prod_{i<p} m_i$. For each i we get at least $m_i - 1$ boundaries that contain x, so that $n \geqslant \sum_{i<p}(m_i - 1)$. From this it easy to deduce that $2^n \geqslant \prod_{i<p} m_i$, so that this map has fibers of size at most 2^n as well.

4.1. Universality

The proofs of Hurewicz and Kuratowski show that if a space is compact, metrizable, n-dimensional and without isolated points then it is an at most $n + 1$-to-1 continuous image of the Cantor set. It is also well known that every compact and metrizable space is a continuous image of the Cantor set, see for example [3, 1.3.D]. Thus the Cantor set is universal in the class of compact metrizable spaces in the sense of continuous onto mappings and even in a parametrized fashion if dimension is taken into account.

Parovičenko [9] proved that, under CH, every compact Hausdorff space of weight $\mathfrak{c}$ is a continuous image of $\mathbb{N}^*$, the remainder in the Čech–Stone compactification of $\mathbb{N}$. This all suggests that, still under CH, the space $\mathbb{N}^*$ should also have this parametrized universality property.

The next result shows that this is not the case. The space $E2^\omega$, mentioned in the following proposition is the *absolute* or *projective cover* of the Cantor set 2^ω — this is the unique (up to homeomorphism) extremally disconnected compact space that admits a perfect irreducible map onto 2^ω, see [2, Problem 6.3.19].

Proposition 4.3. *Let X be a compact F-space that admits a finite-to-one map onto $E2^\omega$. Then X has a nonempty clopen subset that is homeomorphic to $E2^\omega$.*

Proof. Let $f : X \to E2^\omega$ be a continuous surjection whose fibers are finite. Because $E2^\omega$ is zero-dimensional it follows that f is constant on every connected subset of X and this implies that these sets must be finite and hence consist of one point only. This implies that X is zero-dimensional.

There is a closed subspace A of X such that the restriction $f \restriction A \to E2^\omega$ is irreducible. Since $E2^\omega$ is extremally disconnected, it follows that $f \restriction A$ is a homeomorphism [2, Problem 6.3.19(c)]. Hence we may as well assume that $E2^\omega$ is a subspace of X and that f is a retraction from X onto $E2^\omega$. We claim $E2^\omega$ has nonempty interior in X, which clearly suffices. Striving for a contradiction, assume that $E2^\omega$ is nowhere dense in X. Let $\{E_n\colon n < \omega\}$ be a π-base for $E2^\omega$ consisting of clopen sets. For every n the preimage $f^{-1}[E_n]$ is a clopen subset of X such that $f^{-1}[E_n] \cap E2^\omega = E_n$. As we are assuming that $E2^\omega$ is nowhere dense in X we can find a pairwise disjoint family $\{U_{n,i}\colon i, n < \omega\}$ of clopen sets such that for all n we have,

$$\bigcup_{i<\omega} U_{n,i} \subseteq f^{-1}[E_n] \setminus E2^\omega. \tag{4.1}$$

To see that this is possible let $\prec$ well-order ω^2 in type ω and observe that at each stage $f^{-1}[E_n] \setminus \bigcup_{\langle m,j\rangle \prec \langle n,i\rangle} U_{m,j}$ is always clopen and nonempty, and hence never a subset of $E2^\omega$.

For all i put $V_i = \bigcup_{n<\omega} U_{n,i}$. Since X is a compact F-space the closures $\operatorname{cl} V_i$ form a pairwise disjoint family. By (4.1), $f[\operatorname{cl} V_i] = E2^\omega$ for all i, which contradicts f being finite-to-one. $\square$

It is well known that $\mathbb{N}^*$ is not separable and that every nonempty clopen subset of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$ itself. As the space $E2^\omega$ is separable this implies that $\mathbb{N}^*$ does not admit a finite-to-one continuous map onto $E2^\omega$.

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