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On the *G*-compactifications of the rational numbers

Jan van Mill

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Abstract We show that for any sufficiently homogeneous metrizable compactum X there is a Polish group G acting continuously on the space of rational numbers $\mathbb{Q}$ such that X is its unique G-compactification. This allows us to answer Problem 995 in the 'Open Problems in Topology II' book in the negative: there is a one-dimensional Polish group G acting transitively on $\mathbb{Q}$ for which the Hilbert cube is its unique G-completion.

Keywords G-compactification · Rational numbers · Countable dense homogeneous

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1 Introduction

All spaces under discussion are Tychonoff.

By an *action* of a topological group on a space we will always mean a *continuous* action. Let G be a topological group acting on a space X. As usual, we call X a G-*space*. A compactification γX of X is a G-*compactification* if the action of G extends to γX. If G is locally compact, then X has a G-compactification [23]. Similarly if G is $\aleph_0$-bounded and acts transitively on the Baire space X [22]. For more results see Megrelishvili and Scarr [15]. Even if both G and X are Polish, then X need not have a G-compactification [12]. But if X has a G-compactification, then it has a *largest* G-compactification (in the usual order of compactifications), denoted by $\beta_G X$. Not

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J. van Mill (✉)
Faculty of Sciences, Department of Mathematics, VU University Amsterdam,
De Boelelaan 1081[a], 1081 HV Amsterdam, The Netherlands
e-mail: vanmill@few.vu.nl

much is known about these G-spaces. If X is the space of rational numbers $\mathbb{Q}$, or the space of irrational numbers $\mathbb{P}$, and G is the full autohomeomorphism group of X endowed with the discrete topology, then βX is its unique G-compactification, as was shown by van Douwen [5]. Here βX is the Čech-Stone compactification of X. Other concrete examples can be found for example in Smirnov and Stojanov [19], Stojanov [21] and Smirnov [18]. The aim of this note is among other things to provide many other concrete examples.

Recall that a space X is *countable dense homogeneous* (abbreviated: CDH) if for all countable dense subsets $D, E \subseteq X$ there is a homeomorphism $f: X \to X$ with $f(D) = E$.

Let X be a subspace of the space Y. If the topological group G acts on Y and X is invariant under the action of G, then Y is called a *G-extension* of X.

Theorem 1.1 *Let X be a metrizable compactum with the following properties:*

(1) *X is homogeneous,*
(2) *$X\backslash\{x\}$ is* CDH *for every $x \in X$.*

Then there is a Polish group G such that if D is any countable dense subset of X, then G admits a transitive action on D and X is the unique Čech-complete G-extension of D in which D is dense (hence X is the unique G-compactification of D).

The reason that we are not only interested in G-compactifications but also in Čech-complete G-extensions will be explained later.

Observe that Theorem 1.1 is clearly true if X is finite. So from now on we will implicitly assume that X is infinite. Observe that by homogeneity this implies that X is dense in itself. Hence D is a countable, dense in itself metrizable space and so by Sierpiński [17], D and $\mathbb{Q}$ are homeomorphic and so every 'sufficiently homogeneous' metrizable compactum X is the unique G-compactification of $\mathbb{Q}$ for some Polish group G.

A space X is *strongly locally homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f: X \to X$ which is supported on B (that is, f is the identity outside B) and moves x to y. Most of the familiar homogeneous spaces are SLH: the Hilbert cube, homogeneous zero-dimensional spaces (such as the Cantor set), manifolds without boundaries, universal Menger continua, etc. The pseudoarc and the solenoids are examples of homogeneous continua that are not SLH. Not every SLH-space is homogeneous, as the topological sum of the one-sphere and the two-sphere demonstrates. Observe that every open subspace of an SLH-space is again SLH.

It was shown by Bennett [2] that every locally compact separable metrizable SLH-space is CDH. This was generalized by de Groot to Polish spaces in [9]. Hence by Theorem 1.1 we get:

Corollary 1.2 *For every homogeneous metrizable* SLH*-compactum X there exist a Polish group G and a transitive action of G on X such that X is (homeomorphic to) the unique G-compactification of $\mathbb{Q}$.*

We will also comment that Megrelishvili's Example quoted above provides an example of a Polish group H acting on $\mathbb{Q}$ for which there is no H-compactification.

It is well-known that every metrizable space X has a metrizable completion that preserves both the dimension and the weight of X, [7, 4.1.20]. It is natural to ask whether a similar result can be proved for G-spaces. The answer is 'yes' for actions of locally compact and σ-compact groups G on metrizable spaces, as was shown by Megrelishvili [13]. We will show in contrast that for X the Hilbert cube, the group G in Theorem 1.1 is at most one-dimensional. From this we conclude that there is a one-dimensional Polish group G acting transitively on $\mathbb{Q}$ for which the Hilbert cube is its unique G-completion. This answers Problem 995 in Megrelishvili [14] in the negative which asks for the existence of dimension preserving G-completions.

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2 Preliminaries

If X is a space, then $\mathcal{H}(X)$ denotes its group of homeomorphisms. Moreover, if $D \subseteq X$, then $\mathcal{H}(X|D)$ denotes the subgroup

$$\{f \in \mathcal{H}(X) : f(D) = D\}.$$

of $\mathcal{H}(X)$.

If X is compact, then the so-called *compact-open topology* on $\mathcal{H}(X)$ makes it a topological group. A subbasis for the compact-open topology on $\mathcal{H}(X)$ consists of all sets of the form

$$[K, U] = \{f \in \mathcal{H}(X) : f(K) \subseteq U\},$$

where K and U are arbitrary subsets of X with K compact and U open. It is not difficult to prove that the function $(g, x) \mapsto g(x)$ is a continuous action of $\mathcal{H}(X)$ on X. It is well-known, and easy to prove that if X is compact and metrizable, then $\mathcal{H}(X)$ is Polish. Here *Polish* means a separable and completely metrizable space.

A topological group G is said to be *Polishable*, if it admits a finer Polish group topology. An obvious necessary condition for the Polishability of G is that G is a Borel group (that is a topological group which is homeomorphic to a Borel subspace of the Hilbert cube), see [10, 15.2]. But this condition is not sufficient, see Becker and Kechris [1, p. 12]. If a group is Polishable, then its Polish group topology is unique. Hence Polishability is an 'intrinsic' property. For more information on Polishable groups, see e.g., Solecki [20].

In [16], the following simple criterion was formulated that guarantees the Polishability of certain subgroups of Polish groups.

Theorem 2.1 ([16, Sect. 6(A)]) *Let G be a Polish group. In addition, let H be a subgroup of G containing a countable collection $\mathcal{B}$ of subgroups such that*

(A) *every $B \in \mathcal{B}$ is closed in H,*

(B) *for every $B \in \mathcal{B}$ there are countable subsets $A_B, A'_B \subseteq H$ such that*

$$H = \bigcap_{B\in\mathcal{B}} A_B\overline{B} \cap \bigcap_{B\in\mathcal{B}} \overline{B}A'_B$$

(*here closure means closure in G*).

Then H is Polishable.

Corollary 2.2 *Let G be a Polish group acting on a space X. Let D be a countable subset of X on which the subgroup*

$$H = \{g \in G : gD = D\}$$

of G acts transitively. Then H is Polishable.

Proof Enumerate D faithfully as $\{d_n : n < \omega\}$, and for every n put

$$B_n = \{h \in G : hd_n = d_n\}.$$

It is clear that B_n is a closed subgroup of G. Since H acts transitively on D, we may fix for all $n, m < \omega$ an element $f_{n,m} \in H$ such that $f_{n,m}d_n = d_m$. Let F be the subgroup of H generated by $\{f_{n,m} : n, m < \omega\}$. Observe that for every n we have $F(B_n \cap H) = H = (B_n \cap H)F$. Indeed, pick arbitrary $g \in H$ and $n < \omega$. Let $d_m = gd_n$ and $d_k = g^{-1}d_n$. Then $f_{n,m}^{-1}g \in B_n$ and $gf_{k,n}^{-1} \in B_n$, proving that $g \in F(B_n \cap H) \cap (B_n \cap H)F$.

We will prove that $H = \bigcap_{n<\omega} FB_n \cap \bigcap_{n<\omega} B_nF$. An application of Theorem 2.1 then finishes the proof.

By the above it suffices to prove that $\bigcap_{n<\omega} FB_n \cap \bigcap_{n<\omega} B_nF \subseteq H$. Indeed, pick an arbitrary $g \in \bigcap_{n<\omega} FB_n \cap \bigcap_{n<\omega} B_nF$, and fix $n < \omega$. There are $f, f' \in F$ and $\xi, \eta \in B_n$ such that $g = f\xi = \eta f'$. Clearly, $\xi d_n = \eta^{-1}d_n = d_n$. Hence $gd_n = (f\xi)d_n = fd_n \in D$, and, similarly, $g^{-1}d_n = (f'^{-1}\eta^{-1})d_n = f'^{-1}d_n \in D$. Hence $g \in H$, and so we are done. □

Remark 2.3 Let G be a Polishable group acting on a space X. Let $\hat{G}$ denote G with its finer Polish group topology. Observe that since the action $G \times X \to X$ is continuous, so is the action $\hat{G} \times X \to X$.

For a space X we let βX denote its Čech-Stone compactification. We will use the well-known fact that every homeomorphism $f: X \to X$ can be extended to a homeomorphism $\beta f: \beta X \to \beta X$. For details, see [6, Sect. 3.6]. A space X is *Čech-complete* if it is a G_δ-set in any compactification of X.

3 Proof of Theorem 1.1

Throughout this section, let X be an infinite metrizable compactum with the following properties:

(†) X is homogeneous,
(‡) $X\backslash\{x\}$ is CDH for every $x \in X$.

Observe that by (†), X is uncountable and has no isolated points, which implies by (‡) that X is CDH. Simply observe that since X is compact, X is the Alexandrov one-point compactification of $X\backslash\{\mathrm{pt}\}$. Hence every $\xi \in \mathcal{H}(X\backslash\{\mathrm{pt}\})$ extends to a homeomorphism $\bar{\xi} \in \mathcal{H}(X)$ such that $\bar{\xi}(\mathrm{pt}) = \mathrm{pt}$.

Let D be an arbitrary countable dense subset of X.

For a countable dense subset E of X we let G_E denote the group $\mathcal{H}(X|E)$. Since X is CDH, for all countable dense subsets E and F of X we have that G_E and G_F are topologically isomorphic (being conjugate). So the topological group G_E is 'independent of the choice of E'. It will be convenient to denote G_D by G.

For $X = \mathbb{S}^2$, G is homeomorphic to the first category 'rational Hilbert space', as was shown by Dijkstra and van Mill [3]. Hence G need not be Polish. We will first show that it *is* Polishable.

Lemma 3.1 *If E is a countable dense subset of X, and either $x, y \in E$ or $x, y \in X\backslash E$, then there exists $f \in \mathcal{H}(X|E)$ such that $f(x) = y$.*

Proof Assume first that $x, y \in E$. By (†) we may pick $h \in \mathcal{H}(X)$ such that $h(x) = y$. Hence by (‡) there exists $\xi \in \mathcal{H}(X)$ such that

(1) $\xi(y) = y$,
(2) $\xi\,(h(E\backslash\{x\})) = E\backslash\{y\}$.

Then for $f = \xi \circ h$ we clearly have $f(x) = y$ and

$$f(E) = \xi(h(E\backslash\{x\}) \cup \{h(x)\}) = (E\backslash\{y\}) \cup \{y\} = E,$$

as required.

Next assume that $x, y \in X\backslash E$. Pick by (†) an element $h \in \mathcal{H}(X)$ such that $h(x) = y$. Then both E and $h(E)$ are countable dense subsets of $X\backslash\{y\}$. Hence by (‡) we may pick $\xi \in \mathcal{H}(X)$ such that $\xi(y) = y$ and $\xi\,(h(E)) = E$. It is easy to see that $f = \xi \circ h$ is as required. □

We conclude that $G = \mathcal{H}(X|D)$ acts transitively on D, and so G is Polishable by Corollary 2.2. We claim that $\hat{G}$ is the desired group. By Remark 2.3, $\hat{G}$ acts transitively on D and X is a $\hat{G}$-compactification of D.

(A) Čech-complete $\hat{G}$-extensions. Let Y be a Čech-complete G-extension of D in which D is dense. We will first show that Y is compact. We will then proceed to show that Y is homeomorphic to X by a homeomorphism that restricts to the identity on D. Hence Y and X are equivalent G-extensions of D.

Observe that βD, βY and X are compactifications of D. There consequently exist continuous functions $\pi_1 : \beta D \to \beta Y$ and $\pi_2 : \beta D \to X$ that both restrict to the identity on D.

It will be convenient to fix some notation. Fix $g \in G$. The homeomorphism $d \mapsto gd$ of D will be denoted by g_D. It extends to the homeomorphism $x \mapsto gx$ of X; this homeomorphism will be denoted by g_X. It also extends to the homeomorphism

$y \mapsto gy$ of Y; this homeomorphism will be denoted by g_Y. It also extends to the homeomorphism βg_D of βD. Finally, g_Y extends to the homeomorphism βg_Y of βY. Observe that the homeomorphism βg_Y, βg_D and g_X all restrict to g_D on D. It consequently follows that the diagrams

$$\begin{array}{ccc} \beta D & \xrightarrow{\beta g_D} & \beta D \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \beta Y & \xrightarrow[\beta g_Y]{} & \beta Y \end{array} \qquad \begin{array}{ccc} \beta D & \xrightarrow{\beta g_D} & \beta D \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow[g_X]{} & X \end{array} \tag{$*$}$$

commute.

Proposition 3.2 *Y is compact.*

Proof Put $S = \beta D \backslash \pi_1^{-1}(Y)$. Then S is a σ-compact subset of βD which misses D. As a consequence, $\pi_2(S)$ is a σ-compact subset of X which misses D. Since D is not Čech-complete, being homeomorphic to $\mathbb{Q}$, there consequently exists an element $a \in X \backslash (D \cup \pi_2(S))$. So we conclude that $\pi_2^{-1}(a) \subseteq \pi_1^{-1}(Y)$. Take an arbitrary $b \in X \backslash D$. By Lemma 3.1 there exists $g \in G$ such that $g_X(a) = b$. Observe that by the second commutative diagram in $(*)$ we get

$$\beta g_D\left(\pi_2^{-1}(a)\right) = \pi_2^{-1}(b). \tag{1}$$

Since $g_Y(Y) = Y$ it follows that $\beta g_Y(Y) = Y$ and hence by the first commutative diagram in $(*)$,

$$\beta g_D\left(\pi_1^{-1}(Y)\right) = \pi_1^{-1}(Y). \tag{2}$$

Since $\pi_2^{-1}(a) \subseteq \pi_1^{-1}(Y)$, by (1) and (2) we conclude that $\pi_2^{-1}(b) \subseteq \pi_1^{-1}(Y)$. Since $b \in X \backslash D$ was arbitrarily chosen, this gives us that $\pi_2^{-1}(X \backslash D) \subseteq \pi_1^{-1}(Y)$. But $\pi_1^{-1}(D) = D = \pi_2^{-1}(D)$, hence $\pi_2^{-1}(D) \subseteq \pi_1^{-1}(Y)$. So we conclude that $\beta D \subseteq \pi_1^{-1}(Y)$, i.e., Y is compact. □

So we conclude that Y is a $\hat{G}$-compactification of D. We will proceed to show X is the unique $\hat{G}$-compactification of D which finishes the proof.

(B) $\hat{G}$-compactifications of D. Let γD be a $\hat{G}$-compactification of D such that $\gamma D \geq X$. (It is not assumed that γD is metrizable.) Let $\pi : \gamma D \to X$ be the unique continuous surjection that restricts to the identity on D. Our aim is to show that π is one-to-one, i.e., γD and X are equivalent compactifications of D.

Striving for a contradiction, assume that there exists $b \in X$ such that $\pi^{-1}(b)$ contains at least two points, say p and q. Observe that $b \in X \backslash D$. Let $\xi : \gamma D \to \mathbb{I}$ be a Urysohn function such that $\xi(p) = 0$ and $\xi(q) = 1$. Let H be a countable dense subset of $\hat{G}$. We list H as $\{\alpha_n : n < \omega\}$ such that every $h \in H$ is listed infinitely often. For every $g \in G$ denote the homeomorphism $z \mapsto gz$ of γD by $\hat{g}$. For all $n \geq 1$, put

$$U_n = \{x \in X : \operatorname{diam}\left(\xi \circ \hat{\alpha}_n\right)\left(\pi^{-1}(x)\right) < {}^1\!/_n\}.$$

It is clear that U_n is open and contains D since for every $d \in D$ we have that $\pi^{-1}(d) = \{d\}$ (use that π is closed). Put

$$S = \bigcap_{n \geq 1} U_n.$$

Then S is a G_δ-subset of X containing D. Since D is not complete, being homeomorphic to $\mathbb{Q}$, we may pick an element $a \in S \backslash D$. Since every $h \in H$ is listed infinitely often, it follows that for every $h \in H$ we have that $\xi \circ \hat{h}$ is constant on $\pi^{-1}(a)$. By Lemma 3.1, we may pick $g \in G$ such that $ga = b$. Observe that the diagram

$$\begin{array}{ccc}
\gamma D & \xrightarrow{\pi} & X \\
{\scriptstyle z \mapsto gz}\downarrow & & \downarrow{\scriptstyle x \mapsto g(x)} \\
\gamma D & \xrightarrow[\pi]{} & X
\end{array}$$

commutes since π restricts to the identity on D. Hence $\hat{g}\left(\pi^{-1}(a)\right) = \pi^{-1}(b)$. Pick $p', q' \in \pi^{-1}(a)$ such that $gp' = p$ and $gq' = q$.

Let $(h_i)_i$ be a sequence of elements of H converging to g. Observe that $h_i p' \to gp' = p$ and $h_i q' \to gq' = q$. Since $\xi(p) = 0$ and $\xi(q) = 1$, we may consequently pick an index $i(0)$ so large that

$$\xi(h_{i(0)}p') < {}^1\!/_4, \qquad \xi(h_{i(0)}q') > {}^3\!/_4.$$

But this contradicts the fact that the function $\xi \circ \hat{h}_{i(0)}$ is constant on $\pi^{-1}(a)$. This proves that X is the maximal $\hat{G}$-compactification of D.

We will now prove that X is a minimal $\hat{G}$-compactification of D. This implies that up to equivalence, X is the unique $\hat{G}$-compactification of D since we already know that $\beta_{\hat{G}} D = X$.

To this end, let νD be a (necessarily metrizable) $\hat{G}$-compactification of D such that $\nu D \leq X$. Let $\phi\colon X \to \nu D$ be the unique continuous surjection which restricts to the identity on D. Our aim is to show that ϕ is one-to-one, i.e., νD and X are identical compactifications of D.

Lemma 3.3 *There is an element $p \in \nu D \backslash D$ such that $\phi^{-1}(p)$ is a single point.*

Proof For all $n \geq 1$, put

$$U_n = \{x \in \nu D : \operatorname{diam} \phi^{-1}(x) < {}^1\!/_n\}.$$

It is clear that U_n is open and contains D since for every $d \in D$ we have that $\phi^{-1}(d) = \{d\}$. Put $S = \bigcap_{n \geq 1} U_n$. Then S is a G_δ-subset of νD containing D. Since D is not complete, being homeomorphic to $\mathbb{Q}$, we may pick an element $p \in S \backslash D$. Then p is clearly as required. $\square$

Striving for a contradiction, assume that there is a point $q \in \nu D \setminus D$ such that $\phi^{-1}(q)$ contains two distinct points, say r and s. By Lemma 3.1 we may pick $h \in G = \mathcal{H}(X|D)$ such that $h(r)$ is the unique point t in $\phi^{-1}(p)$, where p is such as in Lemma 3.3. Observe that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \nu D \\ {\scriptstyle x \mapsto h(x)} \downarrow & & \downarrow {\scriptstyle z \mapsto hz} \\ X & \xrightarrow[\phi]{} & \nu D \end{array}$$

commutes since ϕ restricts to the identity on D. As a consequence, $hq = p$. Let $(d_n)_n$ be a sequence in D such that $d_n \to s$ (in X). Clearly, $d_n \to q$ (in νD), hence $hd_n \to p$ (in νD) and so $h(d_n) \to t$ (in X). But this is a contradiction since $h(s) \neq h(r) = t$.

Remark 3.4 In the light of Theorem 1.1, it is natural to ask whether for every Polish group G acting on $\mathbb{Q}$ there is a G-compactification $\gamma\mathbb{Q}$ of $\mathbb{Q}$. The answer to this question is in the negative. The counterexample of Megrelishvili [12] mentioned in Sect. 1 is a Polish group G acting on the so-called hedgehog $J(\aleph_0)$ of spininess $\aleph_0$. Consider its subspace $\mathbb{Q}(\aleph_0)$ consisting of all ‘rational’ points. By using the same method as in Megrelishvili [12] it can be shown that the group $H = \{g \in G : g\mathbb{Q}(\aleph_0) = \mathbb{Q}(\aleph_0)\}$ acts on $\mathbb{Q}(\aleph_0)$ and $\mathbb{Q}(\aleph_0)$ has no H-compactification. The group H can be shown to be Polishable by the same technique as was used in the proof of Corollary 2.2. The details of checking this are left to the reader.

The groups in Theorem 1.1 act transitively, while the group H in Remark 3.4 does not. This suggests the following question.

Question 3.5 *Let G be a Polish group G acting transitively on $\mathbb{Q}$. Does $\mathbb{Q}$ admit a G-compactification?*

This is a special case of the following question due to Furstenberg and Scarr: if G acts transitively on X, does X admit a G-compactification? See Question 2.6 in Megrelishvili [14].

Remark 3.6 Our results show that for many G-spaces X, the compactification $\beta_G X$ is metrizable. A similar result is due to Stojanov [21]. He proved that for X the unit sphere in Hilbert space ℓ^2 and $G = U(\ell^2)$ the unitary group endowed with the strong operator topology, $\beta_G X$ is equivalent to the natural inclusion of X into the weakly compact unit ball of ℓ^2. See also [4, Sect. 7.6]. For more metrizable $\beta_G X$ see Smirnov and Stojanov [19] and Kozlov and Chatyrko [11].

4 Dimension

Let X be the Hilbert cube Q with countable dense subset D. As in Sect. 3, let $G = \mathcal{H}(Q|D)$. Then G is homeomorphic to the first category ‘rational Hilbert space’, as was shown by Dijkstra and van Mill [3], which is one-dimensional by Erdős [8]. We showed that G is Polishable by using the simple criterion in Theorem 2.1. Let $\hat{G}$ be G with its Polish group topology. An inspection of the proof in van Mill [16, Sect. 6(A)] shows that $\hat{G}$ is a subspace of $G \times \mathbb{N}^\omega$. Since $\dim G = 1$ and $\dim \mathbb{N}^\omega = 0$, it follows that $\dim \hat{G} \leq 1$. Hence by Theorem 1.1, $\hat{G}$ is an at most one-dimensional Polish group acting transitively on $\mathbb{Q}$ such that the strongly infinite-dimensional Hilbert cube is its unique $\hat{G}$-completion. This answers Problem 995 in Megrelishvili [14] in the negative which asks for dimension preserving G-completions. It may not be so easy to determine the dimension of $\hat{G}$. It is very probable equal to 1. Verifying this is interesting in its own right but irrelevant for our discussion here because if $\dim \hat{G} = 0$ then we can replace $\hat{G}$ if we aim for a one-dimensional group by $\mathbb{R} \times \hat{G}$. These observations suggest the following:

Question 4.1 *Is there is zero-dimensional Polish group G acting transitively on $\mathbb{Q}$ such that the Hilbert cube is its unique G-completion?*

If every Polish group is a continuous homomorphic image of a zero-dimensional Polish group, then we are done of course by what we obtained above. But this is not known as far as I know.

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