ISRAEL JOURNAL OF MATHEMATICS **165** (2008), 133–159
DOI: 10.1007/s11856-008-1007-0

HOMOGENEOUS SPACES AND TRANSITIVE ACTIONS BY POLISH GROUPS

BY

Jan van Mill

Faculty of Sciences, Department of Mathematics
Vrije Universiteit De Boelelaan 1081[a]
1081 HV Amsterdam, The Netherlands
e-mail: vanmill@cs.vu.nl

ABSTRACT

We prove that for every homogeneous and strongly locally homogeneous Polish space X there is a Polish group admitting a transitive action on X. We also construct an example of a homogeneous Polish space which is not a coset space and on which no separable metrizable topological group acts transitively.

1. Introduction

In this paper we are, among other things, interested in interesting topological spaces X that admit an action of an interesting topological group G. Since the action is usually required to be transitive, the topological spaces X we are mostly interested in are **homogeneous**. That is, for all $x, y \in X$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$. For a homogeneous space X its group of homeomorphisms $\mathcal{H}(X)$ endowed with the discrete topology acts transitively on X. But the discrete topology is not interesting. The compact-open topology on $\mathcal{H}(X)$ is better but only works well if X is compact (or if X is locally compact, by using its Alexandrov one-point compactification).

We were motivated by the Effros Theorem on actions of Polish groups on Polish spaces (Effros [8]; see also [2], [14] and [25]). This theorem implies among other things that if G is a Polish group that acts transitively on a Polish

Received October 9, 2006

space X, then X is a coset space of G. Hence, as observed by Ungar [36], every locally compact separable metrizable homogeneous space is a coset space. Ford [11] gave an example of a homogeneous space that is not a coset space but his example is not metrizable. In [24], a separable metrizable **σ-compact** space was constructed that is not a coset space but on which some separable metrizable topological group acts transitively. This left open the question whether every homogeneous Polish space is a coset space, preferably of some Polish group. This is related to Question 3 in Ancel [2]. He asked whether for every homogeneous Polish space X there is an admissible topology on its homeomorphism group $\mathcal{H}(X)$ which makes X a coset space of $\mathcal{H}(X)$.

If H is a closed subgroup of a topological group G, then G acts transitively on the coset space $G/H = \{xH : x \in G\}$ and the natural projection map $\pi\colon G \to G/H$ is open. If G acts transitively on X, then the closed subgroup $G_x = \{g \in G : gx = x\}$ of G is called the **stabilizer** of $x \in X$. It is well-known, and easy to prove, that G/G_x is homeomorphic to X if for every open neighborhood of the neutral element e of G and for some $x \in X$ (equivalently: for every $x \in X$) we have that Ux is open. If the action has this property, then it is called **micro-transitive**. So for a space X to be a coset space it is necessary and sufficient that there is a topological group G acting transitively and micro-transitively on X. The Effros Theorem implies that if both G and X are Polish and the action is transitive, then it is micro-transitive.

The main result of this paper is the construction of the following example.

THEOREM 1.1: *There is a homogeneous Polish space Z with the following property. If G is a topological group acting on Z, then there are an element $z \in Z$ and a neighborhood U of the neutral element e of G such that Uz is meager in Z.*

So an arbitrary homogeneous Polish space X need not be a coset space, since no action on Z by a topological group is micro-transitive. This answers Question 3 in Ancel [2] in the negative.

A topological group G is called **$\aleph_0$-bounded** provided that for every neighborhood U of the identity e there is a countable subset F of G such that $G = FU$. It was proved by Guran that a topological group G is $\aleph_0$-bounded if and only if it is topologically isomorphic to a subgroup of a product of separable metrizable groups. For a proof, see Uspenskiĭ [37].

COROLLARY 1.2: *If G is an $\aleph_0$-bounded topological group acting on Z, then there is an element $z \in Z$ such that its orbit Gz is meager in Z.*

(In fact, all our results hold for actions that are merely **separately continuous**. That is, for every $g \in G$ the bijection $x \mapsto gx$ is a homeomorphism of X, and for every (fixed) $p \in G$, the function $\gamma_p \colon G \to X$ defined by $\gamma_p(g) = gx$ is continuous.)

It was asked by the author in [26, Question 4.2] whether for every homogeneous Polish space X there is a separable metrizable topological group acting transitively on X. Hence, by 1.2, Z is a counterexample to this question. It was also asked by Aarts and Oversteegen [1] whether every homogeneous Polish space admits a product structure. This question was answered in the negative in [27] by using highly nontrivial results of Bing and Jones [6] and Lewis [19]. We will show that Z is a much better (and simpler) counterexample. So Z is a counterexample to several natural questions on homogeneity in the literature.

A space X is **strongly locally homogeneous** (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for every $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f \colon X \to X$ which is supported on B (that is, f is the identity outside B) and moves x to y. This notion is due to Ford [11]. The topological sum of the spheres $\mathbb{S}^1$ and $\mathbb{S}^2$ is SLH, but not homogeneous. It is not hard to prove that a **connected** SLH-space is homogeneous. Most of the well-known homogeneous continua are strongly locally homogeneous: the Hilbert cube (Keller [17]), the universal Menger continua (Bestvina [5]) and manifolds without boundaries. The pseudo-arc is an example of a homogeneous continuum which is not SLH. Ford [11] essentially proved that every Tychonoff homogeneous and SLH-space X is a coset space (see also [31, Theorem 3.2]). The proof goes as follows. As usual, βX denotes the Čech-Stone compactification of X. The subgroup

$$G = \{g \in \mathcal{H}(\beta X) : g(X) = X\}$$

of the homeomorphism group of βX endowed with the compact-open topology acts transitively on X, and by strong local homogeneity, it acts microtransitively as well.

In [26, Theorem 1.1] it was shown that if X is a separable metrizable, homogeneous SLH-space, then X is a coset space of some separable metrizable topological group. So this is Ford's Theorem in the class of separable metrizable spaces. 1.1 motivates the question whether this can also be proved for the class of Polish spaces.

THEOREM 1.3: *Let X be a Polish, homogeneous* SLH*-space. Then X is a coset space of a Polish group.*

The group G we get from the proof of [26, Theorem 1.1] is, in general, not Polish. However, we will prove that it can be chosen to be **Polishable**, i.e., it has a stronger Polish group topology.

So for separable metrizable spaces and separable metrizable groups, we have the following implications:

$$\begin{array}{lcccc} \text{coset space} & \stackrel{\not\leftarrow}{\longrightarrow} & \text{transitive action} & \stackrel{\not\leftarrow}{\longrightarrow} & \text{homogeneous space} \\ \quad\updownarrow & & & & \\ \text{transitive and micro-transitive action} & & & & \end{array} .$$

If we restrict our attention to Polish spaces that are both homogeneous and SLH, then the implications all reverse. Interestingly, the groups involved can then be chosen to be Polish as well.

2. Notation

A space is **Polish** if it is separable and its topology is generated by a complete metric. If A is a subset of a topological space, then $\overline{A}$ and $\operatorname{int} A$ denote its **closure** and **interior**, respectively. A subset A of a topological space X is **meager** in X if it is a countable union of nowhere dense sets.

All actions by topological groups on topological spaces considered in this paper are assumed to be separately continuous (see §1). So for every $g \in G$ the functions $x \mapsto gx$ and $x \mapsto g^{-1}x$ are continuous, hence they are homeomorphisms of X. Under mild conditions that allow the Baire Category Theorem to do its job, separately continuous actions are continuous (see e.g., Kechris [16, Theorem 9.14] for more information). All the notions introduced in the introduction for continuous actions, are adopted for separately continuous actions.

Let G be a topological group acting on a space X. The action is **transitive** if for all $a, b \in X$ there is an element $g \in G$ such that the homeomorphism $x \mapsto gx$ of X takes a onto b. Hence if G acts transitively on X, then X is homogeneous.

If $p \in X$, then $\gamma_p\colon G \to X$ is continuous, and surjective if and only if G acts transitively.

The **identity function** on a set X is denoted by 1_X.

For a space X we let $\mathcal{H}(X)$ denote the homeomorphism group of X. If $A \subseteq X$ then $\mathcal{H}(X|A) = \{h \in \mathcal{H}(X) : h(A) = A\}$. Observe that $\mathcal{H}(X|A)$ is a subgroup of $\mathcal{H}(X)$. The **natural action** of $\mathcal{H}(X)$ on X is defined by the formula

$$(h, x) \mapsto h(x) : \mathcal{H}(X) \times X \to X.$$

A topology on $\mathcal{H}(X)$ is called **admissible** if it makes $\mathcal{H}(X)$ a topological group and makes the natural action of $\mathcal{H}(X)$ on X continuous. If X is compact and metrizable, then the **compact-open** topology on $\mathcal{H}(X)$ is admissible and Polish. If ϱ is an admissible metric on X then the formulas

$$\hat{\varrho}(f, g) = \max\big\{\varrho\big(f(x), g(x)\big) : x \in X\big\}$$
$$\sigma(f, g) = \hat{\varrho}(f, g) + \hat{\varrho}(f^{-1}, g^{-1})$$

define metrics on $\mathcal{H}(X)$ that generate the compact-open topology. The metric $\hat{\varrho}$ is in general not complete but is right-invariant. The metric σ is complete but is in general neither left- nor right-invariant.

Theorem 2.1 (The Inductive Convergence Criterion): *Let (X, ϱ) be a compact metric space, and for each $n \in \mathbb{N}$, let $h_n \in \mathcal{H}(X)$. If for each n we have $\hat{\varrho}(h_{n+1}, h_n) < 2^{-n}$ and*

$$\hat{\varrho}(h_{n+1}, h_n) < 3^{-n} \cdot \min\big\{\min\{\varrho\big(h_i(x), h_i(y)\big) : \varrho(x, y) \geq 1/n\} : 1 \leq i \leq n\big\},$$

then $h = \lim_{n\to\infty} h_n$ is a homeomorphism of X.

This useful theorem is due to Fort [12] and was rediscovered by Anderson [3].

Let X be a compact space and let $(h_n)_n$ be a sequence in $\mathcal{H}(X)$. It is clear that for each $n \in \mathbb{N}$ the function $f_n = h_n \circ \cdots \circ h_1$ belongs to $\mathcal{H}(X)$. If $f = \lim_{n\to\infty} f_n$ exists then it will be denoted by

$$\lim_{n\to\infty} h_n \circ \cdots \circ h_1$$

and is called the **infinite left product** of the sequence $(h_n)_n$. It is easy to find conditions on the sequence $(h_n)_n$ which ensure that $\lim_{n\to\infty} h_n \circ \cdots \circ h_1$ exists and belongs to $\mathcal{H}(X)$. Indeed, first observe that if $f, g, h \in \mathcal{H}(X)$ then $\hat{\varrho}(h \circ f, g \circ f) = \hat{\varrho}(h, g)$. This implies that

$$\hat{\varrho}(h_{n+1} \circ h_n \circ \cdots \circ h_1, h_n \circ \cdots \circ h_1) = \hat{\varrho}(h_{n+1}, 1_X).$$

Hence if the sequence $\big(\hat{\varrho}(h_n, 1_X)\big)_n$ converges rapidly to 0, then the infinite left product of the sequence $(h_n)_n$ exists and is a homeomorphism of X by 2.1.

If f is a function, then $\operatorname{dom}(f)$ denotes its **domain**. Similarly, $\operatorname{range}(f)$ denotes its **range**. We let $\mathbb{I}$ denote the closed unit interval $[0,1]$.

A space is **hereditarily disconnected** if all of its (nonempty) connected subspaces are singletons. In addition, a space X is **totally disconnected** if all distinct points of X have disjoint clopen neighborhoods. It is clear that a totally disconnected space is hereditarily disconnected, but the converse is not true. See [9, §1.4] for more information on these notions.

3. The example

In this section we present the construction of our main example.

(A) The construction. Let $\triangle$ be the Cantor set in $\mathbb{I}$, and put $X = \triangle \times \mathbb{I}$. Let $\pi_1 \colon X \to \triangle$ and $\pi_2 \colon X \to \mathbb{I}$ be the projection maps. If $x \in X$, then x_1 abbreviates $\pi_1(x)$. Similarly for the second coordinate. On X we use the admissible metric

$$\varrho(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

If ϕ is a function such that $\operatorname{dom}(\phi) \subseteq \triangle$ and $\operatorname{range}(\phi) \subseteq (0,1)$, then

$$G(\phi) = \big\{\big(x, \phi(x)\big) : x \in \operatorname{dom}(\phi)\big\}$$

denotes its graph.

Put

$$\mathcal{H} = \big\{f \in \mathcal{H}(X) : f(\triangle \times \{0\}) = \triangle \times \{0\}\big\}.$$

Observe that $\mathcal{H}$ is a closed subgroup of $\mathcal{H}(X)$, and that for every $f \in \mathcal{H}$, $f(\triangle \times \{1\}) = \triangle \times \{1\}$. Hence f is 'order preserving' on every component of X.

Lemma 3.1: *If $x, y, a, b, p, q \in X$ are such that $0 < x_2 < a_2 < p_2 < 1$, $0 < y_2 < b_2 < q_2 < 1$, $x_1 = a_1 = p_1$ and $y_1 = b_1 = q_1$, then there is an element $h \in \mathcal{H}$ such that $h(x) = y$, $h(a) = b$ and $h(p) = q$. If in addition $\varrho(x,y) < \varepsilon$, $\varrho(a,b) < \varepsilon$ and $\varrho(p,q) < \varepsilon$, then we may choose h in such a way that $\hat{\varrho}(h, 1_X) < \varepsilon$. Finally, if C is a clopen subset of $\triangle$ that contains both $x_1 = a_1 = p_1$ and $y_1 = b_1 = q_1$, then we may choose h to be supported on $C \times \mathbb{I}$.*

Proof. The lemma is a triviality, so we will be brief. Let C be a clopen subset of $\triangle$ containing both x_1 and y_1, and let $f \in \mathcal{H}(\triangle)$ be supported on C and $f(x_1) = y_1$. Moreover, let $g \in \mathcal{H}(\mathbb{I})$ be the unique homeomorphism that sends $[0, x_2]$ linearly onto $[0, y_2]$, $[x_2, a_2]$ linearly onto $[y_2, b_2]$, $[a_2, p_2]$ linearly onto $[b_2, q_2]$ and $[p_2, 1]$ linearly onto $[q_2, 1]$. Let h be $f \times g$ on $C \times \mathbb{I}$, and the identity on its complement. Then h is clearly as required. ∎

Let Φ be the collection of all pairs of functions $\langle \phi, \phi' \rangle$ having the following properties:

(1) $\mathrm{dom}(\phi) = \mathrm{dom}(\phi')$ is a countable dense subset of $\triangle$,

(2) $\mathrm{range}(\phi) \cup \mathrm{range}(\phi') \subseteq (0, 1)$,

(3) $\phi \ll \phi'$, i.e., $\phi(d) < \phi'(d)$ for every $d \in \mathrm{dom}(\phi) = \mathrm{dom}(\phi')$,

(4) if $x, y \in X$, $x_1 = y_1$ and $x_2 < y_2$, then for every $\varepsilon > 0$ there exists $d \in \mathrm{dom}(\phi) = \mathrm{dom}(\phi')$ such that

$$|x_1 - d| = |y_1 - d| < \varepsilon, \quad |x_2 - \phi(d)| < \varepsilon, \quad |y_2 - \phi'(d)| < \varepsilon.$$

(Equivalently, $\varrho\big(x, (d, \phi(d))\big) < \varepsilon$ and $\varrho\big(y, (d, \phi'(d))\big) < \varepsilon$.)

Observe that $\mathrm{range}(\phi) = \mathrm{range}(\phi')$ is a countable dense subset of $(0, 1)$.

Lemma 3.2: $\Phi \neq \emptyset$.

Proof. Let $\{D_n : n \in \mathbb{N}\}$ be a pairwise disjoint collection of countable dense subsets of $\triangle$. In addition, let $\{(r_n, s_n) : n \in \mathbb{N}\}$ enumerate all pairs of rational numbers (r, s) in $\mathbb{I}$ such that $r < s$. Now define $\phi, \phi' \colon D = \bigcup_{n \in \mathbb{N}} D_n \to (0, 1)$ by

$$\phi(d) = r_n \Leftrightarrow d \in D_n, \quad \phi'(d) = s_n \Leftrightarrow d \in D_n \quad (n \in \mathbb{N}).$$

Then, clearly, $\langle \phi, \phi' \rangle \in \Phi$. ∎

If $\langle \phi, \phi' \rangle \in \Phi$, and $D = \mathrm{dom}(\phi)$ $\big(= \mathrm{dom}(\phi')\big)$, then

$$U\langle \phi, \phi' \rangle = \bigcup_{x \in \triangle \setminus D} \{x\} \times (0, 1) \cup \bigcup_{d \in D} \{d\} \times \big(\phi(d), \phi'(d)\big).$$

We will now show that 'all elements of Φ' are topologically equivalent.

Proposition 3.3: *Let $\langle \phi, \phi' \rangle, \langle \psi, \psi' \rangle \in \Phi$. Then for every compact set*

$$K \subseteq \triangle \setminus \big(\mathrm{dom}(\phi) \cup \mathrm{dom}(\psi)\big)$$

there is an arbitrarily close to the identity element $h \in \mathcal{H}$ such that

(1) *h restricts to the identity on $K \times \mathbb{I}$,*

(2) *$h\big(G(\phi)\big) = G(\psi)$, $h\big(G(\phi')\big) = G(\psi')$, and hence $h(U\langle\phi,\phi'\rangle) = U\langle\psi,\psi'\rangle$.*

Proof. Assume first that $K = \emptyset$, and let $D = \mathrm{dom}(\phi) = \mathrm{dom}(\phi')$ and $E = \mathrm{dom}(\psi) = \mathrm{dom}(\psi')$, respectively. In addition, let $\{d_n : n \in \mathbb{N}\}$ and $\{e_n : n \in \mathbb{N}\}$ be faithful enumerations of D and E, respectively. For every n, let $S_n = \{d_n\} \times [\phi(d_n), \phi'(d_n)]$ and $T_n = \{e_n\} \times [\psi(e_n), \psi'(e_n)]$. Finally, let $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ and $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$. Observe that both $\mathcal{S}$ and $\mathcal{T}$ are pairwise disjoint 'dense' collections of vertical segments.

Using the Inductive Convergence Criterion (Theorem 2.1), we construct a sequence $(h_n)_n$ in $\mathcal{H}$ such that its infinite left product h is a homeomorphism and the following conditions are satisfied:

(3) $h_n \circ \cdots \circ h_1(S_i) = h_{2i} \circ \cdots \circ h_1(S_i) \in \mathcal{T}$ for each i and $n \geq 2i$,

(4) $(h_n \circ \cdots \circ h_1)^{-1}(T_i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(T_i) \in \mathcal{S}$ for each i and each $n \geq 2i+1$.

Let $h_1 = 1_X$, assume $h_1, \ldots, h_{2i-1}$ have been defined for certain i, and put $\alpha = h_{2i-1} \circ \cdots \circ h_1$.

We claim first that if $\pi_1\big(\alpha(S_i)\big) \cap \pi_1\big(T_1 \cup \cdots \cup T_{i-1}\big) \neq \emptyset$ then $\alpha(S_i) \in \{T_1, \ldots, T_{i-1}\}$. For assume that for some $j \leq i-1$ we have $\pi_1\big(\alpha(S_i)\big) = \pi_1(T_j) = \{e_j\}$. Observe that by (4),

$$\alpha^{-1}(T_j) = (h_{2i-1} \circ \cdots \circ h_1)^{-1}(T_j) = (h_{2j+1} \circ \cdots \circ h_1)^{-1}(T_j) \in \mathcal{S},$$

and that $\alpha^{-1}(T_j)$ and S_i are both contained in the component $\{d_i\} \times \mathbb{I}$ of X. Hence $\alpha^{-1}(T_j) = S_i$.

If $\alpha(S_i) \in \{T_1, \ldots, T_{i-1}\}$, take $h_{2i} = 1_X$. So assume otherwise; let $\{a\} = \pi_1\big(\alpha(S_i)\big)$, and observe that by the above $\{a\} \times \mathbb{I}$ misses

$$B = (T_1 \cup \cdots \cup T_{i-1}) \cup \alpha(S_1 \cup \cdots \cup S_{i-1}).$$

So there is a clopen neighborhood U_{2i} of a in $\triangle$ such that $U_{2i} \times \mathbb{I}$ misses B. Since $\mathcal{T}$ is a 'dense' collection of vertical segments, there is an index $k \geq i$ such that T_k is contained in $U_{2i} \times \mathbb{I}$, and closely approximates $\alpha(S_i)$. We may choose the segment T_k so close to $\alpha(S_i)$ that there exists a 'small' homeomorphism $\xi \in \mathcal{H}$ such that $\xi\big(\alpha(S_i)\big) = T_k$ and ξ is supported on $U_{2i} \times \mathbb{I}$ (Lemma 3.1). Then $h_{2i} = \xi$ is clearly as required.

Put $\beta = h_{2i} \circ \cdots \circ h_1$. If $\pi_1(T_i) \cap \pi_1(\beta(S_1 \cup \cdots \cup S_i)) \neq \emptyset$ then $T_i \in \{\beta(S_1), \ldots, \beta(S_i)\}$. Assume that for some $j \leq i$ we have $\pi_1\big(\beta(S_j)\big) = \pi_1(T_i) = \{e_i\}$. Observe that by (3),

$$\beta(S_j) = h_{2i} \circ \cdots \circ h_1(S_j) = h_{2j} \circ \cdots \circ h_1(S_j) \in \mathcal{T},$$

and that both $\beta(S_j)$ and T_i are contained in $\{e_i\} \times \mathbb{I}$. Hence $\beta(S_j) = T_i$.

If $T_i \in \{\beta(S_1), \ldots, \beta(S_i)\}$, take $h_{2i+1} = 1_X$. So assume otherwise; by the above we have that $\{e_i\} \times \mathbb{I}$ misses

$$B' = (T_1 \cup \cdots \cup T_{i-1}) \cup \beta(S_1 \cup \cdots \cup S_i).$$

So there is a clopen neighborhood U_{2i+1} of e_i in $\triangle$ such that $U_{2i+1} \times \mathbb{I}$ misses B'. Since $\beta(\mathcal{S})$ is just as $\mathcal{S}$ a 'dense' collection of vertical segments in X, we may choose a segment $\beta(S_\ell)$ for some $\ell > i$ such that $\beta(S_\ell) \subseteq U_{2i+1} \times \mathbb{I}$. We may choose the segment $\beta(S_\ell)$ so close to T_i that there exists a 'small' homeomorphism $\eta \in \mathcal{H}$ such that $\eta\big(\beta(S_\ell)\big) = T_i$ and η is supported on $U_{2i+1} \times \mathbb{I}$ (Lemma 3.1). Then $h_{2i+1} = \eta$ is clearly as required.

If the approximations are chosen small enough, the conditions of the Inductive Convergence Criterion (Theorem 2.1) are satisfied so that $h = \lim_{i\to\infty} h_i \circ \cdots \circ h_1$ exists and is a homeomorphism of X. In addition, (3) and (4) easily imply that $h(\bigcup \mathcal{S}) = \bigcup \mathcal{T}$. Since $h \in \mathcal{H}$, we also get $h\big(G(\phi)\big) = G(\psi)$, $h\big(G(\phi')\big) = G(\psi')$ and hence $h(U\langle \phi\phi' \rangle) = U\langle \psi\psi' \rangle$.

If $K \neq \emptyset$, then it is clear that we may construct the sequence $(h_i)_i$ in such a way that every h_i is supported on $X \setminus (K \times \mathbb{I})$. Then h is supported on $X \setminus (K \times \mathbb{I})$ as well. ∎

Now fix $\langle \phi, \phi' \rangle \in \Phi$, let $D = \mathrm{dom}(\phi) = \mathrm{dom}(\phi')$, and consider $Z = U\langle \phi, \phi' \rangle$. We will prove that Z is the example we are looking for. It is clear that Z is a G_δ-subset of X and hence is a Polish space, [23, A.6.3]. Observe that all components of Z are homeomorphic to $(0,1) \approx \mathbb{R}$ but that they are irregularly placed in Z.

(B) Homogeneity properties of Z. We will now prove that Z is homogeneous. Our geometric intuition says that in X, the space Z has 'two types of points': points of which the component of Z containing it has length 1, and points of which the component of Z containing it has length less than 1. We will prove that these points are topologically equivalent in Z, thereby demonstrating that Z is homogeneous. It is clear that for that we need homeomorphisms of Z that cannot be extended to homeomorphisms of X.

Lemma 3.4: *(1) If $h \in \mathcal{H}$ and $\langle \bar{\varphi}, \bar{\varphi}' \rangle \in \Phi$, then $h(U\langle \bar{\varphi}, \bar{\varphi}' \rangle)$ is of the form $U\langle \psi, \psi' \rangle$ for some $\langle \psi, \psi' \rangle \in \Phi$.*

(2) If $x, y \in Z$ are such that $x_1, y_1 \notin D$, then there is an element $h \in \mathcal{H}$ such that $h(x) = y$ and $h(Z) = Z$.

(3) If $x, y \in Z$ are such that $x_1, y_1 \in D$, then there is an element $h \in \mathcal{H}$ such that $h(x) = y$ and $h(Z) = Z$.

Proof. Since h permutes the components of X, 1 is trivial.

For (2), first observe that by Lemma 3.1 there exists an element $h_1 \in \mathcal{H}$ such that $h_1(x) = y$. By (1), pick $\langle \psi, \psi' \rangle \in \Phi$ such that $h_1\bigl(Z\bigr) = U\langle \psi, \psi' \rangle$. Clearly, $y_1 \notin \operatorname{dom}(\phi) \cup \operatorname{dom}(\psi)$. By Proposition 3.3 there is an element $h_2 \in \mathcal{H}$ such that $h_2\bigl(U\langle \psi, \psi' \rangle\bigr) = Z$ while moreover $h_2(y) = y$. It is now clear that $h = h_2 \circ h_1$ is as required.

For (3), let $a = \bigl(x_1, \phi(x_1)\bigr)$, $p = \bigl(x_1, \phi'(x_1)\bigr)$, $b = \bigl(y_1, \phi(y_1)\bigr)$ and $q = \bigl(y_1, \phi'(y_1)\bigr)$, respectively. By Lemma 3.1 there is an element $h_1 \in \mathcal{H}$ such that $h_1(a) = b$, $h_1(x) = y$ and $h_1(p) = q$. Let $\varphi = \phi \restriction \operatorname{dom}(\phi) \setminus \{x_1\}$ and $\varphi' = \phi' \restriction \operatorname{dom}(\phi') \setminus \{x_1\}$. Then $\langle \varphi, \varphi' \rangle \in \Phi$. In addition, let $\mu = \phi \restriction \operatorname{dom}(\phi) \setminus \{y_1\}$ and $\mu' = \phi' \restriction \operatorname{dom}(\phi') \setminus \{y_1\}$. Similarly, $\langle \mu, \mu' \rangle \in \Phi$. By (1), pick $\langle \psi, \psi' \rangle \in \Phi$ such that $U\langle \psi, \psi' \rangle = h_1\bigl(U\langle \varphi, \varphi' \rangle\bigr)$. By Proposition 3.3, there is an element $h_2 \in \mathcal{H}$ such that h_2 restricts to the identity on $\{y_1\} \times \mathbb{I}$ and maps $U\langle \psi, \psi' \rangle$ onto $U\langle \mu, \mu' \rangle$. It is clear that $h = h_2 \circ h_1$ is as required. ∎

We will now show that all components in Z are topologically the same.

Lemma 3.5: *If $x \in D$ and $y \in \triangle \setminus D$, then there is an element $h \in \mathcal{H}(Z)$ such that $h\bigl(\{x\} \times \bigl(\phi(x), \phi'(x)\bigr)\bigr) = \{y\} \times (0, 1)$.*

Proof. Let $t = \phi(x)$ and $t' = \phi'(x)$. Observe that $0 < t < t' < 1$. We let $f \colon \mathbb{I} \to \mathbb{I}$ be the continuous surjection that maps $[0, t]$ onto 0, $[t', 1]$ onto 1 and $[t, t']$ linearly onto $[0, 1]$, i.e.,

$$f(s) = \begin{cases} 0 & (0 \le s \le t), \\ \frac{s}{t'-t} - \frac{t}{t'-t} & (t \le s \le t'), \\ 1 & (t' \le s \le 1). \end{cases}$$

Then f is approximable by homeomorphisms of $\mathbb{I}$. Indeed, let $(t_n)_n$ be a sequence in $(0,t)$ such that $t_n \nearrow t$, and $(t'_n)_n$ a sequence in $(t',1)$ such that $t'_n \searrow t'$. For every n, let f_n be the unique homeomorphism of $\mathbb{I}$ mapping $[0,t_n]$ linearly onto $[0,1/n]$, $[t_n,t'_n]$ linearly onto $[1/n,1-1/n]$ and $[t'_n,1]$ linearly onto $[1-1/n,1]$. Then, clearly, $f_n \to f$. Write $\triangle \setminus \{x\}$ as the union of a pairwise disjoint sequence of nonempty clopen sets $(C_n)_n$ and define $q\colon X \to X$ as follows:

$$q(z,s) = \begin{cases} \big(z, f_n(s)\big) & (z \in C_n), \\ \big(z, f(s)\big) & (z = x). \end{cases}$$

q is a continuous surjection and it maps the complement of $\{x\} \times ([0,t] \cup [t',1])$ homeomorphically onto the complement of $\{(x,0),(x,1)\}$. Define

$$\psi, \psi' \colon D \setminus \{x\} \to (0,1)$$

so that for every $z \in D \setminus \{x\}$ we have that $q\big(z,\phi(z)\big) = \big(z,\psi(z)\big)$ and $q\big(z,\phi'(z)\big) = \big(z,\psi'(z)\big)$. Observe that $\langle \psi,\psi'\rangle \in \Phi$ and $q(Z) = U\langle\psi,\psi'\rangle$. By Proposition 3.3 there is an element $h \in \mathcal{H}$ such that $h(U\langle\psi,\psi'\rangle) = U\langle\phi,\phi'\rangle = Z$. Then $\xi = h \circ q$ maps the complement of $\{x\} \times ([0,t] \cup [t',1])$ homeomorphically onto the complement of $\{h(x,0), h(x,1)\}$, hence $\{x\} \times (t,t')$ homeomorphically onto $\{h(x,0)\} \times (0,1)$ and

$$\xi(Z) = h\big(q(Z)\big) = h(U\langle\psi,\psi'\rangle) = U\langle\phi,\phi'\rangle = Z,$$

as required. ∎

By Lemma 3.4, this evidently yields the following

Theorem 3.6: *Z is a homogenous, Polish space.*

4. Actions on Z

Since Z is a homogeneous topological space by Theorem 3.6, it is natural to ask whether there are actions of topological groups on Z that are more interesting than the natural action of the discrete group $\mathcal{H}(Z)$ on Z.

(A) An action on $\mathbf{Z}$ from its compactification $\mathbf{X}$. We first note that X gives us a natural action of a topological group on Z.

THEOREM 4.1: *The natural action of the separable and metrizable group* $\mathcal{G} = \{h \in \mathcal{H} : h(Z) = Z\}$ *on* Z *has exactly two orbits, one of which is Polish and the other one of which is meager.*

Proof. This is trivial. Simply observe that $\mathcal{G}$ is a separable metrizable topological group, being a subgroup of the Polish group $\mathcal{H}(X)$ (see §2) and that by Lemma 3.4 it follows that $(\triangle \setminus D) \times (0,1)$ (which is Polish) and $\bigcup_{d\in D}\{d\} \times \big(\phi(d), \phi'(d)\big)$ (which is meager) are the $\mathcal{G}$-orbits of Z. ∎

It is easy to prove from Proposition 3.3 that $\mathcal{G}$ is not closed in $\mathcal{H}(X)$, hence $\mathcal{G}$ is not Polish. This 'defect' will be cured in §6 where we will show that $\mathcal{G}$ is Polishable, i.e., it has a stronger Polish group topology. This means that Z admits an 'interesting' action of a Polish group having exactly two orbits. Since any action of a Polish group on Z will always have one meager orbit by 4.5 below, this is in a sense 'best possible'.

(B) GROUPS THAT DO NOT ACT ON **Z**. We will now prove that several topological groups do not admit interesting actions on Z.

THEOREM 4.2: *If* G *is a topological group that acts on* Z *by a separately continuous action, then there are an element* $z \in Z$ *and a neighborhood* U *of the neutral element* e *of* G *such that* Uz *is meager in* Z.

Proof. It will be convenient to denote $\mathcal{U}_e$ the collection of all open neighborhoods of e in G. Striving for a contradiction, assume that for every $z \in Z$ and every $U \in \mathcal{U}_e$ we have that Uz is not meager in Z.

Take $x \in \triangle \setminus D$ and consider the point $p = (x, 1/2)$. Let V be a closed neighborhood of p in Z of diameter less than 1. Let $W = \gamma_p^{-1}(V)$ and let U be a symmetric open neighborhood of e such that $U^2 \subseteq W$. Then $\overline{Up}$ is not nowhere dense by our assumption, i.e., it has nonempty interior.

We claim that if $q \in \overline{Up}$, then $Uq \subseteq V$. This is easy. Indeed, if $h \in U$, then

$$hq \in h\big(\overline{Up}\big) = \overline{hUp} \subseteq \overline{U^2p} \subseteq \overline{Wp} \subseteq \overline{V} = V,$$

as required.

There is a point $d \in D$ such that $\{d\} \times \big(\phi(d), \phi'(d)\big) \subseteq \operatorname{int} \overline{Up}$. Pick an arbitrary element $t \in \big(\phi(d), \phi'(d)\big)$ and consider the point $q_0 = (d, t)$. By assumption, Uq_0 is not meager in Z. Since $\{z \in Z : z_1 \in D\}$ is meager in Z, this implies that there are elements $x' \in \triangle \setminus D$ and $t' \in (0, 1)$ such that

$(x',t') \in Uq_0$. Pick $\eta \in U$ such that $\eta q_0 = (x',t')$. Observe that

$$\eta\big(\{d\} \times \big(\phi(d), \phi'(d)\big)\big) = \{x'\} \times (0,1) \tag{1}$$

since η permutes the components of Z. Since $\{d\} \times \big(\phi(d), \phi'(d)\big) \subseteq \overline{Up}$, by the above we get

$$\eta\big(\{d\} \times \big(\phi(d), \phi'(d)\big)\big) \subseteq \eta(\overline{Up}) \subseteq V. \tag{2}$$

But (1) and (2) contradict, since the diameter of $\{x'\} \times (0,1)$ is 1 and the diameter of V is less than 1. ∎

Corollary 4.3: *No topological group G acts both transitively and micro-transitively on Z by a separately continuous action, hence Z is a homogeneous Polish space that is not a coset space.*

Corollary 4.4: *The homeomorphism group $\mathcal{H}(Z)$ cannot be given an admissible group topology making the natural action on Z micro-transitive.*

This answers Question 3 in Ancel [2] in the negative.

Theorem 4.5: *If G is an $\aleph_0$-bounded topological group that acts on Z by a separately continuous action, then there is an element $z \in Z$ such that its orbit Gz is meager in Z.*

Proof. It will again be convenient to let $\mathcal{U}_e$ denote the collection of all open neighborhoods of e in G. Let $z \in Z$ and $U \in \mathcal{U}_e$ be as in 4.2. Since G is $\aleph_0$-bounded, there is a countable set $F \subseteq G$ such that $FU = G$. Observe that

$$\bigcup\{fUz : f \in F\} = FUz = Gz.$$

If $f \in F$, then function $p \mapsto fp$ is a homeomorphism of Z, so fUz is meager since Uz is. Hence Gz is meager as well, F being countable. ∎

Since Z is second category in itself, this yields:

Corollary 4.6: *Z is a homogeneous Polish space on which no $\aleph_0$-bounded topological group acts transitively by a separately continuous action.*

Since every separable metrizable topological group is clearly $\aleph_0$-bounded, this answers Question 4.2 in van Mill [26] in the negative.

Remark 4.7: There are models of set theory in which no Polish space is the union of fewer than $\mathfrak{c}$ (= continuum) many nowhere dense subsets. This is true, for example, under Martin's Axiom, see Kunen [18] for more details. The proof of Theorem 4.5 shows that in such a model a topological group G acting on Z by a separately continuous action has very bad covering properties: it has a neighborhood U of its neutral element such that fewer than $\mathfrak{c}$ many translates of U do not cover G. Observe that there are nondiscrete metrizable groups G that act transitively on Z. Let G for example be the product of $\mathcal{H}(Z)$ with the discrete topology and any infinite compact group. So such a group may locally be very nice, but, as we showed, its global covering properties are bad.

Remark 4.8: Let G be an $\aleph_0$-bounded topological group acting continuously on Z. In Theorem 4.5 we proved that there is an element $z \in Z$ such that its orbit Gz is meager in Z. As Theorem 4.1 demonstrates, it is not true that every orbit is meager.

Remark 4.9: As was observed by Mati Rubin, the proof of Theorem 4.2 can be applied in some other cases as well.

Let G be an abstract group endowed with an $\aleph_0$-bounded topology. By this we mean again that the topology on G has the property that for every neighborhood U of the identity e of G there is a countable subset F in G such that $FU = G$. One has to be careful, since translations are not assumed to be continuous, hence the open set U may be moved to something terrible by a translation of G. Since $U^{-1}F^{-1} = G$ but U^{-1} need not be open, it is also not clear that there exists a countable subset F' of X such that $UF' = G$. Even if the topology on G is Lindelöf, it is not automatic that it is $\aleph_0$-bounded.

Assume that there is a continuous action $G \times X \to X$. We claim that again there are a neighborhood U of the neutral element e of G and a point $z \in Z$ such that Uz is meager in Z. Again take $x \in \triangle \setminus D$ and consider the point $p = (x, 1/2)$. Let W be a closed neighborhood of p in Z of diameter less than 1. Let U be a neighborhood of e in G and V a neighborhood of p in Z such that $UV \subseteq W$ (here we use that the action is continuous). There is a point $d \in D$ such that $\{d\} \times \big(\phi(d), \phi'(d)\big) \subseteq V$. Pick an arbitrary element $t \in \big(\phi(d), \phi'(d)\big)$ and consider the point $q_0 = (d, t)$. By assumption, Uq_0 is not meager in Z. Since $\{z \in Z : z_1 \in D\}$ is meager in Z, this implies that there are elements $x' \in \triangle \setminus D$ and $t' \in (0, 1)$ such that $(x', t') \in Uq_0$. Pick $\eta \in U$ such that $\eta q_0 = (x', t')$. Observe that $\eta\big(\{d\} \times \big(\phi(d), \phi'(d)\big)\big) = \{x'\} \times (0, 1)$ since η permutes the components of Z. Since $\{d\} \times \big(\phi(d), \phi'(d)\big) \subseteq V$ and $\eta \in U$, by the above we get $\eta(\{d\} \times (\phi(d), \phi'(d))) \subseteq W$. This is the same contradiction as in the proof of Theorem 4.2.

The proof of Theorem 4.5 can now be repeated to conclude that the orbit Gz is meager in Z.

5. More applications

In this section we will present some more applications of the space Z.

(A) Equivariant compactifications. Let G be a topological group acting on a space X by a continuous action. By a result of de Vries [39], X admits a compactification γX such that the action of G on X can be extended to an action of G on γX if and only if the right-uniformly continuous functions on X separate the points and the closed subsets of X (such a compactification is called **equivariant**). Here, a continuous real-valued function f on X is **right-uniformly continuous** if for every $\varepsilon > 0$ there exists a neighborhood U of the neutral element of G such that for all $g \in U$ and all $x \in X$ we have $|f(gx) - f(x)| < \varepsilon$. Observe that for an equivariant compactification γX we have that for every $g \in G$ the homeomorphism $x \mapsto gx$ of X can be extended to the homeomorphism $y \mapsto gy$ of γX. For locally compact G acting on X an equivariant compactification of X exists, see de Vries [39]. Similarly if the action is transitive, the group is $\aleph_0$-bounded and the space is of the second category. See Uspenskiĭ [38] for details. As was shown by Megrelishvili [20], not all actions can be 'equivariantly compactified', even if the group and the space under consideration are both Polish.

Theorem 5.1: *Let X be a second category, separable metrizable space. Then the following statements are equivalent:*

- (a) *There is a metrizable compactification γX of X such that for all $x, y \in X$ there is a homeomorphism $f \colon \gamma X \to \gamma X$ such that $f(x) = y$ and $f(X) = X$.*
- (b) *X admits a transitive continuous action by a separable metrizable topological group.*
- (c) *X admits a transitive continuous action by an $\aleph_0$-bounded topological group.*

Proof. For (a) ⇒ (b), simply observe that the group $G = \mathcal{H}(\gamma X|X)$ acts transitively and that the compact-open topology on $\mathcal{H}(\gamma X)$ is admissible and separable metrizable, see §2. Since (b) ⇒ (c) is trivial, all we need to observe is that (c) ⇒ (a) is a consequence of Uspenskiĭ [38, Proposition 8] and Megrelishvili [21, Theorem 2.13] (see also [40]). ∎

So by Theorems 4.5 and 5.1, Z does not have a metrizable compactification in which it is 'homogeneously' imbedded. Interestingly, $X = \triangle \times \mathbb{I}$ is a metrizable compactification of Z in which Z has only two types of points (Lemma 3.4). So at first glance, one would guess that by making only one extra homeomorphism of Z extendable (such a homeomorphism is provided by Lemma 3.5), one would be able to imbed Z 'homogeneously' in some metrizable compactification. But, as we demonstrated, this is impossible. That there is a homogeneous separable metrizable space not having a metrizable compactification in which it is 'homogeneously' imbedded, answers Question 3.2 in [24] in the negative. In addition, Z also answers Question 2.3 in [24] in the negative.

(B) Products. An interesting and unexpected consequence of the Effros Theorem [8] is that every homogeneous locally compact separable metrizable space is a product of two spaces, one of which is connected and the other of which is zero-dimensional. This result is for the compact case due to Mislove and Rogers [29, 30] and for the general case to Aarts and Oversteegen [1]. It was asked by Aarts and Oversteegen whether every homogeneous Polish space is the product of one of its quasi-components and a totally disconnected space. To put this question into perspective, observe that there are homogeneous, totally disconnected, 1-dimensional Polish spaces. An example of such a space is the so-called **complete Erdős space** E_c, that is, the set of vectors in Hilbert space ℓ^2 all coordinates of which are irrational. See Kawamura, Oversteegen and Tymchatyn [15] for more information. So the product $E_c \times \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the 1-sphere, is a homogeneous Polish space of which the components form an upper semi-continuous decomposition whose quotient space is not zero-dimensional (but is totally disconnected). This shows that for Polish spaces one should aim at totally disconnected (or hereditarily disconnected) instead of zero-dimensional factors. This question was answered in the negative in van Mill [27] by using highly nontrivial results of Bing and Jones [6] and Lewis [19]: the 'complete Erdős space of pseudo-arcs' is a counterexample. We will now show that Z is a much better (and simpler) counterexample. Observe that the components of Z coincide with its quasi-components. Hence the following result proves our claim.

THEOREM 5.2: *If A is hereditarily disconnected and B is connected, then Z and $A \times B$ are not homeomorphic.*

Proof. The components of $A \times B$ are precisely the sets in the collection $\mathcal{S} = \big\{\{a\} \times B : a \in A\big\}$, and the components of Z are precisely the sets in the collection

$$\mathcal{T} = \big\{\{x\} \times (0,1) : x \in \triangle \setminus D\big\} \cup \big\{\{d\} \times \big(\phi(d), \phi'(d)\big) : d \in D\big\}.$$

This clearly implies that $Z \not\approx A \times B$ since in the collection $\mathcal{T}$ there are sequences converging to a singleton, while in the collection $\mathcal{S}$ this is not the case. ∎

(C) QUASI-COMPONENTS VERSUS COMPONENTS. As we observed above, every locally compact separable metrizable homogeneous space is is a product of two spaces, one of which is connected and the other of which is zero-dimensional. This implies that in a locally compact homogeneous separable metrizable space the component of a point coincides with the quasi-component of that point. Aarts and Oversteegen [1, p. 4] constructed a homogeneous separable metrizable space in which components and quasi-components do not coincide. But their example is not Polish.

Question 5.3: Let X be a homogeneous Polish space. Do components and quasi-components coincide in X?

6. Proof of 1.3 and more

A topological group is **Polishable** if it admits a stronger Polish topology that is compatible with its group structure. An obvious necessary condition for the Polishability of G is that G is a Borel group. But this condition is not sufficient, see Becker and Kechris [4, p. 12] for details. If such a Polish topology exists, then it is unique; see Kechris [16, Theorem 9.10]. The reader can find more information on Polishable groups for example in Solecki [34].

We will first formulate a rather special condition that guarantees the Polishability of certain subgroups of Polish groups. This will then be applied in the framework of homeomorphism groups. Related results were obtained by Solecki [34, Theorem 2.1].

(A) A criterion for Polishability. Let G be a topological group with admissible complete metric ϱ. In addition, let H be a subgroup of G containing a countable collection $\mathcal{B}$ of subgroups such that

(A) every $B \in \mathcal{B}$ is closed in H,

(B) for every $B \in \mathcal{B}$ there are countable subsets $A_B, A'_B \subseteq H$ such that
$$H = \bigcap_{B \in \mathcal{B}} A_B \overline{B} \cap \bigcap_{B \in \mathcal{B}} \overline{B} A'_B$$
(here closure means closure in G).

It is clear that H is an $F_{\sigma\delta}$-subset of G and hence is a Borel group. We will prove that it is Polishable.

Let A be the (countable) subgroup of H generated by $\bigcup_{B \in \mathcal{B}} A_B \cup A'_B$. Put
$$\mathcal{F} = \{aBa^{-1} : a \in A, B \in \mathcal{B}\},$$
and observe that $\mathcal{B} \subseteq \mathcal{F}$.

Lemma 6.1:

(1) *If $y \in H$ and $F \in \mathcal{F}$, then $yFy^{-1} \in \mathcal{F}$.*

(2) *If $F \in \mathcal{F}$, then $AF = H = FA$.*

(3) $H = \bigcap_{F \in \mathcal{F}} A\overline{F} \cap \bigcap_{F \in \mathcal{F}} \overline{F}A = \bigcap_{F \in \mathcal{F}} H\overline{F} \cap \bigcap_{F \in \mathcal{F}} \overline{F}H.$

Proof. For (1), write F in the form aBa^{-1}, where $a \in A$ and $B \in \mathcal{B}$. Since $H = A_B B$, there are $v \in A_B \subseteq A$ and $b \in B$ such that $ya = vb$. But then, clearly,
$$yFy^{-1} = yaBa^{-1}y^{-1} = vbBb^{-1}v^{-1} = vBv^{-1} \in \mathcal{F},$$
For (2), notice that if $F \in \mathcal{F}$, say $F = aBa^{-1}$ for certain $a \in A$ and $B \in \mathcal{B}$, then
$$AF = AaBa^{-1} = ABa^{-1} \supseteq A_B Ba^{-1} = Ha^{-1} = H;$$
similarly, $FA = H$.

For (3), notice that it is clear that
$$\bigcap_{B \in \mathcal{B}} A\overline{B} \cap \bigcap_{B \in \mathcal{B}} \overline{B}A \subseteq \bigcap_{B \in \mathcal{B}} H\overline{B} \cap \bigcap_{B \in \mathcal{B}} \overline{B}H.$$
Pick an arbitrary $x \in \bigcap_{B \in \mathcal{B}} H\overline{B} \cap \bigcap_{B \in \mathcal{B}} \overline{B}H$ and fix $B \in \mathcal{B}$. There is an element $y \in H$ such that $x \in y\overline{B}$. Since $y \in A_B B$, we can pick $a \in A_B$ and $b \in B$ such that $y = ab$. But then
$$x \in y\overline{B} = ab\overline{B} = a\overline{b\overline{B}} = a\overline{B} \subseteq A_B\overline{B}$$

since B is a subgroup of G. Hence $x \in \bigcap_{B\in\mathfrak{B}} A_B\overline{B}$ and, similarly, $x \in \bigcap_{B\in\mathfrak{B}} \overline{B}A'_B$. So we conclude that $x \in H$. Hence by (2) we obtain,

$$H \subseteq \bigcap_{F\in\mathfrak{F}} A\overline{F} \cap \bigcap_{F\in\mathfrak{F}} \overline{F}A \subseteq \bigcap_{F\in\mathfrak{F}} H\overline{F} \cap \bigcap_{F\in\mathfrak{F}} \overline{F}H \subseteq \bigcap_{B\in\mathfrak{B}} H\overline{B} \cap \bigcap_{B\in\mathfrak{B}} \overline{B}H \subseteq H,$$

as required. ∎

For every $F \in \mathfrak{F}$ let $H/F = \{xF : x \in H\}$ respectively $H\backslash F = \{Fx : x \in H\}$ be the collections of left- and right-cosets of F in H. Observe that H/F and $H\backslash F$ are countable by Lemma 6.1(2). We endow both H/F and $H\backslash F$ by the discrete topology and denote the natural functions $G \to H/F$ and $G \to G\backslash F$ by λ_F respectively ρ_F.

Define a function $\phi\colon H \to \prod_{F\in\mathfrak{F}}(H/F\times H\backslash F)$ by $\phi(x) = \big(\lambda_F(x), \rho_F(x)\big)_{F\in\mathfrak{F}}$.

We endow $\Xi = \prod_{F\in\mathfrak{F}}(H/F \times H\backslash F)$ by the standard product topology. Observe that Ξ is Polish being a product of countably many countable discrete spaces.

Proposition 6.2: *The graph $G(\phi) = \big\{\big(x, \phi(x)\big) : x \in H\big\}$ of ϕ is a closed subset of $G\times\Xi$ and its subspace topology is compatible with the group structure on H.*

Proof. Let $\big(x_n, \phi(x_n)\big)_n$ be a sequence in $G(\phi)$ converging to an element $(p, q) \in G \times \Xi$.

Claim 1: For every $F \in \mathfrak{F}$ there exists N such that for all $n, m \geq N$ we have $x_n^{-1}x_m \in F$ and $x_nx_m^{-1} \in F$.

Proof. Since $H/F \times H\backslash F$ is discrete and $\phi(x_n) \to q$, there exists N such that for all $n, m \geq N$ we have $(x_nF, Fx_n) = \phi(x_n)_F = q_F = \phi(x_m)_F = (x_mF, Fx_m)$, i.e., $x_n^{-1}x_m \in F$ and $x_nx_m^{-1} \in F$. ∎

Claim 2: $p \in H$ and $\phi(x_n) \to \phi(p)$ (hence $\phi(p) = q$).

Proof. Pick an arbitrary $F \in \mathfrak{F}$ and let N be as in Claim 1 for F. Fix $M \geq N$ for a moment. Observe that for all $m \geq N$ we have $x_m \in x_MF$. Since $x_m \to p$ and $x_M\overline{F}$ is closed in G and contains x_MF, we get $p \in x_M\overline{F} \subseteq H\overline{F}$. Similarly, $p \in \overline{F}x_M \subseteq \overline{F}H$. Hence, $p \in H$ by Lemma 6.1(3).

Observe that from the fact that $p \in H$ and the above calculation it follows that $p \in x_MF \cap Fx_M$ for all $M \geq N$. As a consequence, for such M,

$\phi(p)_F = (pF, Fp) = (x_M F, F x_M) = \phi(x_M)_F$. So we conclude that $\phi(x_n)_F \to \phi(p)_F$ for every $F \in \mathcal{F}$ and hence that $\phi(x_n) \to \phi(p)$. ∎

We will now prove that the topology on $G(\phi)$ is compatible with the group structure on H. To this end, let $\big(x_n, \phi(x_n)\big)_n$ and $\big(y_n, \phi(y_n)\big)_n$ be sequences in $G(\phi)$ converging to $\big(x, \phi(x)\big)$ and $\big(y, \phi(y)\big)$ respectively in $G(\phi)$.

Claim 3: $\big(x_n y_n^{-1}, \phi(x_n y_n^{-1})\big) \to \big(xy^{-1}, \phi(xy^{-1})\big)$.

Proof. We need to prove that $\phi(x_n y_n^{-1}) \to \phi(xy^{-1})$ or, equivalently, that for every $F \in \mathcal{F}$ there exists N such that for all $n \geq N$ we have $\phi(x_n y_n^{-1})_F = \phi(xy^{-1})_F$. So fix $F \in \mathcal{F}$. By Claim 1, we may pick M_1 such that $y_n y^{-1} \in F$ for every $n \geq M_1$. Since $y^{-1}Fy \in \mathcal{F}$ (Lemma 6.1(1)), again by Claim 1 we may pick M_2 such that $x_n^{-1}x \in y^{-1}Fy$ for every $n \geq M_2$. This means that for $n \geq \max(M_1, M_2)$ we have

$$y_n x_n^{-1} x y^{-1} \in y_n y^{-1} F y y^{-1} = y_n y^{-1} F = F,$$

i.e., $xy^{-1} \in x_n y_n^{-1} F$. So $\phi(x_n y_n^{-1})_F$ and $\phi(xy^{-1})_F$ have the same first coordinates. Similarly, for the second coordinates. ∎

This completes the proof. ∎

So $G(\phi)$ is Polish being a closed subspace of a Polish space. This means that H is indeed Polishable.

Remark 6.3: From the proof of Proposition 6.2 it is possible to obtain an explicit metric generating the Polish group topology on H. Indeed, let ϱ be a complete admissible metric on G. Enumerate $\mathcal{F}$ as $\{F_i : i \in \mathbb{N}\}$. For every i define $\varphi_i \colon H \to \{0,1\}$ by the formula

$$\varphi_i(x) = \begin{cases} 0 & (x \in F_i), \\ 1 & (x \notin F_i). \end{cases}$$

Define $\sigma \colon H \times H \to \mathbb{R}$ by

$$\sigma(x,y) = \varrho(x,y) + \sum_{i=1}^{\infty} 2^{-i} \varphi_i(x^{-1}y) + \sum_{i=1}^{\infty} 2^{-i} \varphi_i(xy^{-1}).$$

Then σ is a complete metric generating the Polish group topology on H. It is possible to prove this directly, but the essential ingredients of that proof are identical to the ones above.

(B) Application to homeomorphism groups. Let X be homogeneous, Polish and SLH. Using the above criterion, we will prove that there is a Polish group admitting a transitive action on X.

By van Mill [26, Corollary 3.2], there are a metrizable compactification γX of X and a countable collection $\mathcal{W}$ open subsets of γX and a countable subgroup F of $\mathcal{H}(\gamma X|X)$ such that

(1) $\gamma X \in \mathcal{W}$ and $\mathcal{W} \restriction X$ is a base of X,
(2) $\mathcal{W}$ is invariant under F,
(3) for all $W \in \mathcal{W}$, $x, y \in W \cap X$ and $\varepsilon > 0$, there exist $A, B \in \mathcal{W}$ and $f \in F$ with
  (i) $\overline{A \cup B} \subseteq W$, $x \in A$, $y \in B$,
  (ii) $\operatorname{diam} A < \varepsilon$, $\operatorname{diam} B < \varepsilon$,
  (iii) f is supported on W and $f(A) = B$.

Since X is Polish, there is a countable family of compacta $\mathcal{A}$ with $\bigcup \mathcal{A} = \gamma X \setminus X$. Put $\mathcal{B} = \{f(A) : f \in F\}$ and observe that $\bigcup \mathcal{B} = \gamma X \setminus X$. Let

$$H = \{f \in \mathcal{H}(\gamma X) : (\forall\, B \in \mathcal{B})(\exists\, g, h \in F)(f \restriction B = g \restriction B \;\&\; f^{-1} \restriction B = h \restriction B)\}.$$

Observe that if $f \in H$ and $B \in \mathcal{B}$, then $f(B), f^{-1}(B) \in \mathcal{B}$. In addition, $F \subseteq H$.

Lemma 6.4: *H is a subgroup of $\mathcal{H}(\gamma X|X)$ and for all $x, y \in X$ there exists $f \in H$ such that $f(x) = y$.*

Proof. That H is a subgroup of $\mathcal{H}(\gamma X|X)$ is trivial since F is a subgroup of $\mathcal{H}(\gamma X|X)$ and every $f \in F$ permutes $\mathcal{B}$.

Now take arbitrary $x, y \in X$. By [26, Lemma 3.4] there are a sequence $(g_n)_n$ in F and a decreasing neighborhood base $(A_n)_n$ of x in γX such that

(1) The infinite left-product $f = \lim_{n\to\infty} g_n \circ \cdots \circ g_1$ is a homeomorphism of γX such that $f(x) = y$,
(2) $f(X) = X$,
(3) if $p \notin A_n$, then $f(p) = g_n \circ \cdots \circ g_1(p)$.

We claim that $f \in H$. To prove this, take an arbitrary $B \in \mathcal{B}$. There exists i such that $B \cap A_i = \emptyset$ since $B \subseteq \gamma X \setminus X$ and $x \in X$. Hence by (3), $f \restriction B = g \restriction B$ for some $g \in F$. There also exists j such that $f^{-1}(B) \cap A_j = \emptyset$. Hence, again by (3), $f \restriction f^{-1}(B) = h \restriction f^{-1}(B)$ for some $h \in F$. But this means that $f^{-1} \restriction B = h^{-1} \restriction B$. ∎

It is not difficult to prove that H is an $F_{\sigma\delta}$-subgroup of $\mathcal{H}(\gamma X)$. In addition, it acts transitively on X by 6.4. Unfortunately, it need not be Polish. To see this, let $X = \mathbb{S}^2 \setminus D$, where D is any countable dense subset of the 2-sphere $\mathbb{S}^2$. It is not hard to show that we can arrange H to be the space of all homeomorphisms of $\mathbb{S}^2$ that map D onto D. But that space was shown by Dijkstra and van Mill [7] to be homeomorphic to Erdős space, [10], i.e., the first category 'rational' Hilbert space.

Our 'plan' is to prove that H is Polishable. This suffices since H with its new Polish topology acts transitively on X as well. We will show that H satisfies the special conditions that were considered earlier in this section.

For every $B \in \mathcal{B}$ we let

$$H_B = \{h \in H : h{\restriction}B = 1_B\},$$

where 1_B denotes the identity on B. It is clear that H_B is a closed subgroup of H.

Lemma 6.5: $H = \bigcap_{B\in\mathcal{B}} F\overline{H}_B \cap \bigcap_{B\in\mathcal{B}} \overline{H}_B F$.

Proof. Indeed, let $h \in H$ and $B \in \mathcal{B}$ be arbitrary. There is by assumption an element $f \in F$ such that $h{\restriction}B = f{\restriction}B$. Hence $h \in fH_B \subseteq F\overline{H}_B$, i.e., $H \subseteq \bigcap_{B\in\mathcal{B}} F\overline{H}_B$. Similarly, $H \subseteq \bigcap_{B\in\mathcal{B}} \overline{H}_B F$.

Conversely, let $h \in \bigcap_{B\in\mathcal{B}} F\overline{H}_B$ be arbitrary. Fix $B \in \mathcal{B}$. There exist $f \in F$ and $g \in \overline{H}_B$ such that $h = f \circ g$. Clearly, $g{\restriction}B = 1_B$, hence $h{\restriction}B = f{\restriction}B$. Similarly, if $h \in \bigcap_{B\in\mathcal{B}} \overline{H}_B F$, then for every $B \in \mathcal{B}$, $h^{-1}{\restriction}B = f^{-1}{\restriction}B$ for certain $f \in F$. We conclude that $h \in H$. ∎

This completes the proof of the following result.

Theorem 6.6: *Every homogeneous, Polish and* SLH*-space X admits a transitive continuous action by a Polish group.*

Remark 6.7: The referee pointed out the following approach to Theorem 6.6. Let X be a homogeneous Polish SLH-space. By [26] there is a separable metrizable group H acting transitively on X. Consider the completion $\hat{H}$ of H. It is a Polish group, so if the action of H on X could be extended to an action of $\hat{H}$ on X, then Theorem 6.6 follows. But this approach unfortunately does not work. The group H we get from the proof of Theorem 1.1 in [26] is of the form $\mathcal{H}(\gamma X|X)$ for some metrizable compactification γX of X. Hence there are many cases in which H is dense in $\mathcal{H}(\gamma X)$. For example, let X be the pseudo-interior of the Hilbert cube Q and let γX be Q. Then $\hat{H}$ is $\mathcal{H}(Q)$ which does not act on X.

(C) Application to $\mathbf{Z}$. Let us return to Theorem 4.1, where we showed that the natural action of the separable and metrizable group $\mathcal{G} = \mathcal{H}(X|Z)$ on Z has exactly two orbits, one of which is Polish and the other one is meager. We will prove that our criterion implies that $\mathcal{G}$ is Polishable.

To this end, enumerate $D = \mathrm{dom}(\phi) = \mathrm{dom}(\phi')$ as $\{d_n : n \in \mathbb{N}\}$. By Lemma 3.4(3) we may fix a countable subgroup F of $\mathcal{G}$ such that for all $n, m \in \mathbb{N}$ there exists $f \in F$ such that

$$f(\{d_n\} \times [\phi(d_n), \phi'(d_n)]) = \{d_m\} \times [\phi(d_m), \phi'(d_m)]$$

(any $f \in \mathcal{G}$ that maps an arbitrary point from $\{d_n\} \times \big(\phi(d_n), \phi'(d_n)\big)$ to an arbitrary point from $\{d_m\} \times \big(\phi(d_m), \phi'(d_m)\big)$ will do). Now for every n, put

$$\mathcal{G}_n = \{f \in \mathcal{G} : f(\{d_n\} \times [\phi(d_n), \phi'(d_n)]) = \{d_n\} \times [\phi(d_n), \phi'(d_n)]\}.$$

Then, clearly, $\mathcal{G}_n$ is closed in $\mathcal{G}$. A straightforward calculation, cf., the proof of Lemma 6.5, shows that

$$\mathcal{G} = \bigcap_{n\in\mathbb{N}} F\overline{\mathcal{G}}_n \cap \bigcap_{n\in\mathbb{N}} \overline{\mathcal{G}}_n F$$

(here closure means closure in the Polish group $\mathcal{H}(X)$). Hence $\mathcal{G}$ is Polishable by our criterion. This completes the proof of the following result.

Theorem 6.8: *There is a Polish group G acting on Z having precisely two G-orbits, one of which is Polish and the other one of which is meager.*

Observe that by Corollary 4.6, this is ‘best possible’.

7. Questions

In this section we state some open problems that were motivated by the results in this paper.

(A) Complexity of Polish group actions in terms of the number of orbits. Corollary 4.6 and Theorem 6.8 suggest the following.

Definition 7.1: Let X be a space and $n \in \mathbb{N}$. Say that $\mathbb{G}\text{-}\dim(X) \leq n$ if there is a Polish group G acting continuously on X such that X has at most n G-orbits. Moreover, let $\mathbb{G}\text{-}\dim(X) = n$ if $\mathbb{G}\text{-}\dim(X) \leq n$ but $\mathbb{G}\text{-}\dim(X) \not\leq n-1$. Finally, let $\mathbb{G}\text{-}\dim(X) = \infty$ if $\mathbb{G}\text{-}\dim(X) \not\leq n$ for every $n \in \mathbb{N}$.

Every Polish group G clearly has $\mathbb{G}\text{-}\dim(G) = 1$, and the celebrated Effros Theorem from [8] implies the following characterization result, see §1.

PROPOSITION 7.1: *Let X be a second category separable metrizable space. Then the following statements are equivalent:*

(a) $\mathbb{G}\text{-}\dim(X) = 1$,
(b) *X is a coset space of some Polish group.*

Observe that Z is a homogeneous Polish space with $\mathbb{G}\text{-}\dim(Z) = 2$. This motivates the following problem:

Question 7.2: Is there for every $n \geq 3$ a homogeneous Polish space Z_n for which $\mathbb{G}\text{-}\dim(Z_n) = n$?

(B) COMPLEXITY OF POLISH GROUP ACTIONS IN TERMS OF ISOMETRY GROUPS. It is a well-known result of Teleman [35] that every topological group G is isomorphic to a subgroup of the isometry group $\operatorname{Iso} B$ of some Banach space B (here $\operatorname{Iso} B$ is endowed with the strong operator topology). Megrelishvili [22] proved that if G is the (Polish) group of all orientation preserving homeomorphisms of the closed unit interval, then Teleman's B cannot be chosen to be reflexive (this was strengthened recently in Glasner and Megrelishvili [13]: B can not even be Asplund). This motivates the following problem.

Question 7.3: Let X be a homogeneous Polish space on which some Polish group acts transitively. Is there a Polish group acting transitively on X which is isomorphic to a subgroup of $\operatorname{Iso} H$ for some Hilbert space H?

It is very likely that the answer to this question is in the negative, but there do not seem to be known techniques that can be used to solve it. The referee noted that a positive answer would follow from a positive answer to the following question due to Kechris [32, Question 17 in §5.2]: *Is every Polish group a topological factor-group of a subgroup of the unitary group $U(\ell^2)$ with the strong topology?*

(C) Improving group topologies. A separable and metrizable space is **analytic** if it is a continuous image of the space of irrational numbers. Every absolute Borel set is analytic, but the converse is not true. In [28], an example was constructed of a homogeneous analytic space on which some separable and metrizable group acts transitively, but on which no **analytic** group acts transitively. This suggests the following problem.

Question 7.4: Let X be a homogeneous Polish space on which some separable metrizable group acts transitively. Is there a Polish (analytic) group that acts transitively on X?

(D) Reconstructing Z. Under mild conditions, locally compact spaces can be reconstructed from the algebraic properties of their groups of homeomorphisms (Rubin [33]). This suggests the following problem.

Question 7.5: Is it possible to reconstruct Z from the algebraic properties of its group of homeomorphisms $\mathcal{H}(Z)$?

References

[1] J. M. Aarts and L. G. Oversteegen, *The product structure of homogeneous spaces*, Indagationes Mathematicae (New Series) **1** (1990), 1–5.

[2] F. D. Ancel, *An alternative proof and applications of a theorem of E. G. Effros*, The Michigan Mathematical Journal **34** (1987), 39–55.

[3] R. D. Anderson, *On topological infinite deficiency*, The Michigan Mathematical Journal **14** (1967), 365–383.

[4] H. Becker and A. S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press, Cambridge, 1996.

[5] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Memoirs of the American Mathematical Society **71** (1988), no. 380, vi+110.

[6] R. H. Bing and F. B. Jones, *Another homogeneous plane continuum*, Transactions of the American Mathematical Society **90** (1959), 171–192.

[7] J. J. Dijkstra and J. van Mill, *Homeomorphism groups of manifolds and Erdős space*, Electronic Research Announcments of the American Mathematical Society **10** (2004), 29–38.

[8] E. G. Effros, *Transformation groups and C^*-algebras*, Annals of Mathematics **81** (1965), 38–55.

[9] R. Engelking, *Theory of Dimensions Finite and Infinite*, Heldermann Verlag, Lemgo, 1995.

[10] P. Erdős, *The dimension of the rational points in Hilbert space*, Annals of Mathematics **41** (1940), 734–736.

[11] L. R. Ford, Jr., *Homeomorphism groups and coset spaces*, Transactions of the Amrecian Mathematical Society **77** (1954), 490–497.

[12] M. Fort, *Homogeneity of infinite products of manifolds with boundary*, Pacific Journal of Mathematics **12** (1962), 879–884.

[13] E. Glasner and M. Megrelishvili, *Some new algebras of functions on toopological groups arising from G-spaces*, 2006, preprint.

[14] A. Hohti, *Another alternative proof of Effros' theorem*, Topology Proceedings **12** (1987), 295–298.

[15] K. Kawamura, L. G. Oversteegen and E. D. Tymchatyn, *On homogeneous totally disconnected* 1*-dimensional spaces*, Fundamenta Mathematicae **150** (1996), 97–112.

[16] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.

[17] O. H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Mathematische Annalen **105** (1931), 748–758.

[18] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, Studies in Logic and the foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, 1980.

[19] W. Lewis, *Continuous curves of pseudo-arcs*, Houston Journal of Mathematics **11** (1985), 91–99.

[20] M. G. Megrelishvili, *A Tikhonov G-space admitting no compact Hausdorff G-extension or G-linearization*, Russian Mathematical Surveys **43** (1988), 177–178.

[21] M. G. Megrelishvili, *Compactification and factorization in the category of G-spaces*, in *Categorical Topology and its Relation to Analysis, Algebra and Combinatorics* (Prague, 1988), World Sci. Publishing, Teaneck, NJ, 1989, pp. 220–237.

[22] M. G. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial*, Semigroup Forum **63** (2001), 357–370.

[23] J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland Publishing Co., Amsterdam, 2001.

[24] J. van Mill, *A note on Ford's Example*, Topology Proceedings **28** (2004), 689–694.

[25] J. van Mill, *A note on the Effros Theorem*, The American Mathematical Monthly **111** (2004), 801–806.

[26] J. van Mill, *Strong local homogeneity and coset spaces*, Proceedings of the American Mathematical Society **133** (2005), 2243–2249.

[27] J. van Mill, *Not all homogeneous Polish spaces are products*, Houston Journal of Mathematics **32** (2006), 489–492.

[28] J. van Mill, *Homogeneous spaces and transitive actions by analytic groups*, The Bulletin of the London Mathematical Society **39** (2007), 329–336.

[29] M. W. Mislove and J. T. Rogers, Jr., *Local product structures on homogeneous continua*, Topology and its Applications **31** (1989), 259–267.

[30] M. W. Mislove and J. T. Rogers, Jr., *Addendum: "Local product structures on homogeneous continua"*, Topology and its Applications **34** (1990), 209.

[31] P. S. Mostert, *Reasonable topologies for homeomorphism groups*, Proceedings of the American Mathematical Society **12** (1961), 598–602.

[32] V. Pestov, *Dynamics of Infinite-dimensional Groups and Ramsey-type Phenomena*, Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2005.

[33] M. Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Transactions of the Amrecian Mathematical Society **312** (1989), 487–538.

[34] S. Solecki, *Polish group topologies*, in *Sets and proofs*, London Math. Soc. Lecture Note Series, vol. 258 (S. B. Cooper and J. K. Truss, eds.), Cambridge University Press, Cambridge, 1999, pp. 339–364.

[35] S. Teleman, *Sur la représentation linéaire des groupes topologiques*, Annales Scientifiques de l'École Normale Supérieure **74** (1957), 319–339.

[36] G. S. Ungar, *On all kinds of homogeneous spaces*, Transactions of the Amrecian Mathematical Society **212** (1975), 393–400.

[37] V. V. Uspenskiĭ, *Why compact groups are dyadic*, in *General Topology and its Relations to Modern Analysis and Algebra, VI* (Prague, 1986), Research and Exposition in Mathematics, vol. 16, Heldermann, Berlin, 1988, pp. 601–610.

[38] V. V. Uspenskiĭ, *Topological groups and Dugundji compact spaces*, Matematicheskiĭ Sbornik **180** (1989), 1092–1118.

[39] J. de Vries, *On the existence of G-compactifications*, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques **26** (1978), 275–280.

[40] J. de Vries, *Linearization, compactification and the existence of non-trivial compact extensors for topological transformation groups*, in *Topology and Measure III*, Ernst-Moritz-Arndt-Universität zu Greifswald, 1982, pp. 339–346.