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On the supremum of the pseudocompact group topologies

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Abstract

P is the class of pseudocompact Hausdorff topological groups, and $\mathbf{P}'$ is the class of groups which admit a topology $\mathcal{T}$ such that $(G,\mathcal{T}) \in \mathbf{P}$. It is known that every $G = (G,\mathcal{T}) \in \mathbf{P}$ is totally bounded, so for $G \in \mathbf{P}'$ the supremum $\mathcal{T}^{\vee}(G)$ of all pseudocompact group topologies on G and the supremum $\mathcal{T}^{\#}(G)$ of all totally bounded group topologies on G satisfy $\mathcal{T}^{\vee} \subseteq \mathcal{T}^{\#}$.

The authors conjecture for abelian $G \in \mathbf{P}'$ that $\mathcal{T}^{\vee} = \mathcal{T}^{\#}$. That equality is established here for abelian $G \in \mathbf{P}'$ with any of these (overlapping) properties. (a) G is a torsion group; (b) $|G| \leqslant 2^{\mathfrak{c}}$; (c) $r_0(G) = |G| = |G|^{\omega}$; (d) $|G|$ is a strong limit cardinal, and $r_0(G) = |G|$; (e) some topology $\mathcal{T}$ with $(G,\mathcal{T}) \in \mathbf{P}$ satisfies $w(G,\mathcal{T}) \leqslant \mathfrak{c}$; (f) some pseudocompact group topology on G is metrizable; (g) G admits a compact group topology, and $r_0(G) = |G|$. Furthermore, the product of finitely many abelian $G \in \mathbf{P}'$, each with the property $\mathcal{T}^{\vee}(G) = \mathcal{T}^{\#}(G)$, has the same property.

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1. Introduction

All topological spaces here (in particular, all topological groups) are assumed to satisfy the Hausdorff separation axiom.

The term *pseudocompact* was introduced by Hewitt [24] to refer to those spaces on which each realvalued continuous function is bounded. It is not difficult to see, as in [13] (1.1), that a pseudocompact topological group $(G,\mathcal{T})$ is *totally bounded* (some authors prefer the term *precompact*) in the sense that for every nonempty $U \in \mathcal{T}$ there is $F \in [G]^{<\omega}$ such that $G = FU$. According to a theorem of Weil [27], a topological group G is totally bounded iff G is a dense topological subgroup of a compact topological group. This latter, the *Weil completion* of G, is unique in an obvious sense; we denote it by the symbol $\overline{G}$.

We use the following notation throughout this paper.

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Notation 1.1.

(a) **T** is the class of totally bounded (Hausdorff) topological groups;
(b) **P** is the class of pseudocompact (Hausdorff) topological groups;
(c) for an abelian group G, $\mathcal{S}(G)$ is the set of point-separating subgroups of $\mathrm{Hom}(G, \mathbb{T})$;
(d) for an abelian group G and $A \in \mathcal{S}(G)$, $\mathcal{T}_A$ is the topology induced on G by A.

For abelian G the topology $\mathcal{T}^{\#}(G)$ is defined in the abstract; evidently we have $\mathcal{T}^{\#}(G) = \mathcal{T}_A$ with $A = \mathrm{Hom}(G, \mathbb{T})$. The notation $\mathcal{T}^{\#}$ is in the tradition of van Douwen [20] and many subsequent workers, who have used the symbol $G^{\#}$ to denote G with the topology $\mathcal{T}^{\#}(G)$. When in addition $G \in \mathbf{P}'$—that is, when G admits a pseudocompact group topology—the supremum of all such topologies on G is denoted $\mathcal{T}^{\vee}(G)$. When no ambiguity can arise, we denote $\mathcal{T}^{\#}(G)$ and $\mathcal{T}^{\vee}(G)$ simply by $\mathcal{T}^{\#}$ and $\mathcal{T}^{\vee}$, respectively.

When G and A are as in 1.1(d), the map $e_A : G \to \mathbb{T}^A$ defined by $e_A(x) = h(x)$ $(x \in G,\ h \in A)$ is an isomorphism onto its range, and (identifying G with its isomorph $e_A[G]$) the topology $\mathcal{T}_A$ is the topology inherited by G from $\mathbb{T}^A$.

The following two theorems are basic in our considerations. The first affords a useful criterion for determining which (not necessarily abelian) totally bounded groups are pseudocompact, and the second describes explicitly how totally bounded group topologies arise on an abelian group.

Theorem 1.2. *(See [13].)*

(a) *A totally bounded topological group G is pseudocompact iff G is G_δ-dense in $\overline{G}$ (in the sense that G meets each nonempty G_δ-subset of $\overline{G}$); and*
(b) *a dense subgroup of a pseudocompact group is itself pseudocompact iff it is G_δ-dense.*

Theorem 1.3. *(See [12].) Let G be an abelian group. Then*

(a) $A \in \mathcal{S}(G) \Rightarrow (G, \mathcal{T}_A) \in \mathbf{T}$;
(b) $[A \in \mathcal{S}(G),\ h \in \mathrm{Hom}(G, \mathbb{T})] \Rightarrow [h$ *is $\mathcal{T}_A$-continuous if and only if* $h \in A]$;
(c) *if $(G, \mathcal{T}) \in \mathbf{T}$ then $\mathcal{T} = \mathcal{T}_A$ with $A := \widehat{(G, \mathcal{T})} \in \mathcal{S}(G)$.*

Remark 1.4. (a) It is known [12] with notation as in Theorem 1.3 and G infinite that $w(G, \mathcal{T}_A) = |A|$. It follows for $(G, \mathcal{T}_A) \in \mathbf{P}$ that there is $\mathcal{T}_B \subseteq \mathcal{T}_A$ such that $(G, \mathcal{T}_B) \in \mathbf{P}$ and $w(G, \mathcal{T}_B) = |B| \leqslant |G|$.

(b) Since a pseudocompact group $(G, \mathcal{T})$ in which $\{0\}$ is a G_δ-set is both metrizable and compact [15] (3.1), such a topology is both maximal and minimal among pseudocompact (Hausdorff) group topologies on G. Thus if A and B are as in (a) with $|A| > \omega$, then necessarily also $|B| > \omega$.

Discussion 1.5. (a) We note for clarity that the topology $\mathcal{T}^{\#} = \mathcal{T}^{\#}(G)$ is defined for every abelian group G, while $\mathcal{T}^{\vee} = \mathcal{T}^{\vee}(G)$ is defined if and only if $G \in \mathbf{P}'$. Readers versed in the theory of topological groups will recognize $\mathcal{T}^{\#}$ as the *Bohr topology* associated with, or derived from, the discrete topology on G; as indicated above, we have $\mathcal{T}^{\#} = \mathcal{T}_H$ with $H = \mathrm{Hom}(G, \mathbb{T})$.

(b) As a notational convenience, for abelian $G \in \mathbf{P}'$ we write

$$\mathcal{H}(G) := \bigcup \{A \in \mathcal{S}(G)\colon\ (G, \mathcal{T}_A) \in \mathbf{P}\}.$$

Thus for $h \in \mathrm{Hom}(G, \mathbb{T})$ we have $h \in \mathcal{H}(G)$ if and only if there is $A \in \mathcal{S}(G)$ such that $(G, \mathcal{T}_A) \in \mathbf{P}$ and h is $\mathcal{T}_A$-continuous.

We noted in the Abstract that since $\mathbf{P} \subseteq \mathbf{T}$ we have $\mathcal{T}^{\vee} \subseteq \mathcal{T}^{\#}$ for every (abelian) $G \in \mathbf{P}'$. Here is a simple condition for equality.

Theorem 1.6. *Let $G \in \mathbf{P}'$. Then*

(a) $\mathcal{T}^{\vee} = \mathcal{T}_{\langle \mathcal{H}(G) \rangle}$; *and*
(b) $\mathcal{T}^{\vee}(G) = \mathcal{T}^{\#}(G)$ *if and only if* $\langle \mathcal{H}(G) \rangle = \mathrm{Hom}(G, \mathbb{T})$.

Proof. (a) The topology $\mathcal{T}_{\langle\mathcal{H}(G)\rangle}$ contains every pseudocompact group topology on G, so $\mathcal{T}^\vee \subseteq \mathcal{T}_{\langle\mathcal{H}(G)\rangle}$. Every pseudocompact group topology $\mathcal{T}_A$ on G satisfies $\mathcal{T}_A \subseteq \mathcal{T}^\vee$, so $\mathcal{T}_{\langle\mathcal{H}(G)\rangle} \subseteq \mathcal{T}^\vee$.

(b) is immediate from (a) and Theorem 1.3. □

In Example 5.1 we give an example of an abelian group $G \in \mathbf{P}'$ such that $(G, \mathcal{T}^\vee) = (G, \mathcal{T}^\#)$, while $\mathcal{H}(G) \neq \mathrm{Hom}(G, \mathbb{T})$; indeed in that example, $\mathcal{H}(G)$ is not a subgroup of $\mathrm{Hom}(G, \mathbb{T})$.

Corollary 1.7. *The product of finitely many $G \in \mathbf{P}'$, each satisfying $\mathcal{T}^\vee(G) = \mathcal{T}^\#(G)$, has the same property.*

Proof. Let $G = G_0 \times G_1$ with G_i as hypothesized. There is a topology $\mathcal{T}_i$ on G_i such that $(G_i, \mathcal{T}_i) \in \mathbf{P}$, so $(G, \mathcal{T}_0 \times \mathcal{T}_1) \in \mathbf{P}$ by [13] and hence $G \in \mathbf{P}'$. Now let

$$h = (h_0, h_1) \in \mathrm{Hom}(G, \mathbb{T}) = \mathrm{Hom}(G_0, \mathbb{T}) \times \mathrm{Hom}(G_1, \mathbb{T})$$

and let $\mathbf{c}_i$ be the constant (trivial) homomorphism from G_i to $\mathbb{T}$. Clearly $\mathbf{c}_1 \in \mathcal{H}(G_1)$, and $h_0 \in \langle\mathcal{H}(G_0)\rangle$ by Theorem 1.6(b); hence $(h_0, \mathbf{c}_1) \in \langle\mathcal{H}(G)\rangle$. Similarly $(\mathbf{c}_0, h_1) \in \langle\mathcal{H}(G)\rangle$. Thus

$$h = (h_0, h_1) = (h_0, \mathbf{c}_1) + (\mathbf{c}_0, h_1) \in \big\langle\mathcal{H}(G)\big\rangle$$

and another appeal to Theorem 1.6(b) completes the proof. □

The authors do not know whether "finitely many" may be legitimately replaced by "arbitrarily many" in the statement of Corollary 1.7. Remark 3.6 below provides additional perspective on Question 3.7.

The present paper is an investigation into the following conjecture.

Conjecture 1.8. *Every abelian group $G \in \mathbf{P}'$ satisfies $(G, \mathcal{T}^\vee) = (G, \mathcal{T}^\#)$.*

Remark 1.9. (a) It is known that for every abelian $(G, \mathcal{T}) \in \mathbf{P}$ with $w(G, \mathcal{T}) > \omega$ there is a topology $\mathcal{U}$ on G, strictly larger than $\mathcal{T}$, such that $(G, \mathcal{U}) \in \mathbf{P}$ [6]. It is known, further, that if $(G, \mathcal{T}) \in \mathbf{P}$ with $w(G, \mathcal{T}) = \omega$, then although $\mathcal{T}$ is maximal among pseudocompact group topologies on G there is a topology $\mathcal{U}$ on G such that $(G, \mathcal{U}) \in \mathbf{P}$ and $w(G, \mathcal{U}) > \omega$ (cf. [9]). It follows that no infinite abelian group has a largest pseudocompact group topology, so the relation $(G, \mathcal{T}^\vee) \in \mathbf{P}$ fails for each such G.

(b) The weaker statement for abelian G that $(G, \mathcal{T}^\#) \in \mathbf{P}$ if and only if G is finite has been known for some years [14] (2.2).

Roadmap 1.10. Always with an eye on Conjecture 1.8, we proceed in this paper as follows. In Section 2 we verify that $(G, \mathcal{T}^\vee) = (G, \mathcal{T}^\#)$ for abelian torsion groups in $\mathbf{P}'$. Then in Theorem 3.5 we show that a certain mild and natural "fragmentation" condition on a group $G \in \mathbf{P}$ suffices to ensure that $(G, \mathcal{T}^\vee) = (G, \mathcal{T}^\#)$, and in Section 4 we show that a vast array of groups do satisfy that condition, thus completing the proof of the theorem stated in the Abstract. An example in Section 5 concludes the paper.

2. $\mathcal{T}^\vee = \mathcal{T}^\#$: The torsion case

For groups A and B we write $A =_{\mathrm{alg}} B$ to indicate that A and B are algebraically isomorphic; and $A \subseteq_{\mathrm{alg}} B$ means that B contains an isomorphic copy of A. These relations are blind to, and independent of, any topological structure which A or B may carry.

Discussion 2.1. Which groups admit a pseudocompact group topology—that is, which G are in $\mathbf{P}'$? This question, not fully answered even for abelian groups, has attracted considerable attention in the literature. We cite some relevant facts which provide helpful background.

(i) The algebraic classification of the abelian groups which admit a compact group topology is complete. The full story is given in [25] (§25).

(ii) Every infinite $G \in \mathbf{P}'$ satisfies $|G| \geqslant \mathfrak{c}$. See [10] or [2] (6.13)(a) for an explicit proof, and see [1], [19] (1.3) for earlier, more general results.

(iii) The fact that every pseudocompact topological space satisfies the conclusion of the Baire Category Theorem has two consequences useful to us: (1) If $G \in \mathbf{P}'$, then the cardinal number $|G| = \kappa$ cannot be a strong limit cardinal with $\mathrm{cf}(\kappa) = \omega$ [19]; and (2) every abelian torsion group in $\mathbf{P}'$ is of bounded order [11] (7.4).

In this section we establish Conjecture 1.8 for torsion groups. An ingredient essential to the proof is the fact that every subgroup of finite index in an abelian torsion group in $\mathbf{P}'$ is itself in $\mathbf{P}'$. We give two proofs of that theorem, quite different in flavor. Each requires some background.

Theorem 2.6 = Theorem 2.10: Proof 1

Discussion 2.2. Observation 2.1(iii)(1) of van Douwen suggests the following terminology. A cardinal γ is *admissible* if there is a topological group $G \in \mathbf{P}$ such that $|G| = \gamma$. (The term was introduced explicitly in this context in [16], though for technical reasons the authors there applied it only to infinite cardinals.) From 2.1 (i) and (ii) we see that ω is not admissible, and that every γ of the form $\gamma = 2^\kappa$ is admissible. The classification of the admissible cardinals need not occupy our attention here, but it is worthwhile to note that, even for abelian torsion groups G of bounded order, the condition that $|G|$ is admissible does not guarantee that $G \in \mathbf{P}'$. It was noted in [8] (3.14) and [18], for example, that if p is prime and κ is a strong limit cardinal of countable cofinality, then

$$\bigoplus_{2^\kappa} \mathbb{Z}(p^2) \oplus \bigoplus_{\kappa} \mathbb{Z}(p) \in \mathbf{P}' \quad \text{while} \quad \bigoplus_{2^\kappa} \mathbb{Z}(p) \oplus \bigoplus_{\kappa} \mathbb{Z}(p^2) \notin \mathbf{P}'.$$

That example suggests the following criterion, one of several given in [18] (6.2). See also [16,17,8], for similar conditions on an abelian torsion group G necessary and sufficient that $G \in \mathbf{P}'$.

As usual for an abelian group G and $m \in \mathbb{Z}$, we write $mG := \{mx\colon x \in G\}$.

Theorem 2.3. *(See [18].) Let G be an abelian torsion group of bounded order. Then $G \in \mathbf{P}'$ if and only if $|mG|$ is admissible for each $m \in \mathbb{Z}$.*

The following lemma will be useful. (We omit obvious generalizations, for example, to the nonabelian case with H normal in G, since we have no need of these.)

Lemma 2.4. *Let G be an abelian group and ϕ a surjective homomorphism from G onto a group A. Let H be a subgroup of G and let $B = \phi[H]$. Then $|G/H| \geqslant |A/B|$.*

Proof. The map $G/H \to A/B$ given by $x + H \to \phi(x) + B$ is (well defined and) surjective. □

Corollary 2.5. *Let G be an abelian group and H a subgroup, and let $m \in \mathbb{Z}$. Then $|mG/mH| \leqslant |G/H|$.*

Proof. Define ϕ on G by $\phi(x) = mx$, and set $A := \phi[G]$. Then Lemma 2.4 applies. □

Theorem 2.6. *Let G be an abelian group in $\mathbf{P}'$ and let H be a subgroup of G such that $|G/H| < \omega$. Then $H \in \mathbf{P}'$.*

Proof. Let $\mathcal{T}$ be a group topology for G such that $(G, \mathcal{T}) \in \mathbf{P}$. Since G (and hence H) is of bounded order by Observation 2.1(iii)(2), it suffices by Theorem 2.3 to show for each $m \in \mathbb{Z}$ that the cardinal number $|mH|$ is admissible. Given such m, surely $|mG|$ is admissible, since mG (in the topology inherited from $(G, \mathcal{T})$) is the image of G under the $\mathcal{T}$-continuous homomorphism $x \to mx$. Then from $|mG/mH| \leqslant |G/H| < \omega$ it follows that either $|mH| < \omega$ or $|mH| = |mG|$. □

Theorem 2.6 = Theorem 2.10: Proof 2

We begin this proof with a result weaker than Theorem 2.6 (which is proved, however, by a direct, transparent argument not using the concept of an admissible cardinal).

Lemma 2.7. *Let G be an abelian group in $\mathbf{P}'$ and let H be a subgroup of G such that $|G/H| < \omega$ and H is algebraically a direct summand of G. Then $H \in \mathbf{P}'$.*

Proof. Let $\mathcal{T}$ be a group topology for G such that $(G, \mathcal{T}) \in \mathbf{P}$.

We have algebraically $G = F \oplus H$ with $|F| < \omega$. Let $\pi : G \twoheadrightarrow H =_{\mathrm{alg}} G/F$ be the natural projection and give H the quotient topology $\mathcal{T}_q$ determined by π and $\mathcal{T}$. Since $\ker(\pi) = F \times \{0\}$ is finite, it is $\mathcal{T}$-closed in G, so $(H, \mathcal{T}_q)$ is a (Hausdorff) topological group (cf. [25] (5.16 and 5.21)). The map $\pi : (G, \mathcal{T}) \twoheadrightarrow (H, \mathcal{T}_q)$ is continuous, so indeed $(H, \mathcal{T}_q)$ is pseudocompact. □

Remark 2.8. It may be noted in connection with Theorem 2.6 and Lemma 2.7 that a finite-index subgroup H of a topological group $(G, \mathcal{T}) \in \mathbf{P}$ may fail to be pseudocompact in the inherited topology, even when G is an abelian group of bounded order and H is algebraically a direct summand. For an example, let $G_0 = \{0, 1\}^\omega$ and $G_1 = \{0, 1\}^\kappa$ with $\kappa > \omega$, let H_0 be a maximal proper dense subgroup of G_0, and let $H = H_0 \times G_1$ and $G = G_0 \times G_1$. Then:

(a) $|G/H| = |G_0/H_0| = 2$ and H is a direct summand of G, since if $x \in G \backslash H$ then $\langle x \rangle \cap H = \{0\}$ so G has algebraically the structure $G = \langle x \rangle \oplus H$; and
(b) H is dense in G but not G_δ-dense in G (since H_0 is not G_δ-dense in G_0), so by Theorem 1.2 H is not pseudocompact in the topology inherited from G.

It follows in particular that the quotient topology $\mathcal{T}_q$ imposed on H in the proof of Lemma 2.7 may fail to coincide with the topology which H inherits from $(G, \mathcal{T})$.

The following lemma shows in effect that in a torsion abelian group G of bounded order, every subgroup of finite index shares with G a common direct summand of finite index in G. The formulation and the proof of this lemma are due to James D. Reid, and are offered here with his kind permission.

Lemma 2.9. *Let G be an abelian torsion group of bounded order and let H be a subgroup of G such that $|G/H| < \omega$. Then there are a subgroup H_0 of G and finite subgroups F_0 and F_1 of G such that $H = F_0 \oplus H_0$ and $G = F_1 \oplus H_0$.*

Proof. Let X be a selection set for the coset space G/H (that is, $|X \cap (x + H)| = 1$ for each $x \in G$). Like every abelian torsion group of bounded order, G (and similarly H) can be expressed as a direct sum of (finite) cyclic groups [21] (17.2). Thus since $\langle X \rangle$ is finite there is a finite direct summand F of G such that $\langle X \rangle \subseteq F$, and similarly there is a finite direct summand F_0 of H such that $F \cap H \subseteq F_0$; we write $H = F_0 \oplus H_0$. We have

$$G = \langle X \rangle + H \subseteq F + H = F + (F_0 \oplus H_0) \subseteq G,$$

so it suffices to show, setting $F_1 := F + F_0$, that $F_1 \cap H_0 = \{0\}$ (for then $G = F_1 \oplus H_0$, as required). To that end let $a + b = c \in H_0$ with $a \in F$ and $b \in F_0$. From $b \in F_0 \subseteq H$ and $c \in H_0 \subseteq H$ follows $a = c - b \in F \cap H \subseteq F_0$, so indeed $c \in F_0 \cap H_0 = \{0\}$. □

Theorem 2.10. *Let G be an abelian torsion group in $\mathbf{P}'$ and let H be a subgroup of G such that $|G/H| < \omega$. Then $H \in \mathbf{P}'$.*

Proof. Let H_0, F_0 and F_1 be as given by Lemma 2.9. Since $G \in \mathbf{P}'$ we have $H_0 \in \mathbf{P}'$ by Lemma 2.7, and it is obvious (upon giving the finite group F_0 the discrete topology) that $H = F_0 \times H_0 \in \mathbf{P}'$. □

With Theorem 2.6 = 2.10 at our disposal, we conclude the proof that $(G, \mathcal{T}^\vee) = (G, \mathcal{T}^\#)$ for every abelian torsion group $G \in \mathbf{P}'$.

Lemma 2.11. *Let G be an abelian torsion group in $\mathbf{P}'$. Then $\mathcal{H}(G) = \mathrm{Hom}(G, \mathbb{T})$ (that is: for every $h \in \mathrm{Hom}(G, \mathbb{T})$ there is a topology $\mathcal{W} = \mathcal{W}(h)$ on G such that $(G, \mathcal{W}) \in \mathbf{P}$ and h is $\mathcal{W}$-continuous).*

Proof. Let $\mathcal{T}$ be a group topology for G such that $(G, \mathcal{T}) \in \mathbf{P}$.

The group G is of bounded order by Observation 2.1(iii)(2), so with $H := \ker(h)$ we have $|G/H| = |h[G]| < \omega$; thus by Theorem 2.6 = 2.10 there is a pseudocompact group topology $\mathcal{U}$ on H. Now give G the topology $\mathcal{W}$ defined by the requirement that H and each of its translates is $\mathcal{W}$-open. (More precisely: a subset W of G is $\mathcal{W}$-open iff $(-x + W) \cap H \in \mathcal{U}$ for each $x \in W$.) It is easy to check that $\mathcal{W}$ is a group topology for G, so $(G, \mathcal{W})$, like every space which is the union of finitely many pseudocompact topological spaces, is pseudocompact.

It remains to see that h is $\mathcal{W}$-continuous. This is obvious, since h is constant on each of the cosets $x + W$ $(x \in G)$, and each such set is $\mathcal{W}$-clopen in G. □

Theorem 2.12. *Let G be an abelian torsion group such that $G \in \mathbf{P}'$. Then $(G, \mathcal{T}^{\vee}) = (G, \mathcal{T}^{\#})$.*

Proof. This is immediate from Lemma 2.11. □

3. $\mathcal{T}^{\vee} = \mathcal{T}^{\#}$: The case $r_0(G) > 0$

For a subgroup A of a group H (typically here of the form $H = \mathrm{Hom}(G, \mathbb{T})$) and for $h \in H$, we denote the subgroup $\langle A \cup \{h\}\rangle$ of H by the abbreviated symbol $A(h)$.

Theorem 3.1. *(See [22,4].) Let G be an abelian group and $A \in \mathcal{S}(G)$, and let $h \in \mathrm{Hom}(G, \mathbb{T})$ satisfy $\langle h\rangle \cap A = \{0\}$. Then conditions* (i) *and* (ii) *are equivalent.*

(i) $(G, \mathcal{T}_{A(h)}) \in \mathbf{P}$;
(ii) (1) $(G, \mathcal{T}_A) \in \mathbf{P}$; (2) $h[G]$ *is a closed subgroup of* $\mathbb{T}$; *and* (3) $\ker(h)$ *is* G_δ*-dense in* $(G, \mathcal{T}_A)$.

In view of Theorem 1.3, the condition $\langle h\rangle \cap A = \{0\}$ in Theorem 3.1 is equivalent to the condition that if $n \in \mathbb{Z}$ and nh is $\mathcal{T}_A$-continuous, then $nh \equiv 0$.

Definitions and Notation 3.2. Let G be an abelian group.

(a) A set $\{H_i\colon i \in I\}$ of subgroups of G is *independent* in G if it satisfies either of the following two (equivalent) conditions:
  (1) $\langle \bigcup_{i\in I} H_i\rangle = \bigoplus_{i\in I} H_i$;
  (2) if $[F \in [I]^{<\omega}$ and $\sum_{i\in F} k_i x_i = 0$ with $k_i \in \mathbb{Z}$ and $x_i \in H_i$ for $i \in F$, then each $k_i x_i = 0$.
(b) A subset X of G is *independent* if $\{\langle x\rangle\colon x \in X\}$ is an independent set of subgroups of G.
(c) Definitions (a) and (b) follow the convention favored by Fuchs [21] (§16). We note, as there, that if X is independent in G and $X \cap \mathrm{tor}(G) = \emptyset$, then $\langle X\rangle =_{\mathrm{alg}} \bigoplus_{x\in X} \mathbb{Z}_x$.
(d) We refer the reader to [21] or [25] for the definition and the basic properties of the torsion-free rank $r_0(G)$ of an abelian group G; for us it is enough to know that $r_0(G) \geqslant \kappa$ if and only if $G \supseteq_{\mathrm{alg}} \bigoplus_\kappa \mathbb{Z}$. As to groups which admit a pseudocompact group topology, the following simple fact is basic.

Theorem 3.3. *(See [5] (2.17), [17] (2.17 and 3.17), [18] (3.8), [4] (4.3).) Every abelian group G such that $G \in \mathbf{P}'$ and $r_0(G) > 0$ satisfies $r_0(G) \geqslant \mathfrak{c}$.*

The following familiar result is proved, for example, in [25] (A.7), [21] (21.1).

Lemma 3.4. *Let G and D be abelian groups with D divisible and let $h \in \mathrm{Hom}(H, D)$ with H a subgroup of G. Then there is $\bar{h} \in \mathrm{Hom}(G, D)$ such that $h \subseteq \bar{h}$.*

This next result is our principal tool for establishing the relation $\mathcal{T}^\vee = \mathcal{T}^\#$ when $r_0(G) > 0$. Much of what follows thereafter in this paper is devoted to uncovering conditions on a group $(G, \mathcal{T}) \in \mathbf{P}$ sufficient to guarantee that the hypotheses of Theorem 3.5 are satisfied.

Theorem 3.5. *Let G be an abelian group with $(G, \mathcal{T}) \in \mathbf{P}$. Suppose there are four subgroups H_k $(0 \leqslant k \leqslant 3)$ of G such that*

(1) *the set $\{H_k\colon\ 0 \leqslant k \leqslant 3\}$ is independent*;
(2) *there are surjective homomorphisms $h_0 \in \mathrm{Hom}(H_0, \mathbb{T})$ and $h_1 \in \mathrm{Hom}(H_1, \mathbb{T})$; and*
(3) *H_2 and H_3 are G_δ-dense in $(G, \mathcal{T})$.*

Then

(a) *for every $f \in \mathrm{Hom}(G, \mathbb{T})$ there are $\bar{g}, \bar{h} \in \mathcal{H}(G)$ such that $f = \bar{g} + \bar{h}$; and*
(b) $(G, \mathcal{T}^\vee) = (G, \mathcal{T}^\#)$.

Proof. By Theorem 1.3 there is $A \in \mathcal{S}(G)$ such that $\mathcal{T} = \mathcal{T}_A$.

It suffices to prove (a), since (b) is then immediate. We show more, namely that for $f \in \mathrm{Hom}(G, \mathbb{T})$ there exist $\bar{g}$, $\bar{h}$ as indicated such that $(G, \mathcal{T}_{A(\bar{g})}), (G, \mathcal{T}_{A(\bar{h})}) \in \mathbf{P}$; it will then follow not only that $\mathcal{T}^\# = \mathcal{T}^\vee$ but even that $\mathcal{T}^\# = \bigvee\{\mathcal{U} \supseteq \mathcal{T}\colon\ (G, \mathcal{U}) \in \mathbf{P}\}$. To this end, set $H := \bigoplus_{0 \leqslant k \leqslant 3} H_k$ and define $h : H \to \mathbb{T}$ by

$$h \restriction H_0 = h_0, \qquad h \restriction H_1 = f - h_1, \qquad h \restriction H_2 \equiv 0, \quad \text{and} \quad h \restriction H_3 \equiv f.$$

Then h is well defined on H by (1), and $h[H] = \mathbb{T}$ by (2). Lemma 3.4 gives a homomorphism $\bar{h}$ such that $h \subseteq \bar{h} \in \mathrm{Hom}(G, \mathbb{T})$.

The set $\ker(\bar{h})$ contains the G_δ-dense set H_2. Thus conditions (ii) of Theorem 3.1 are satisfied, so to prove $(G, \mathcal{T}_{A(\bar{h})}) \in \mathbf{P}$ (and hence $\bar{h} \in \mathcal{H}(G)$) it suffices by Theorem 3.1 to verify that $\langle \bar{h} \rangle \cap A = \emptyset$. If $n \in \mathbb{Z}$ and $n\bar{h} \in A$ then $\ker(n\bar{h})$ is $\mathcal{T}_A$-closed in G, so from $\ker(n\bar{h}) \supseteq \ker(\bar{h}) \supseteq H_2$ (with H_2 dense) it follows that $\ker(n\bar{h}) = G$ and hence $n\bar{h} \equiv 0$, as required.

Now define $\bar{g} := f - \bar{h}$. The argument just given, *mutatis mutandis*, shows $(G, \mathcal{T}_{A(\bar{g})}) \in \mathbf{P}$ (so $\bar{g} \in \mathcal{H}(G)$), and the decomposition $f = \bar{g} + \bar{h}$ is as required. $\square$

Remark 3.6. Associated with this study are three classes of groups closed under the formation of (arbitrary) products: the class $\mathbf{P}$, the class $\mathbf{P}'$, and the class of $G \in \mathbf{P}$ satisfying the hypotheses of Theorem 3.5. (To check this last assertion let $(G_i)_{i \in I}$ be a set of groups in $\mathbf{P}$, each with a set of subgroups $H_{i,k}$ $(0 \leqslant k \leqslant 3)$ satisfying (1), (2) and (3) of Theorem 3.5, and set $H_k := \prod_{i \in I} H_{i,k}$; it is immediate that $\{H_k\colon\ 0 \leqslant k \leqslant 3\}$ satisfies (1), (2) and (3) of Theorem 3.5 for $G = \prod_{i \in I} G_i$.) We are led naturally to the following question (to which Corollary 1.7 provides only a partial answer).

Question 3.7. (a) Is the class of abelian groups $G \in \mathbf{P}'$ such that $(G, \mathcal{T}^\vee) = (G, \mathcal{T}^\#)$ closed under arbitrary products?

(b) If the answer to (a) is "No", what is the least cardinal κ for which there is a family $\{G_\eta\colon\ \eta < \kappa\} \subseteq \mathbf{P}'$ of abelian groups, each satisfying $(G_\eta, \mathcal{T}^\vee) = (G_\eta, \mathcal{T}^\#)$, such that $G := \prod_{\eta < \kappa} G_\eta$ satisfies $(G, \mathcal{T}^\vee) \neq (G, \mathcal{T}^\#)$?

Of course if Conjecture 1.8 is correct, then the answer to Question 3.7(a) is "Yes".

Discussion 3.8. In some early versions of this manuscript circulated informally to colleagues, we speculated that every abelian group $G = (G, \mathcal{T}) \in \mathbf{P}$ with $w(G) > \omega$ and $r_0(G) \geqslant \mathfrak{c}$ satisfies the hypotheses of Theorem 3.5. Simple examples noted subsequently and independently by one of the authors and by Dikranjan defeat this conjecture: Let $\kappa > \omega$ and let $G := \mathbb{T} \times \{0, 1\}^\kappa$ with the usual compact group topology, and suppose that there are G_δ-dense subgroups G_0, G_1 of G such that $G_0 \cap G_1 = \{0_G\}$. The map $\phi : G \to G$ given by $\phi(x) = 2x$ is continuous, and since $\phi[G_i]$ is G_δ-dense in $\mathbb{T} \times \{0\}$ we have

$$\mathbb{T} \times \{0\} = \phi[G_0] \cap \phi[G_1] \subseteq G_0 \cap G_1 = \{0_G\},$$

a contradiction.

Such considerations give rise naturally to the following question, which seems worthy of study quite independent of its relevance to the present paper.

Question 3.9. Which groups $G \in \mathbf{P}$ (with $w(G) > \omega$) admit G_δ-dense subgroups G_0, G_1 such that $G_0 \cap G_1 = \{0\}$? What about the case $r_0(G) > \omega$?

We turn now to the task of finding many groups G which satisfy the hypothesis of Theorem 3.5.

4. Concerning "Fragmentation"

Discussion 4.1. (a) Given a group $G \in \mathbf{P}$, let $\mathfrak{m}(G)$ be the least cardinal number such that some dense, pseudocompact subgroup H of G satisfies $|H| = \mathfrak{m}(G)$. After reading the work of Cater, Erdős and Galvin [1], the authors of [10] showed that compact groups K and K' with $w(K) = w(K')$ satisfy $\mathfrak{m}(K) = \mathfrak{m}(K')$; that is, for compact groups K the number $\mathfrak{m}(K)$ depends solely on $w(K)$ and is independent of algebraic properties of K. That justifies the following definition.

[10] Let K be a compact group such that $w(K) = \alpha \geqslant \omega$. Then $\mathfrak{m}(\alpha) := \mathfrak{m}(K)$.

For frequent use below we remark that $\mathfrak{m}(\alpha) \leqslant (\log \alpha)^\omega$ for all $\alpha \geqslant \omega$ (cf. [1] and [10]). We use also this fact:

If $\omega \leqslant \alpha \leqslant 2^{\mathfrak{c}}$ then $\mathfrak{m}(\alpha) = \mathfrak{c}$.

(**Proof.** From Observation 2.1(c)(ii) we have $\mathfrak{m}(\alpha) \geqslant \mathfrak{c}$ for all $\alpha \geqslant \omega$. Thus

$$\mathfrak{c} \leqslant \mathfrak{m}(\alpha) \leqslant (\log \alpha)^\omega \leqslant \left(\log(2^{\mathfrak{c}})\right)^\omega \leqslant \mathfrak{c}^\omega = \mathfrak{c}. \qquad \square)$$

(b) The Singular Cardinals Hypothesis, a consequence of GCH, implies that $\mathfrak{m}(\alpha) = (\log \alpha)^\omega$ for all α [1,10]); it is known however that equality can fail for some α in some models of ZFC [23].

(c) It is easy to see that a dense subgroup G of a topological group G' satisfies $w(G) = w(G')$. It follows that if $G \in \mathbf{P}'$, say with $|G|$ the admissible cardinal γ, then if $(G, \mathcal{T}) \in \mathbf{P}$ with $w(G, \mathcal{T}) = \alpha$ we have $\mathfrak{m}(\alpha) \leqslant \gamma \leqslant 2^\alpha = |\overline{(G, \mathcal{T})}|$ (with $\overline{(G, \mathcal{T})}$ denoting as usual the Weil completion of $(G, \mathcal{T})$).

(d) The works [7] and [18] and others by the same authors give this result for many specific groups S: "If some group of cardinality γ admits a pseudocompact group topology of weight α, then so does S." (Among the groups S so treated are $S = \bigoplus_\gamma F$ with F finite abelian, $S = \bigoplus_\gamma \mathbb{Q}$, S the free group on γ-many generators [18], and S the free abelian group $S = \bigoplus_\gamma \mathbb{Z}$ [18] (5.13).)

Theorem 4.4 extends and develops the argument of [7] (4.4), which shows that a group K^α as in Theorem 4.4 contains a G_δ-dense copy of the free abelian group $\bigoplus_{\mathfrak{m}(\alpha)} \mathbb{Z}$. A result parallel to [7] (4.4) in the more general context of a "variety of groups" is given in [18] (4.3). Our preliminary Lemmas 4.2 and 4.3 are as in [7]. See also [3] for an early, less incisive version of Theorem 4.4.

Lemma 4.2. *Let K be an abelian group and H a subgroup of K such that $|H| < r_0(K)$. Then there is $y \in K \backslash \mathrm{tor}(K)$ such that $\langle H \cup \{y\} \rangle = H \oplus \langle y \rangle$.*

Proof. From $r_0(K) = r_0(H) + r_0(K/H)$ (cf. [21] (Ex. 16.3(d)) follows $r_0(K/H) > 0$. It is enough to choose $y \in K$ such that $y + H \notin \mathrm{tor}(K/H)$. $\square$

Lemma 4.3. *Let $\{X_i \colon i \in I\}$ be a set of spaces with each $|X_i| > 1$ and with $|I| > \omega$. Let $D = \{p(\eta) \colon \eta < \gamma\}$ and $E = \{x(\eta) \colon \eta < \gamma\}$ be subsets of $\prod_{i \in I} X_i$ such that*

(i) *D is G_δ-dense in $\prod_{i \in I} X_i$, and*
(ii) *$\eta' < \eta < \gamma \Rightarrow \{i \in I \colon x(\eta')_i \neq p(\eta')_i\} \cap \{i \in I \colon x(\eta)_i \neq p(\eta)_i\} = \emptyset$.*

Then E is G_δ-dense in $\prod_{i \in I} X_i$.

Proof. The routine proof is given in [7] (4.1). □

Theorem 4.4. *Let α and ρ be cardinals such that $\mathfrak{m}(\alpha) \leqslant \alpha$ and $\mathfrak{m}(\alpha) \leqslant \rho \leqslant 2^\alpha$, and let K be a compact abelian group such that $w(K) \leqslant \alpha$ and $K \neq \operatorname{tor}(K)$. Then there is a family $\{X_\zeta\colon \zeta < 2^\alpha\}$ of pairwise disjoint subsets of $K^\alpha \backslash \operatorname{tor}(K^\alpha)$, each G_δ-dense in K^α and with each $|X_\zeta| = \rho$, such that $X := \bigcup_{\zeta<2^\alpha} X_\zeta$ is independent in K^α.*

Proof. We consider two cases.

Case 1. $\rho \leqslant \alpha$. Since $w(K^\alpha) = \alpha \geqslant \rho \geqslant \mathfrak{m}(\alpha)$, there is a G_δ-dense subset $\{p(\eta)\colon \eta < \rho\}$ of K^α.

For $S \subseteq \alpha$ let $\pi_S : K^\alpha \twoheadrightarrow K^S$ be the natural projection.

Let $\mathcal{A} = \{A(\eta)\colon \eta < \rho\}$ be a set of pairwise disjoint subsets of α with each $|A(\eta)| = \alpha$. (One may arrange that $\mathcal{A}$ is a partition of α, but this is not essential.)

For $\eta < \rho$ we have

$$2^\alpha = (2^\alpha)^\alpha = |K^{A(\eta)}| \geqslant r_0(K^{A(\eta)}) \geqslant r_0(\mathbb{Z}^\alpha) = 2^\alpha,$$

so $r_0(K^{A(\eta)}) = 2^\alpha$.

We well-order $2^\alpha \times \rho$ lexicographically: $(\zeta', \eta') < (\zeta, \eta)$ if either $(\zeta' < \zeta)$ or $(\zeta' = \zeta$ and $\eta' < \eta)$. By recursion we will define $y(\zeta, \eta) \in K^{A(\eta)}$ and $x(\zeta, \eta) \in K^\alpha$.

Choose $y(0, 0) \in K^{A(0)} \backslash \operatorname{tor}(K^{A(0)})$ and set $x(0, 0) = (y(0, 0), \pi_{\alpha \backslash A(0)}(p(0)))$.
(That is:

$$x(0,0)_i = \left\{ \begin{array}{ll} y(0,0)_i & \text{if } i \in A(0) \\ p(0)_i & \text{if } i \in \alpha \backslash A(0) \end{array} \right\}.)$$

Now let $(\zeta, \eta) \in 2^\alpha \times \rho$ and suppose that $y(\zeta', \eta')$ and $x(\zeta', \eta')$ have been defined for all $(\zeta', \eta') < (\zeta, \eta)$. Let

$$H(\zeta, \eta) = \bigl\langle \bigl\{ x(\zeta', \eta')\colon (\zeta', \eta') < (\zeta, \eta) \bigr\} \bigr\rangle.$$

Then $|\pi_{A(\eta)}[H(\zeta, \eta)]| \leqslant |H(\zeta, \eta)| < 2^\alpha = r_0(K^{A(\eta)})$, so by Lemma 4.2 (applied to $K^{A(\eta)}$ and $H(\zeta, \eta)$ in place of K and H) there is $y(\zeta, \eta) \in K^{A(\eta)} \backslash \operatorname{tor}(K^{A(\eta)})$ such that

$$\bigl\langle \pi_{A(\eta)}\bigl[H(\zeta, \eta)\bigr] \cup \bigl\{y(\zeta, \eta)\bigr\} \bigr\rangle = \pi_{A(\eta)}\bigl[H(\zeta, \eta)\bigr] \oplus \bigl\langle y(\zeta, \eta) \bigr\rangle. \tag{$*$}$$

We set $x(\zeta, \eta) = (y(\zeta, \eta), \pi_{\alpha \backslash A(\eta)}(p(\eta)))$.
(That is:

$$x(\zeta,\eta)_i = \left\{ \begin{array}{ll} y(\zeta,\eta)_i & \text{if } i \in A(\eta) \\ p(\eta)_i & \text{if } i \in \alpha \backslash A(\eta) \end{array} \right\}.)$$

The definition of $x(\zeta, \eta)$ for all $(\zeta, \eta) \in 2^\alpha \times \rho$ is complete. Since $\pi_{A(\eta)}(x(\zeta, \eta)) = y(\zeta, \eta)$ is nontorsion, also $x(\zeta, \eta)$ is nontorsion. From $(*)$ it follows that

$$\bigl\langle H(\zeta, \eta) \cup \bigl\{x(\zeta, \eta)\bigr\} \bigr\rangle = H(\zeta, \eta) \oplus \bigl\langle x(\zeta, \eta) \bigr\rangle \tag{$**$}$$

(for if $n \in \mathbb{Z}$ and $nx(\zeta, \eta) \in H(\zeta, \eta)$ then $ny(\zeta, \eta) \in \pi_{A(\eta)}[H(\zeta, \eta)]$, so $ny(\zeta, \eta) = 0$ and hence $n = 0$). Then from $(**)$ it follows that the set $X := \{x(\zeta, \eta)\colon (\zeta, \eta) \in 2^\alpha \times \rho\}$ is independent. Now for $\zeta < 2^\alpha$ define $X_\zeta := \{x(\zeta, \eta)\colon \eta < \rho\}$. For fixed $\zeta < 2^\alpha$ and $\eta' < \eta < \rho$ we have

$$\bigl\{i < 2^\alpha\colon x(\zeta, \eta')_i \neq p(\eta')_i\bigr\} \cap \bigl\{i < 2^\alpha\colon x(\zeta, \eta)_i \neq p(\eta)_i\bigr\} \subseteq A(\eta') \cap A(\eta) = \emptyset,$$

so from Lemma 4.3 and the G_δ-density of the set $\{p(\eta)\colon \eta < \rho\}$ in K^α it follows that each set X_ζ is G_δ-dense in K^α, as required.

Case 2. *Case* 1 *fails*. Fix ρ' so that $\mathfrak{m}(\alpha) \leqslant \rho' \leqslant \alpha$ and (as given by Case 1) let X be an independent set of nontorsion elements of K^α with a partition $\{X'_\zeta\colon \zeta < 2^\alpha\}$ such that each $|X'_\zeta| = \rho'$ and each X'_ζ is G_δ-dense in K^α. Now amalgamate: using $\rho \leqslant 2^\alpha$, let $\{A_s\colon s \in 2^\alpha\}$ be a partition of 2^α with each $|A_s| = \rho$, and for $s \in 2^\alpha$ set $X_s := \bigcup_{\zeta \in A_s} X'_\zeta$; then $\{X_s\colon s < 2^\alpha\}$ is as required. □

Remark 4.5. Theorem 4.4 may be profitably compared with and juxtaposed to the work of Itzkowitz and Shakhmatov [26], especially Corollaries 1.9 and 1.10 there. With minimal modifications, those statements and arguments in [26] show that a group like K^α (in Theorem 4.4 above) possesses an independent family of cardinality 2^α of dense pseudocompact subgroups; further, those subgroups may be chosen in each case to be algebraically of the form $\bigoplus_{2^\alpha}\mathbb{Z}$ (or even, slightly modifying the arguments of [26], of the form $\bigoplus_{(w(K))^\omega}\mathbb{Z}$). Our argument in Theorem 4.4, essential in the application 4.6 below, is of the same flavor but different in detail: we need that the dense, pseudocompact subgroups in our large family are of *small* cardinality. Of course, as indicated in the amalgamation argument given in Case 2 of Theorem 4.4, the fact that the set $X=\bigcup_{\zeta<2^\alpha}X_\zeta$ there is independent yields the "large free groups" statement of [26]. In any case it should be emphasized that both results are optimal in the sense that in each case the independent families of dense pseudocompact subgroups are of the maximal size possible, i.e., of size 2^α.

Simple examples show that in Lemma 3.4 the extension $\bar h$ of h cannot in general be chosen to be injective (that is, an isomorphism onto its range) even when h is injective. We see next however that this stronger property of $\bar h$ can be achieved under certain circumstances. Our statement and proof are suggested by the weaker results [7] (4.12) and [18] (4.4).

Theorem 4.6. *Let α be an infinite cardinal such that $\mathfrak{m}(\alpha)\leqslant\alpha$ and let G be an abelian group such that*

$$\mathfrak{m}(\alpha)\leqslant r_0(G)\leqslant|G|\leqslant 2^\alpha.$$

Then there is an isomorphism $\bar h$ of G into $\mathbb{T}^\alpha$ such that $\bar h[G]$ contains an independent set $\{H_\zeta\colon \zeta<r_0(G)\}$ of G_δ-dense subgroups of $\mathbb{T}^\alpha$, with each $H_\zeta=_{\mathrm{alg}}\bigoplus_{r_0(G)}\mathbb{Z}$.

Proof. (1) Taking $\rho:=r_0(G)$ and $K:=\mathbb{T}$ in Theorem 4.4 and discarding if necessary the sets X_ζ with $r_0(G)\leqslant\zeta<2^\alpha$, we have a family $\{X_\zeta\colon \zeta<r_0(G)\}$ of pairwise disjoint subsets of $\mathbb{T}^\alpha\backslash\mathrm{tor}(\mathbb{T}^\alpha)$, each G_δ-dense in $\mathbb{T}^\alpha$ and with each $|X_\zeta|=r_0(G)$, such that the set $\bigcup_{\zeta<r_0(G)}X_\zeta$ (again for simplicity denoted X) is independent in $\mathbb{T}^\alpha$.

Let Y be an independent subset of G maximal with respect to the property $Y\cap\mathrm{tor}(G)=\emptyset$, and let h_0 be an isomorphism from $\langle Y\rangle\subseteq G$ onto $\langle X\rangle\subseteq\mathbb{T}^\alpha$ (this exists since $|Y|=|X|=r_0(G)$ and $\langle Y\rangle=_{\mathrm{alg}}\bigoplus_{r_0(G)}\mathbb{Z}=_{\mathrm{alg}}\langle X\rangle$).

Let h_1 be an isomorphism from $\mathrm{tor}(G)$ into $\mathbb{T}^\alpha$. (This exists because the divisible hull D of $\mathrm{tor}(G)$ satisfies

$$\mathrm{tor}(G)\subseteq D=_{\mathrm{alg}}\bigoplus_p\bigoplus_{r_p(G)}\mathbb{Z}(p^\infty)\subseteq_{\mathrm{alg}}\bigoplus_{2^\alpha}\mathbb{Q}\oplus\bigoplus_p\bigoplus_{2^\alpha}\mathbb{Z}(p^\infty)=_{\mathrm{alg}}\mathbb{T}^\alpha.)$$

From $\langle Y\rangle\cap\mathrm{tor}(G)=\{0\}$ follows $\langle Y\rangle+\mathrm{tor}(G)=\langle Y\rangle\oplus\mathrm{tor}(G)\subseteq G$; similarly $\langle X\rangle+h_1[\mathrm{tor}(G)]=\langle X\rangle\oplus h_1[\mathrm{tor}(G)]\subseteq\mathbb{T}^\alpha$. Thus $h:=h_0\oplus h_1$ is a (well-defined) isomorphism from $\langle Y\rangle\oplus\mathrm{tor}(G)$ into $\mathbb{T}^\alpha$.

Let E be a divisible hull of G and let F be a minimal divisible subgroup of $\mathbb{T}^\alpha$ containing $h[\langle Y\rangle\oplus\mathrm{tor}(G)]$. Then E is also a divisible hull of $\langle Y\rangle\oplus\mathrm{tor}(G)$ [21] (24.3), so the isomorphism h extends to an isomorphism $\bar h$ from E onto F [21] (24.4). Then

$$X\subseteq\bar h[G]\subseteq\bar h[E]=F\subseteq\mathbb{T}^\alpha,$$

as required.

(2) The group $\bar h[G]$ is G_δ-dense in $\mathbb{T}^\alpha$, so $(\bar h[G],\mathcal{T})\in\mathbf{P}$ by Theorem 1.2. The remainder of (2) is now clear. □

Remark 4.7. A pseudocompact group G satisfies $\overline{G}=\beta(G)$ [13], so the topology $\mathcal{T}$ in Theorem 4.6 is connected. The fact that a group G satisfying the hypotheses of Theorem 4.6 admits a connected pseudocompact group topology is not new. See [7] (4.6) for a direct proof, and see [18] (7.1) for a definitive algebraic characterization of those abelian groups which admit a connected pseudocompact group topology.

Corollary 4.8. *Let G be an abelian group. If there is a cardinal number α such that $\mathfrak{m}(\alpha)\leqslant\alpha$ and $\mathfrak{m}(\alpha)\leqslant r_0(G)\leqslant|G|\leqslant 2^\alpha$, then $G\in\mathbf{P}'$ and $(G,\mathcal{T}^\vee)=(G,\mathcal{T}^\#)$.*

Proof. We identity G with its isomorph $\bar h[G]\subseteq\mathbb{T}^\alpha$ given in Theorem 4.6, and from among the 2^α-many G_δ-dense subgroups of $(G,\mathcal{T})\in\mathbf{P}$ given there we select, say, H_0, H_1, H_2 and H_3. Clearly conditions (1) and (3) of Theorem 3.5 are satisfied. To see that (2) also is satisfied it suffices to check that each $r_0(H_k)\geqslant\mathfrak{c}$, for then Lemma 3.4 gives the required surjective homomorphisms $h_k\in\mathrm{Hom}(H_k,\mathbb{T})$; and $r_0(H_k)\geqslant\mathfrak{c}$ is clear since otherwise by Theorem 3.3 and

Observation 2.1(iii)(2) H_k is a torsion group of bounded order and from the density of H_k in $(G, \mathcal{T})$ G itself would be a torsion group (of bounded order). That $(G, \mathcal{T}^{\vee}) = (G, \mathcal{T}^{\#})$ then follows from Theorem 3.5. □

Finally we pull together the essentials of what has preceded to prove the theorem stated in the Abstract.

Theorem 4.9. *Let G be an abelian group in $\mathbf{P}'$. If G satisfies any of the following (overlapping) properties, then $(G, \mathcal{T}^{\vee}) = (G, \mathcal{T}^{\#})$.*

(a) *G is a torsion group*;
(b) $|G| \leqslant 2^{\mathfrak{c}}$;
(c) $r_0(G) = |G| = |G|^{\omega}$;
(d) *$|G|$ is a strong limit cardinal, and $r_0(G) = |G|$*;
(e) *some topology $\mathcal{T}$ with $(G, \mathcal{T}) \in \mathbf{P}$ satisfies $w(G, \mathcal{T}) \leqslant \mathfrak{c}$*;
(f) *some pseudocompact group topology $\mathcal{T}$ on G is metrizable*;
(g) *G admits a compact group topology, and $r_0(G) = |G|$.*

Furthermore, the product of finitely many abelian $G \in \mathbf{P}'$, each with the property $\mathcal{T}^{\vee}(G) = \mathcal{T}^{\#}(G)$, has the same property.

Proof. (a) is Theorem 2.12. We assume in what follows that $r_0(G) > 0$, so $r_0(G) \geqslant \mathfrak{c}$ by Theorem 3.3.

(b) From 4.1(a) we have

$$\mathfrak{m}(\mathfrak{c}) = \mathfrak{c} \leqslant r_0(G) \leqslant |G| \leqslant 2^{\mathfrak{c}},$$

so Corollary 4.8 applies.

(c) Take $\alpha = |G|$. Then

$$\mathfrak{m}(\alpha) \leqslant (\log \alpha)^{\omega} \leqslant \alpha^{\omega} = \alpha = r_0(G) < 2^{\alpha}$$

(using 4.1(a)), so Corollary 4.8 applies.

(d) If $|G|$ is a strong limit cardinal, necessarily with $\mathrm{cf}(|G|) > \omega$ by Observation 2.1(iii)(1), then $|G| = |G|^{\omega}$ so (c) applies.

(e) follows from (b), since every (Hausdorff) space X satisfies $|X| \leqslant 2^{w(X)}$.

(f) is immediate from (b), (c) or (e).

(g) A compact group G satisfies $|G| = 2^{\alpha}$ with $\alpha = w(G)$, so (c) applies.

The final statement of the theorem is given in Corollary 1.7. □

5. An example

In Section 2 we showed that $\mathcal{T}^{\vee}(G) = \mathcal{T}^{\#}(G)$ for abelian torsion groups in $G \in \mathbf{P}'$, indeed for the very strong reason that $\mathcal{H}(G) = \mathrm{Hom}(G, \mathbb{T})$ for such G. The following example is logically inessential to the thrust of this paper, but it helps to establish what can and cannot be shown in the general case: The condition $\mathcal{H}(G) = \mathrm{Hom}(G, \mathbb{T})$ fails for some abelian $G \in \mathbf{P}'$ which satisfy $\mathcal{T}^{\vee}(G) = \mathcal{T}^{\#}(G)$; in fact the subset $\mathcal{H}(G)$ of $\mathrm{Hom}(G, \mathbb{T})$ may fail to be a subgroup.

Example 5.1. We show that there are an abelian group G, and $h_0, h_1 \in \mathrm{Hom}(G, \mathbb{T})$, and pseudocompact group topologies $\mathcal{T}_i$ on G, such that h_i is $\mathcal{T}_i$-continuous but the homomorphism $h := h_0 + h_1$ is not continuous with respect to any pseudocompact group topology on G. Let $G = \mathbb{T} \times \mathbb{Q} \times \mathbb{T}$. Since $\mathbb{T}$ contains algebraically the group $\bigoplus_{\omega} \mathbb{Q}$ as a direct summand (indeed even $\bigoplus_{\mathfrak{c}} \mathbb{Q}$, cf. [21] (p. 105)), we have $G_0 := \mathbb{T} \times \mathbb{Q} =_{\mathrm{alg}} \mathbb{T}$ and $G_1 := \mathbb{Q} \times \mathbb{T} =_{\mathrm{alg}} \mathbb{T}$; thus G_0 and G_1 admit compact metric group topologies. We give G the (compact metric) product topology $\mathcal{T}_0$ thus associated with $G_0 \times \mathbb{T}$, also the product topology $\mathcal{T}_1$ associated with $\mathbb{T} \times G_1$.

Now define $h_i \in \mathrm{Hom}(G, \mathbb{T})$ as follows: If $x = (a, q, b) \in G = \mathbb{T} \times \mathbb{Q} \times \mathbb{T}$, then

$$h_0\big((a, q), b\big) = a + q - b \quad \text{and} \quad h_1\big(a, (q, b)\big) = -a + q + b.$$

Then h_i is continuous with respect to the (pseudo)compact group topology $\mathcal{T}_i$ on G, so $h_i \in \mathcal{H}(G)$, but $h \notin \mathcal{H}(G)$: If h were continuous with respect to some pseudocompact group topology then $h[G] = \{2q\colon q \in \mathbb{Q}\} = \mathbb{Q} \subseteq \mathbb{T}$ would be a countably infinite, pseudocompact (hence compact) subgroup of $\mathbb{T}$, contrary to 2.1(iii).

It is evident from Theorem 4.9 that the group $G = \mathbb{T} \times \mathbb{Q} \times \mathbb{T}$ does satisfy $\mathcal{T}^{\vee}(G) = \mathcal{T}^{\#}(G)$.

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