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# Classes defined by stars and neighbourhood assignments ☆

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**Abstract**

We apply and develop an idea of E. van Douwen used to define $D$-spaces. Given a topological property $\mathcal{P}$, the class $\mathcal{P}^*$ dual to $\mathcal{P}$ (with respect to neighbourhood assignments) consists of spaces $X$ such that for any neighbourhood assignment $\{O_x\colon x \in X\}$ there is $Y \subset X$ with $Y \in \mathcal{P}$ and $\bigcup\{O_x\colon x \in Y\} = X$. We prove that the classes of compact, countably compact and pseudocompact are self-dual with respect to neighbourhood assignments. It is also established that all spaces dual to hereditarily Lindelöf spaces are Lindelöf. In the second part of this paper we study some non-trivial classes of pseudocompact spaces defined in an analogous way using stars of open covers instead of neighbourhood assignments.


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## 0. Introduction

A *neighbourhood assignment* in a space $X$ is a family $\{O_x\colon x \in X\}$ such that $x \in O_x \in \tau(X)$ for any $x \in X$. Neighbourhood assignments are useful for defining covering properties. For example, a space $X$ is Lindelöf if and only if, for any neighbourhood assignment $\{O_x\colon x \in X\}$ there is a countable $Y \subset X$ such that $\bigcup\{O_x\colon x \in Y\} = X$. If we substitute “closed discrete” for “countable” then we obtain the definition of the class of $D$-spaces introduced by van Douwen [6] and studied in [1,3–5,10] and other papers.

We generalize this idea of van Douwen by defining, for any class (or property) $\mathcal{P}$, a dual class $\mathcal{P}^*$ which consists of spaces $X$ such that, for any neighbourhood assignment $\{O_x\colon x \in X\}$ there exists a subspace $Y \subset X$ with $\bigcup\{O_x\colon x \in Y\} = X$ and $Y \in \mathcal{P}$. It turns out that many classical covering properties are self-dual in this sense. For example, compactness, pseudocompactness, countable compactness and the linear Lindelöf property are self-dual with respect to neighbourhood assignments.

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A similar idea gives rise to new classes of spaces determined by stars of open covers. Namely, if $\mathcal{P}$ is a class (or a property) of spaces then $X$ is *star-*$\mathcal{P}$ (*or star determined by* $\mathcal{P}$) if, for any open cover $\mathcal{U}$ of the space $X$, there is a subspace $Y \subset X$ with $\mathrm{St}(Y,\mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon\ U \cap Y \neq \emptyset\} = X$ and $Y \in \mathcal{P}$. This notion and some of its variations were considered in [11–14], for a more detailed treatment see [15]. It was observed in [15] that star-pseudocompactness is equivalent to pseudocompactness; however, the concepts of star-compactness and star-countable-compactness give rise to new non-trivial classes of spaces.

It is well known that being star-finite is equivalent to countable compactness in the class of Hausdorff spaces [9], so studying star properties basically consists of looking at generalizations of the class of countably compact spaces. We show, among other things, that the classes determined by countable metrizable spaces, metrizable spaces, countably compact and compact spaces provide distinct star properties.

The terms "star-compact", "star-Lindelöf" and some others have been also used by some authors in a completely different context. For example, in [7] the term "$n$-starcompact" was defined for any natural $n$; in the paper [16] the relationship between $n$-starcompactness and $n$-pseudocompactness was studied. On the other hand, in the paper [2] the terms "star-compact" and "star-Lindelöf" are equivalent to our terms "star-finite" and "star-countable", respectively. In particular, star-compactness in [2] coincides with countable compactness and hence gives no new class. This shows that there is no established tradition in naming the above-mentioned notions; since the terminology of [15] seems more direct and intuition-friendly to us, we use it systematically in this paper.

## 1. Notation and terminology

All spaces are assumed to be $T_1$. If $X$ is a space then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$; given a set $A \subset X$ let $\tau(A, X) = \{U \in \tau(X)\colon\ A \subset U\}$; we will write $\tau(x, X)$ instead of $\tau(\{x\}, X)$ for any $x \in X$. Given a family $\mathcal{U}$ of subsets of $X$ *the star,* $\mathrm{St}(A,\mathcal{U})$*, of the set $A$ with respect to* $\mathcal{U}$ is the set $\bigcup\{U \in \mathcal{U}\colon\ A \cap U \neq \emptyset\}$.

If $\kappa$ is an infinite cardinal then a space $X$ is $\kappa$-compact if any open cover of $X$ of cardinality at most $\kappa$ has a finite subcover. The *extent*, $\mathrm{ext}(X)$, of a space $X$ is the supremum of cardinalities of closed discrete subspaces of $X$. A space $X$ is called *$\kappa$-Lindelöf* if any open cover of $X$ has a subcover of cardinality at most $\kappa$. We say that $X$ is *linearly Lindelöf* if every nested open cover of $X$ has a countable subcover.

A family $\mathcal{N}$ of (not necessarily open) subsets of a space $X$ is *a network* of $X$ if every open $U \subset X$ is a union of a subfamily of $\mathcal{N}$. The spaces with a countable network are called *cosmic*. The symbols $\mathbb{D}$ stands for the discrete space $\{0, 1\}$ and $\mathbb{N} = \omega\backslash\{0\}$. The rest of our terminology is standard and follows [8].

## 2. Duality with respect to neighbourhood assignments

The notion of neighbourhood assignment makes it possible to define some interesting dual versions of well-known properties. Sometimes it is non-trivial to determine whether the dual version coincides with the original property. The following definition formalizes the main concept of this paper.

**Definition 2.1.** A class $\mathcal{P}^*$ is dual to a class $\mathcal{P}$ (with respect to neighbourhood assignments) if a space $X$ belongs to $\mathcal{P}^*$ if and only if for any neighbourhood assignment $\{O_x\colon\ x \in X\}$ there is a subspace $Y \subset X$ such that $Y \in \mathcal{P}$ and $\bigcup\{O_x\colon\ x \in Y\} = X$. If $X$ is a member of the class $\mathcal{P}^*$, then we say that $X$ is *dually* $\mathcal{P}$.

**Remark 2.2.** The $D$-property in the sense of van Douwen [6] is a subclass of the dual class of discrete spaces. It is evident that the $\kappa$-Lindelöf spaces form the dual class of spaces of cardinality at most $\kappa$. Besides, any compact space is dually finite and hence dually discrete.

**Example 2.3.** The space $\omega_1$ with its order topology is dually discrete. Consequently, a space can be

(a) dually metrizable (and hence dually paracompact) without being metacompact;
(b) dually realcompact but not realcompact;
(c) dually discrete without being a $D$-space.

**Proof.** The space $\omega_1$ is countably compact and non-compact; this, evidently implies that $\omega_1$ is neither realcompact nor a $D$-space. To see that it is dually discrete take any neighbourhood assignment $\{O_\alpha\colon \alpha < \omega_1\}$ of the space $\omega_1$. For any non-isolated point $\alpha \in \omega_1$ there is $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha] \subset O_\alpha$. The pressing-down lemma shows that there is an uncountable $A \subset \omega_1$ and $\beta < \omega_1$ for which $f(\alpha) = \beta$ for any $\alpha \in A$.

The space $\omega_1$ being scattered, there is a discrete uncountable $B \subset A \backslash (\beta + 1)$; it is clear that $\bigcup\{O_\alpha\colon \alpha \in B\} \supset \omega_1 \backslash (\beta + 1)$ and, since the space $\beta + 1$ is compact, we can choose a finite $F \subset \beta + 1$ such that $\bigcup\{O_\alpha\colon \alpha \in F\} \supset \beta + 1$. Then $D = F \cup B$ is a discrete subspace of $\omega_1$ such that $\bigcup\{O_\alpha\colon \alpha \in D\} = \omega_1$. Thus $\omega_1$ is dually discrete; since any discrete space of cardinality $\leqslant \omega_1$ is realcompact, the space $\omega_1$ is dually realcompact as well. $\square$

**Proposition 2.4.** *Given an infinite cardinal $\kappa$ let $E_\kappa$ be the class of spaces $X$ such that* $\mathrm{ext}(X) \leqslant \kappa$. *Then the class $E_\kappa$ is self-dual with respect to neighbourhood assignments.*

**Proof.** Suppose that a space $X$ is in the dual class to $E_\kappa$ and there is a closed discrete $D \subset X$ with $|D| = \kappa^+$. If $x \in X \backslash D$ let $O_x = X \backslash D$; if $x \in D$ then fix $O_x \in \tau(x, X)$ such that $O_x \cap D = \{x\}$. For the neighbourhood assignment $\{O_x\colon x \in X\}$ there exists a subspace $Y \subset X$ such that $\mathrm{ext}(Y) \leqslant \kappa$ and $\bigcup\{O_x\colon x \in Y\} = X$. However, $y \in D$ implies that $y \in O_x$ if and only if $y = x$; this shows that $D \subset Y$ and hence $\mathrm{ext}(Y) \geqslant \kappa^+$ which is a contradiction. $\square$

**Theorem 2.5.** *Compactness, and $\kappa$-compactness are self-dual with respect to neighbourhood assignments for any cardinal $\kappa$.*

**Proof.** Suppose that $X$ is dually ($\kappa$-)compact; if $X$ is not ($\kappa$-)compact then there is an infinite cardinal $\lambda$ ($\leqslant \kappa$) and a cover $\mathcal{U} = \{U_\alpha\colon \alpha < \lambda\} \subset \tau(X)$ of the space $X$ such that $\alpha < \beta$ implies $U_\alpha \subset U_\beta$ and $U_\alpha \neq X$ for any $\alpha < \lambda$. For any $x \in X$ let $O_x = U_\alpha$ where $\alpha$ is the first ordinal with $x \in U_\alpha$.

There exists a ($\kappa$-)compact $K \subset X$ such that $\bigcup\{O_x\colon x \in K\} = X$. Since $\mathcal{U}$ also covers $K$, there is $\alpha < \lambda$ such that $K \subset U_\alpha$. It is clear that $O_x \subset U_\alpha$ for every $x \in K$ so $\bigcup\{O_x\colon x \in K\} \subset U_\alpha \neq X$, a contradiction. $\square$

**Theorem 2.6.** *Pseudocompactness* (*which, for non-Tychonoff spaces we identify with feeble compactness*) *is self-dual with respect to neighbourhood assignments.*

**Proof.** Suppose that a space $X$ is not pseudocompact and take a discrete family $\{U_n\colon n \in \omega\} \subset \tau^*(X)$. Choose a point $x_n \in U_n$ for every $n \in \omega$. The sets $W = X \backslash \{x_n\colon n \in \omega\}$ and $U = \bigcup_{n\in\omega} U_n$ are open in $X$. If $x \in U$ then there is a unique $n \in \omega$ with $x \in U_n$; let $O_x = U_n$. If $x \in X \backslash U$ then let $O_x = W$.

If $X$ is dually pseudocompact then there is a pseudocompact $P \subset X$ such that $\bigcup\{O_x\colon x \in P\} = X$. This, evidently, implies that $P \cap U_n \neq \emptyset$ for any $n \in \omega$, so $\{U_n \cap P\colon n \in \omega\} \subset \tau^*(P)$ is discrete which is a contradiction. $\square$

**Proposition 2.7.** *The class of linearly Lindelöf spaces is self-dual with respect to neighbourhood assignments.*

**Proof.** Suppose that a space $X$ is in the dual class of linearly Lindelöf spaces. If $X$ is not linearly Lindelöf then there is an uncountable regular cardinal $\lambda$ and a cover $\mathcal{U} = \{U_\alpha\colon \alpha < \lambda\} \subset \tau(X)$ of the space $X$ such that $\alpha < \beta$ implies $U_\alpha \subset U_\beta$ and $U_\alpha \neq X$ for any $\alpha < \lambda$. For any $x \in X$ let $O_x = U_\alpha$ where $\alpha$ is the first ordinal with $x \in U_\alpha$.

There exists a linearly Lindelöf $K \subset X$ such that $\bigcup\{O_x\colon x \in K\} = X$. Since $\mathcal{U}$ also covers $K$, there is $\alpha < \lambda$ such that $K \subset U_\alpha$. It is clear that $O_x \subset U_\alpha$ for every $x \in K$ so $\bigcup\{O_x\colon x \in K\} \subset U_\alpha \neq X$, a contradiction. $\square$

**Theorem 2.8.** *If $\kappa$ is an infinite cardinal and a space $X$ is dually hereditarily $\kappa$-Lindelöf then $X$ is $\kappa$-Lindelöf.*

**Proof.** Take an open cover $\mathcal{U} = \{U_\alpha\colon \alpha < \lambda\}$ of the space $X$; given a point $x \in X$ let $\alpha(x) = \min\{\alpha < \lambda\colon x \in U_\alpha\}$ and $O_x = U_{\alpha(x)}$. For the neighbourhood assignment $\{O_x\colon x \in X\}$ there exists a set $Y \subset X$ such that $\mathrm{hl}(Y) \leqslant \kappa$ and $\bigcup\{O_y\colon y \in Y\} = X$. It turns out that

(∗) the set $P = \{\alpha(y)\colon y \in Y\}$ has cardinality at most $\kappa$.

Indeed, assume that $A \subset Y$ is a set such that $|A| = \kappa^+$ and $\alpha(x) \neq \alpha(y)$ for any distinct $x, y \in A$. Define a well-order $\prec$ on the set $A$ by declaring that $x \prec y$ if and only if $\alpha(x) < \alpha(y)$. The space $A$ is right-separated with respect to this order because, for any point $x \in A$, its neighbourhood $U_{\alpha(x)}$ contains no $y \in A$ with $x \prec y$. Since no hereditarily $\kappa$-Lindelöf space can have a right-separated subspace of cardinality $\kappa^+$, we obtained a contradiction thus proving $(*)$.

Apply $(*)$ to choose a set $Z \subset Y$ such that $|Z| \leqslant \kappa$ and $\{\alpha(x)\colon x \in Z\} = P$. If $y \in Y$ then there is $z \in Z$ with $\alpha(z) = \alpha(y)$ and hence $O_y = U_{\alpha(y)} = U_{\alpha(z)} = O_z$. This shows that $\bigcup\{O_z\colon z \in Z\} = \bigcup\{O_y\colon z \in Y\} = X$ so $\{O_z\colon z \in Z\}$ is a subcover of $\mathcal{U}$ of cardinality at most $\kappa$. $\square$

**Corollary 2.9.** *The following conditions are equivalent for any space $X$:*

(a) *$X$ is dually countable*;
(b) *$X$ is dually cosmic*;
(c) *$X$ is dually hereditarily Lindelöf*;
(d) *$X$ is Lindelöf.*

## 3. Star properties and pseudocompactness

In the paper [2] a space $X$ was defined to be star compact (Lindelöf) if, for any open cover $\mathcal{U}$ of the space $X$, there is a finite (countable) set $A \subset X$ such that $\mathrm{St}(A, \mathcal{U}) = X$. In this interpretation, the notion of star compactness coincides with countable compactness. On the other hand, in [15] a definition of the class of star-$\mathcal{P}$ spaces was given for any property/class $\mathcal{P}$. This notion gives, in case when $\mathcal{P} =$ "compact", a new interesting subclass of the class of pseudocompact spaces.

It is part of the folklore (see, e.g., [15]) that a space is pseudocompact if and only if it is star pseudocompact. As a consequence, any star compact space is pseudocompact; since being star-finite is equivalent to countable compactness [9], studying classes star determined by compactness-like properties actually means studying extensions of the class of countably compact spaces. In particular, star compact spaces form a class which lies between pseudocompact spaces and countably compact spaces. The purpose of this section is to give examples which show that the classes star determined by some classical compactness-like properties are distinct whenever the properties are distinct.

**Definition 3.1.** [15] Given a class (or a property) $\mathcal{P}$ of topological spaces say that a space $X$ is star-$\mathcal{P}$ if, for any open cover $\mathcal{U}$ of the space $X$, there is a subspace $Y \subset X$ such that $Y \in \mathcal{P}$ and $\mathrm{St}(Y, \mathcal{U}) = X$. The subspace $Y$ will be called *a kernel* of the cover $\mathcal{U}$. If the name of the class/property $\mathcal{P}$ is long-winded then the term "star-$\mathcal{P}$" becomes even more cumbersome so we use the phrase "star determined by $\mathcal{P}$" as a synonym.

It is natural to outline first some limits of "good" behaviour of star-$\mathcal{P}$ properties. Since we only deal with compact-like properties $\mathcal{P}$, the respective star-$\mathcal{P}$ spaces are all pseudocompact, so all positive results about them are basically the same as positive results about pseudocompact spaces. The proof of the following proposition is straightforward and left to the reader.

**Proposition 3.2.** *If a class $\mathcal{P}$ of spaces is invariant under continuous maps then the class of star-$\mathcal{P}$ spaces is also invariant under continuous maps. In particular, the classes of star compact spaces, spaces star determined by countably compact spaces, spaces star determined by compact metrizable spaces and spaces star determined by compact countable spaces are all preserved by continuous maps.*

**Remark 3.3.** We cannot expect a good behaviour of star-$\mathcal{P}$ properties with respect to products because there exists a countably compact space whose square is not pseudocompact. Therefore, the square of a star-$\mathcal{P}$ space can fail to be star-$\mathcal{P}$ for any compact-like property $\mathcal{P}$. Analogously, a closed subspace of a star-$\mathcal{P}$ space can fail to be star-$\mathcal{P}$ because all non-trivial star-$\mathcal{P}$ spaces are not countably compact so they have an infinite closed discrete subspace which is not even pseudocompact. It is easy to see that star-$\mathcal{P}$ properties are preserved by finite unions; however, any infinite discrete sum of such spaces fails to be pseudocompact if the summands are non-empty. Therefore only finite discrete sums preserve star-$\mathcal{P}$ properties.

**Proposition 3.4.** *There exists a pseudocompact first countable space which is not star determined by countably compact spaces.*

**Proof.** Take any maximal almost disjoint family $\mathcal{M}$ of infinite subsets of $\omega$. Recall that the Mrowka space $M$ which corresponds to $\mathcal{M}$ has the underlying set $\omega \cup \{x_A\colon A \in \mathcal{M}\}$ where $\omega$ is dense in $M$ and all points of $\omega$ are isolated in $M$ while the topology at every $x_A \in M\backslash\omega$ is given by the local base $\{\{x_A\} \cup (A\backslash F)\colon F$ is a finite subset of $A\}$. The space $M$ is pseudocompact and $D = M\backslash\omega$ is a closed discrete subspace of $M$ with $|D| = w(M)$.

In the paper [7] a space $X$ is called 1-starcompact if, for any open cover $\mathcal{U}$ of the space $X$, there is a finite $\mathcal{V} \subset \mathcal{U}$ such that $\mathrm{St}(\mathcal{V}, \mathcal{U}) = X$. It is an easy exercise that any star compact space is 1-starcompact; this, together with Lemma 2.2.4 of [7] shows that $M$ is not 1-starcompact and hence not star compact. Clearly, every countably compact subspace of $M$ is countable and hence compact so $M$ is not star determined by countably compact spaces. □

**Theorem 3.5.** *There exists a non-star compact space $X$ which is* 1*-starcompact in the sense of* [7] (*this is the same as being $1\frac{1}{2}$-starcompact in the sense of* [15]) *while $X$ is star determined by countably compact spaces.*

**Proof.** In the space $\omega^* = \beta\omega\backslash\omega$ every infinite closed set has cardinality $\mathbf{2}^{\mathfrak{c}}$ and there are $\mathbf{2}^{\mathfrak{c}}$-many infinite closed subsets of $\omega^*$. This makes it possible to construct, by a standard transfinite recursion, a set $G \subset \omega^*$ such that every compact subspace of $G$ is finite and every compact subspace of $H = \omega^*\backslash G$ is finite as well. It is easy to see that both sets $G$ and $H$ are countably compact.

Choose a countably infinite discrete subspace $D \subset G$ and let $X = G\backslash(\overline{D}\backslash D)$ (where closures are taken in $\omega^*$). It is clear that $D$ is a closed discrete subspace of $X$ so $X$ is not countably compact. If $X$ is star compact then, for any $\mathcal{U} \subset \tau(X)$ such that $\bigcup\mathcal{U} = X$ there is a compact $K \subset X$ with $\mathrm{St}(K, \mathcal{U}) = X$. However, all compact subspaces of $X$ are finite so $K$ has to be finite which proves that $X$ is star finite and hence countably compact; this contradiction shows that $X$ is not star compact.

We will prove simultaneously that the space $X$ is star determined by countably compact spaces and 1-starcompact; to this end fix an open cover $\mathcal{U}$ of the space $X$. Refining $\mathcal{U}$ if necessary we can assume, without loss of generality, that $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$ where $\overline{U} \cap \overline{D} = \emptyset$ for any $U \in \mathcal{U}_0$ and $\mathcal{U}_1 = \{U_d\colon d \in D\}$ where $U_d \cap D = \{d\}$ for any $d \in D$ and the family $\mathcal{U}_1$ is disjoint. Take a set $O(U) \in \tau(\omega^*)$ such that $O(U) \cap X = U$ for any $U \in \mathcal{U}$ and let $\mathcal{O} = \{O(U)\colon U \in \mathcal{U}\}$.

It is evident that the set $M = \omega^*\backslash(\bigcup\mathcal{O})$ is compact and $F = \overline{D}\backslash D \subset M$. We claim that $M\backslash F$ is finite. Indeed, if this is not true, then we can find a countably discrete infinite set $C \subset M\backslash F$. The subspace $D \cup C$ being discrete we conclude that $\overline{D} \cap \overline{C} = \emptyset$ and therefore $\overline{C}$ is an infinite compact subspace of $H$ which is a contradiction.

Since a countable family of non-empty open sets in $\omega^*$ cannot form a $\pi$-base at a point of $\omega^*$, there is $W \in \tau(M\backslash F, \omega^*)$ such that $U_d\backslash\overline{W} \neq \emptyset$ for any $d \in D$; choose a point $x_d \in U_d\backslash(W \cup \{d\})$. Then $\overline{\{x_d\colon d \in D\}} \cap (M\backslash F) = \emptyset$; the set $D \cup \{x_d\colon d \in D\}$ being discrete and $D \cap \{x_d\colon d \in D\} = \emptyset$ we also have $\overline{D} \cap \overline{\{x_d\colon d \in D\}} = \emptyset$. An immediate consequence is that the compact space $N = \overline{\{x_d\colon d \in D\}}$ is covered by the family $\mathcal{O}$ so there is a finite $\mathcal{U}' \subset \mathcal{U}$ with $N \subset \bigcup\mathcal{U}'$. Since $N$ intersects all elements of $\mathcal{U}_1$, any element of $\mathcal{U}_1$ has to meet an element of $\mathcal{U}'$; as a consequence, $\mathrm{St}(\bigcup\mathcal{U}', \mathcal{U}) \supset \bigcup\mathcal{U}_1$. Observe also that the set $K = \overline{\{x_d\colon d \in D\}} \cap X$ is countably compact and $\mathrm{St}(K, \mathcal{U}) \supset \bigcup\mathcal{U}_1$.

The space $X' = X\backslash D$ is countably compact so there is a finite set $E \subset X'$ for which $\mathrm{St}(E, \mathcal{U}) \supset X\backslash\overline{D}$. Therefore $L = E \cup K$ is a countably compact subspace for which $\mathrm{St}(L, \mathcal{U}) = X$ and hence the space $X$ is star determined by countably compact spaces. If we take a finite family $\mathcal{U}'' \subset \mathcal{U}$ such that $E \subset \bigcup\mathcal{U}''$ then $\mathcal{V} = \mathcal{U}' \cup \mathcal{U}''$ is a finite subfamily of $\mathcal{U}$ such that $\mathrm{St}(\bigcup\mathcal{V}, \mathcal{U}) = X$ so $X$ is 1-starcompact in the sense of [7]. □

**Theorem 3.6.** *There exists a star compact space $X$ which is not star determined by metrizable compact spaces and hence $X$ is not countably compact.*

**Proof.** The proof is an easier version of the proof of Theorem 3.5. Choose a countably infinite discrete subspace $D \subset \omega^* = \beta\omega\backslash\omega$ and let $X = \omega^*\backslash(\overline{D}\backslash D)$ (where again, closures are taken in $\omega^*$). It is clear that $D$ is a closed discrete subspace of $X$ so $X$ is not countably compact. Every metrizable compact subspace of $X$ is finite so the failure of $X$ to be countably compact implies that $X$ is not star determined by metrizable compact spaces.

To see that $X$ is star compact fix an open cover $\mathcal{U}$ of the space $X$. By refining $\mathcal{U}$ if necessary we can assume, without loss of generality, that $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$ where $\overline{U} \cap D = \emptyset$ for any $U \in \mathcal{U}_0$ and $\mathcal{U}_1 = \{U_d\colon d \in D\}$ where $U_d \cap D = \{d\}$ for any $d \in D$ and the family $\mathcal{U}_1$ is disjoint. Choose $x_d \in U_d\backslash\{d\}$ for any $d \in D$; the subspace $D \cup \{x_d\colon d \in D\}$ is discrete and $D \cap \{x_d\colon d \in D\} = \emptyset$. As a consequence, $\overline{D} \cap \overline{\{x_d\colon d \in D\}} = \emptyset$. The set $K = \overline{\{x_d\colon d \in D\}} \cap X$ is compact and $\mathrm{St}(K, \mathcal{U}) \supset \bigcup \mathcal{U}_1$.

As before, the space $X' = X \backslash D$ is countably compact so there is a finite set $E \subset X'$ for which $\mathrm{St}(E, \mathcal{U}) \supset X \backslash \overline{D}$. Therefore $L = E \cup K$ is a compact subspace for which $\mathrm{St}(L, \mathcal{U}) = X$ and hence the space $X$ is star compact. □

**Corollary 3.7.** *The classes of pseudocompact spaces, the spaces star determined by countably compact spaces, star compact spaces and countably compact spaces are all distinct.*

In fact, even the classes star determined by countable and metrizable compact spaces are distinct; we will see this later. Therefore it seems to be an interesting problem whether any two non-homeomorphic metrizable compact spaces determine distinct star classes.

**Proposition 3.8.** *The "Tychonoff plank" is star determined by convergent sequences, i.e., for every open cover $\mathcal{U}$ of the Tychonoff plank $P$ there is a convergent sequence $S \subset P$ such that $\mathrm{St}(S, \mathcal{U}) = P$. Therefore, even a space star determined by convergent sequences need not be countably compact.*

**Proof.** Recall that the Tychonoff plank $P$ is the subspace $(\omega_1 + 1) \times (\omega + 1) \backslash \{(\omega_1, \omega)\}$ of the product $(\omega_1 + 1) \times (\omega + 1)$. If $\mathcal{U}$ is an open cover of $P$, fix a set $U_n \in \mathcal{U}$ such that $(\omega_1, n) \in \mathcal{U}$ for any $n \in \omega$. There exists $\alpha < \omega_1$ such that $(\beta, n) \in U_n$ for any $n \in \omega$ and $\beta \geqslant \alpha$. Clearly, the set $S' = \{\alpha\} \times (\omega + 1)$ is a convergent sequence such that $\mathrm{St}(S', \mathcal{U}) \supset W = \bigcup_{n \in \omega} U_n$.

The set $P \backslash W$ is closed in $\omega_1 \times (\omega + 1)$ and hence countably compact so there is a finite $F \subset P$ for which $\mathrm{St}(F, \mathcal{U}) \supset P \backslash W$. Therefore $S = S' \cup F$ is a convergent sequence with $\mathrm{St}(S, \mathcal{U}) = P$. □

**Proposition 3.9.** *If a space $X$ is not countably compact then, for any $n \in \mathbb{N}$, the space $X \times n = X \times \{0, \ldots, n-1\}$ is not star determined by less than $n$ convergent sequences.*

**Proof.** The set $X_i = X \times \{i\}$ is open in $X \times n$ for any $i < n$; since $X_i$ is not countably compact, it is not star finite [9] and so we can find an open cover $\mathcal{U}_i$ of the space $X_i$ such that there is no finite set $K \subset X_i$ with $\mathrm{St}(K, \mathcal{U}_i) = X_i$.

The family $\mathcal{U} = \bigcup_{i<n} \mathcal{U}_i$ is an open cover of $X \times n$. Suppose that $k < n$ and $\mathcal{U}$ has a kernel $S = \bigcup_{i<k} S_i$ where $S_i$ is a convergent sequence for each $i < k$. There exists $j < n$ such that $X_j$ contains no limit points of $S$ and hence $S' = S \cap X_j$ is finite. It is immediate that the set $\mathrm{St}(S \backslash S', \mathcal{U})$ does not meet $X_j$ so $\mathrm{St}(S', \mathcal{U}) = \mathrm{St}(S', \mathcal{U}_j) = X_j$ which is a contradiction with our choice of the family $\mathcal{U}_j$. □

**Corollary 3.10.** *If $P$ is the Tychonoff plank then for any $n \in \mathbb{N}$ the space $P \times n$ is star determined by $n$-many convergent sequences but not by $(n-1)$-many convergent sequences.*

**Proof.** That the space $P \times n$ is star determined by $n$-many convergent sequences is an easy consequence of Proposition 3.8. The space $P$ is not countably compact so Proposition 3.9 can be applied to see that it is not star determined by less than $n$ convergent sequences. □

**Proposition 3.11.** *Given a cardinal $\kappa > \omega$, let $\Sigma = \{x \in \mathbb{D}^\kappa\colon |x^{-1}(1)| \leqslant \omega\}$ be a $\Sigma$-product in the Cantor cube $\mathbb{D}^\kappa$. Then, for any countable set $A \subset \mathbb{D}^\kappa$, the space $X = A \cup \Sigma$ is star determined by metrizable compact spaces.*

**Proof.** If $\mathcal{U}$ is an open cover of $X$ then there is a countable $\mathcal{U}_0 \subset \mathcal{U}$ such that $A \subset \bigcup \mathcal{U}_0$. Since $\Sigma$ is dense in $X$, there is a countable set $S \subset \Sigma$ which meets every element of $\mathcal{U}_0$. Since $\Sigma$ is countably compact, there is a finite $F \subset \Sigma$ such that $\mathrm{St}(F, \mathcal{U}) \supset \Sigma$. The closure of any countable subset of $\Sigma$ is compact and metrizable so $K = F \cup \overline{S}$ is a metrizable compact subspace of $X$ such that $\mathrm{St}(K, \mathcal{U}) = X$. □

**Theorem 3.12.** *There exists a space $X$ which is star determined by metrizable compact spaces but not by compact countable spaces.*

**Proof.** Let $\pi_0 : \mathbb{D}^{\omega_1} \to \mathbb{D}^{\omega}$ and $\pi_1 : \mathbb{D}^{\omega_1} \to \mathbb{D}^{\omega_1\setminus\omega}$ be the natural projections; denote by $\Sigma$ the subspace $\{x \in \mathbb{D}^{\omega_1}\colon |x^{-1}(1)| \leqslant \omega\}$ of $\mathbb{D}^{\omega_1}$. Take a faithfully indexed sequence $S = \{s_n\colon n \in \omega\} \subset \mathbb{D}^{\omega_1\setminus\omega} \setminus \pi_1(\Sigma)$ such that $S$ converges to a point $s \in \mathbb{D}^{\omega_1\setminus\omega} \setminus (S \cup \pi_1(\Sigma))$. We leave it to the reader to verify that

($*$) there is a disjoint family $\{O_n\colon n \in \omega\} \subset \tau^*(\mathbb{D}^{\omega})$ such that for any set $D \subset \mathbb{D}^{\omega}$ with $D \cap O_n \neq \emptyset$ for all $n \in \omega$, the set $\overline{D}$ is uncountable.

Pick a point $x_n \in \pi_0^{-1}(O_n)$ such that $\pi_1(x_n) = s_n$ for any $n \in \omega$; then the set $D = \{x_n\colon n \in \omega\}$ does not meet $\Sigma$ and it follows from $\pi_1(D) \subset S$ that $\overline{D} \cap \Sigma = \emptyset$. Therefore the set $D$ is a closed discrete subspace of the space $X = \Sigma \cup D$. By Proposition 3.11, the space $X$ is star determined by metrizable compact spaces.

To see that $X$ is not star determined by countable compact spaces observe that $\mathcal{U} = \{\Sigma\} \cup \{\pi_0^{-1}(O_n) \cap X\colon n \in \omega\}$ is an open cover of $X$. If a set $K \subset X$ is a compact kernel of $\mathcal{U}$ then $K \cap \pi_0^{-1}(O_n) \neq \emptyset$ for each $n \in \omega$. As a consequence, $\pi_0(K) \cap O_n \neq \emptyset$ for every $n \in \omega$ so ($*$) implies that $\pi_0(K) = \overline{\pi_0(K)}$ is uncountable. Therefore $K$ is uncountable as well, i.e., $\mathcal{U}$ is a cover which witnesses that $X$ is not star determined by compact countable spaces. □

## 4. Open problems

To show that this topic is far from being exhausted we present the following list of problems which might require new methods for their solution.

**Problem 4.1.** Is Lindelöfness a self-dual property with respect to neighbourhood assignments?

**Problem 4.2.** Recall that a space $X$ is weakly Lindelöf if, for any open cover $\mathcal{U}$ of the space $X$, there is a countable $\mathcal{U}' \subset \mathcal{U}$ such that $\bigcup \mathcal{U}'$ is dense in $X$. Is weak Lindelöfness a self-dual property with respect to neighbourhood assignments?

**Problem 4.3.** Suppose that a space $X$ is dually $\sigma$-compact, i.e., for any neighbourhood assignment $\{O_x\colon x \in X\}$ of the space $X$, there is a $\sigma$-compact subspace $A \subset X$ such that $\bigcup\{O_x\colon x \in A\} = X$? Must $X$ be Lindelöf? How about dually Lindelöf $\Sigma$-spaces?

**Problem 4.4.** Is it true that every Lindelöf space is dually second countable, i.e., for any neighbourhood assignment $\{O_x\colon x \in X\}$ of a Lindelöf space $X$, there is a second countable $A \subset X$ such that $\bigcup\{O_x\colon x \in A\} = X$? This is a weakening of a van Douwen's problem in which it is asked whether the set $A$ can be made closed and discrete.

**Problem 4.5.** Suppose that a space $X$ is dually separable, i.e., for any neighbourhood assignment $\{O_x\colon x \in X\}$ there is a separable subspace $Y \subset X$ such that $\bigcup\{O_x\colon x \in Y\} = X$. Must $X$ be weakly Lindelöf?

**Problem 4.6.** Is any dually second countable (or dually cosmic) space a $D$-space? It is Lindelöf by Corollary 2.9(b) but it is an open problem due to E. van Douwen as to whether every Lindelöf space is a $D$-space.

**Problem 4.7.** Suppose that $X$ is a first countable space such that, for any neighbourhood assignment $\{O_x\colon x \in X\}$ there is a subspace $Y \subset X$ such that $c(Y) = \omega$ and $\bigcup\{O_x\colon x \in Y\} = X$ (i.e., $X$ is in the class dual to the spaces with the Souslin property). Is it true that $|X| \leqslant \mathfrak{c}$? A positive answer would generalize the Hajnal–Juhász inequality $|X| \leqslant 2^{\chi(X)\cdot c(X)}$ for the case when $\chi(X) = c(X) = \omega$.

**Problem 4.8.** Is a star compact space metrizable if it has a $G_\delta$-diagonal?

**Problem 4.9.** Suppose that $K$ and $L$ are metrizable compact spaces such that the classes star determined by $K$ and $L$ coincide. Must $K$ and $L$ be homeomorphic?

**Problem 4.10.** Must every star compact topological group be countably compact?

**Problem 4.11.** Is it true that any Tychonoff space embeds as a closed subspace in a space which is star determined by convergent sequences?

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