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Covering compacta by discrete subspaces ☆

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Abstract

For any space X, denote by $\mathrm{dis}(X)$ the smallest (infinite) cardinal κ such that κ many discrete subspaces are needed to cover X. It is easy to see that if X is any crowded (i.e. dense-in-itself) compactum then $\mathrm{dis}(X) \geqslant \mathfrak{m}$, where $\mathfrak{m}$ denotes the additivity of the meager ideal on the reals. It is a natural, and apparently quite difficult, question whether in this inequality $\mathfrak{m}$ could be replaced by $\mathfrak{c}$. Here we show that this can be done if X is also hereditarily normal.

Moreover, we prove the following mapping theorem that involves the cardinal function $\mathrm{dis}(X)$. If $f : X \to Y$ is a continuous surjection of a countably compact T_2 space X onto a perfect T_3 space Y then $|\{y \in Y\colon\ f^{-1}y \text{ is countable}\}| \leqslant \mathrm{dis}(X)$.

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In this note we use standard topological terminology and notation, as e.g. in [1] or [3]. Our aim is to study the following new cardinal functions.

Definition 1. For any space X we let $\mathrm{dis}(X)$ ($\mathrm{ls}(X)$, respectively $\mathrm{rs}(X)$) denote the smallest infinite cardinal κ such that X can be covered by κ many discrete (left separated, respectively right separated) subspaces.

Since discrete spaces are both left and right separated, we clearly have $\mathrm{ls}(X) \leqslant \mathrm{dis}(X)$ and $\mathrm{rs}(X) \leqslant \mathrm{dis}(X)$. Any crowded (i.e. dense-in-itself) compactum X has a crowded closed subspace Y that maps irreducibly onto the interval $[0, 1]$. Then Y is separable, moreover, as any right separated subspace of Y is nowhere dense in Y, we have

$$\mathrm{dis}(X) \geqslant \mathrm{rs}(X) \geqslant \mathrm{rs}(Y) \geqslant \mathrm{N}(Y) = \mathrm{N}\big([0, 1]\big) = \mathfrak{m}.$$

Here $\mathrm{N}(X)$ is the Novák-number of a space X, i.e. the smallest number of nowhere-dense sets needed to cover X; hence $\mathfrak{m}$ is also known as the additivity of the meager ideal on the reals.

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It is less obvious to see but, as it has been shown in [2], the inequality $\mathrm{ls}(X) \geqslant \mathfrak{m}$ is also valid for any crowded compactum X.

Now, it is a natural question to raise if these inequalities are sharp, in particular the following attractive problem can be formulated and will be examined below. (Of course, here $\mathfrak{c}$ denotes the cardinality of the continuum.)

Problem 2. Is there a crowded compactum X with $\mathrm{dis}(X) < \mathfrak{c}$?

As we have seen above, such a space X has a separable closed subspace $Y \subset X$. Since the weight of a separable compactum is at most $\mathfrak{c}$, any discrete subspace of Y has size $\leqslant \mathfrak{c}$. Consequently, we must have $|Y| = \mathfrak{c}$ and so by the Čech–Pospišil theorem some point of Y must have character $< \mathfrak{c}$. Our next result yields a much stronger statement in this vein.

Theorem 3. *Every compactum X with $\mathrm{rs}(X) > 1$ contains a point p such that*

$$\chi(p, X) < \mathrm{rs}(X).$$

Proof. As we have seen above, if $\mathrm{rs}(X) < \mathfrak{m}$ then X is scattered, hence X contains (a dense set of) isolated points, i.e. points of character 1. So assume now that $\kappa = \mathrm{rs}(X) \geqslant \mathfrak{m} > \omega$. Assume also, indirectly, that $\chi(x, X) \geqslant \kappa$ holds for all $x \in X$.

By definition, there is a sequence $\langle S_\alpha\colon \alpha \in \kappa\rangle$ such that each S_α is a right separated subspace of X and $X = \bigcup\{S_\alpha\colon \alpha \in \kappa\}$. By transfinite recursion on $\alpha \in \kappa$ we shall then define a *decreasing* sequence of non-empty closed sets $H_\alpha \subset X$ such that

$$\psi(H_\alpha, X) = \chi(H_\alpha, X) \leqslant |\alpha| + \omega,$$

moreover $H_\alpha \cap S_\alpha = \emptyset$ holds for each $\alpha \in \kappa$. Since, by the compactness of X, we have $\bigcap\{H_\alpha\colon \alpha \in \kappa\} \neq \emptyset$, this clearly leads to a contradiction.

So let $\alpha \in \kappa$ and assume that H_β has been suitably defined for each $\beta \in \alpha$. Let us set then

$$\widetilde{H_\alpha} = \bigcap\{H_\beta\colon \beta \in \alpha\}.$$

Clearly we have

$$\psi(\widetilde{H_\alpha}, X) = \chi(\widetilde{H_\alpha}, X) \leqslant |\alpha| + \omega,$$

hence if $\widetilde{H_\alpha} \cap S_\alpha = \emptyset$ holds then we may set $H_\alpha = \widetilde{H_\alpha}$. So assume now that $\widetilde{H_\alpha} \cap S_\alpha \neq \emptyset$ and therefore it has an isolated point, say x_α. But x_α cannot be an isolated point of $\widetilde{H_\alpha}$ since otherwise we would have $\chi(x_\alpha, X) \leqslant |\alpha| + \omega < \kappa$. Thus if U is any open neighbourhood of x_α such that $U \cap (\widetilde{H_\alpha} \cap S_\alpha) = \{x_\alpha\}$, then $U \cap (\widetilde{H_\alpha} \setminus \{x_\alpha\}) \neq \emptyset$ and $\psi(U \cap (\widetilde{H_\alpha} \setminus \{x_\alpha\}), X) \leqslant |\alpha| + \omega$. We may then finish by defining H_α as any non-empty closed subset of $U \cap (\widetilde{H_\alpha} \setminus \{x_\alpha\})$ that also satisfies $\psi(H_\alpha, X) \leqslant |\alpha| + \omega$. The existence of such a closed subset is obvious from the regularity of the space X. □

It immediately follows from Theorem 3 that if X is a compactum satisfying $\mathrm{dis}(X) = \omega_1$ then the points of first countability are dense (even G_δ-dense) in X. This leads us to the following weaker version of our main Problem 2:

Problem 4. Is it provable that $\mathrm{dis}(X) = \mathfrak{c}$ for each *first countable* crowded compactum X?

(Note that, by Archangelskiĭ's theorem, we have $\mathrm{dis}(X) \leqslant |X| = \mathfrak{c}$ in this case.) We are sorry to admit that we could not answer this problem, however we do have a partial positive solution to Problem 2 for *hereditarily normal* spaces.

Theorem 5. *If X is a hereditarily normal crowded compactum then either* $\mathrm{ls}(X) \geqslant \mathfrak{c}$ *or* $\mathrm{rs}(X) \geqslant \mathfrak{c}$ *holds, hence surely* $\mathrm{dis}(X) \geqslant \mathfrak{c}$.

Proof. Assume, indirectly, that we have both $\mathrm{ls}(X) < \mathfrak{c}$ and $\mathrm{rs}(X) < \mathfrak{c}$. As we have noted above, we may also assume that X is separable and consequently $\varrho(X)$, the number of regular open subsets of X, is equal to $\mathfrak{c}$. Of course, in this case we also have $|X| = \mathfrak{c}$.

Let us assume now that $\mathfrak{c}$ is a regular cardinal. Then first using $\mathrm{ls}(X) < \mathfrak{c}$ and then $\mathrm{rs}(X) < \mathfrak{c}$ it is easy to find a subset of X of size $\mathfrak{c}$ that is both left and right separated. But then, see e.g. [3, 2.12], there is also a discrete subset $D \subset X$ with $|D| = \mathfrak{c}$. If, on the other hand, $\mathfrak{c}$ is singular then a similar argument yields us for each $\kappa < \mathfrak{c}$ a discrete subset of X of size κ, in particular a discrete $D \subset X$ with $|D| = \mathrm{cf}(\mathfrak{c})$. Thus we have established that X contains a discrete subspace of size $\mathrm{cf}(\mathfrak{c})$ whether or not $\mathfrak{c}$ is regular.

But then the hereditary normality of X implies $\varrho(X) \geqslant 2^{\mathrm{cf}(\mathfrak{c})} = 2^{\omega\cdot\mathrm{cf}(\mathfrak{c})} = \mathfrak{c}^{\mathrm{cf}(\mathfrak{c})} > \mathfrak{c}$, a contradiction. (The finishing argument is widely known as Jones' lemma, see e.g. [1, 2.1.10].) □

Of course, Theorem 5 would be much more esthetic if one could prove that say $\mathrm{ls}(X) \geqslant \mathfrak{c}$ is always valid for a crowded T_5 space X (or the same with $\mathrm{ls}(X)$ replaced with $\mathrm{rs}(X)$).

Now we turn to our last theorem that establishes a rather surprising connection between certain continuous maps and the cardinal function $\mathrm{dis}(X)$. This result also sheds some light on potential counterexamples to Problem 4. A space is *perfect* if all closed sets are G_δ.

Theorem 6. *Let $f : X \to Y$ be a continuous surjection from a countably compact T_2 space X onto a perfect T_3 space Y. Then we have*

$$\left|\left\{y \in Y\colon\ f^{-1}(y) \text{ is countable}\right\}\right| \leqslant \mathrm{dis}(X).$$

Moreover, if $\mathrm{dis}(X) = \omega$ *then we even have* $|Y| \leqslant \omega$.

Proof. Let us start by noting that Y is also countably compact, being the continuous image of X. This implies that Y is first countable because any G_δ point in a countably compact T_3 space has countable character. This in turn implies that f is a closed map. Indeed, if $F \subset X$ is closed then $f[F]$ is a countably compact subset of Y and as such it is closed in Y because Y is first countable and T_2.

Next we show that for any set $A \subset X$ we have

$$\left|A \setminus f^{-1}\left(f[A']\right)\right| \leqslant \omega,$$

where, as usual, A' denotes the derived set of all limit points of the set A. To see this, we first note that by our above remark $f[A']$ is closed and hence a G_δ set in Y. Thus we may write

$$f[A'] = \bigcap\{G_n\colon\ n < \omega\},$$

where each set G_n is open in Y. Consequently, we have

$$f^{-1}\left(f[A']\right) = \bigcap\left\{f^{-1}(G_n)\colon\ n < \omega\right\}.$$

Now, for each $n < \omega$ we have $A' \subset f^{-1}(G_n)$, hence the countable compactness of X implies that $A \setminus f^{-1}(G_n)$ is finite, consequently

$$A \setminus f^{-1}\left(f[A']\right) = \bigcup\left\{A \setminus f^{-1}(G_n)\colon\ n < \omega\right\}$$

is indeed countable.

Let us assume now that Y is uncountable. We claim that in this case for every discrete subspace D of X there is a closed set $F \subset X \setminus D$ such that $f[F]$ is uncountable as well.

To see this, we distinguish two cases. First, if $f[\overline{D}]$ is countable then $Y \setminus f[\overline{D}]$ is an uncountable F_σ-set, hence clearly there is an uncountable closed set Z in Y that is disjoint from $f[\overline{D}]$. Obviously, then $F = f^{-1}(Z)$ is as required. (In this case we have not used that D is discrete.)

If, on the other hand, $f[\overline{D}]$ is uncountable then by the previous observation we have $|D \setminus f^{-1}(f[D'])| \leqslant \omega$, hence $|f[\overline{D}]| = |f[D']| > \omega$. But '$D$ is discrete' just means that $D \cap D' = \emptyset$, hence in this case we may simply set $F = D'$.

Now assume that we have $\mathrm{dis}(X) = \omega$, hence

$$X = \bigcup\{D_n\colon\ n < \omega\}$$

where each D_n is a discrete subspace of X. Our aim is to show that in this case Y is countable. Indeed, if this were false then, using the above claim, we could define by a straightforward recursion a decreasing sequence

$\langle F_n\colon n < \omega\rangle$ of closed subsets of X such that $F_n \cap D_n = \emptyset$ and $f[F_n]$ is uncountable for each $n < \omega$. But then we have $\bigcap\{F_n\colon n < \omega\} \neq \emptyset$ as X is countably compact, contradicting that X is covered by the D_n's. We have thus established the second part of the theorem, namely that $\operatorname{dis}(X) = \omega$ implies $|Y| \leqslant \omega$.

So now we turn to the first (main) part. Let us start by noting that every countable closed subset of X is in fact compact. However, it is well known that any countable compact T_2 space is homeomorphic to a (countable successor) ordinal (taken, of course, with its natural order topology). With this in mind, we introduce here the following piece of notation: If Z is any topological space that is homeomorphic to some ordinal then $\alpha(Z)$ will denote the smallest such ordinal. We are going to make use of the following easy to prove fact: If $\alpha(Z)$ is a successor ordinal then there is a point $z \in Z$ such that

$$\alpha\big(Z \setminus \{z\}\big) < \alpha(Z).$$

Let us now denote by $I(\xi)$ the following statement: For every continuous surjection f from a countably compact T_2 space X onto a perfect T_3 space Y we have

$$\big|\big\{y \in Y\colon \alpha\big(f^{-1}(y)\big) \leqslant \xi\big\}\big| \leqslant \operatorname{dis}(X).$$

Since in this part of the proof we may assume that $\operatorname{dis}(X) > \omega$, it will clearly suffice to prove that $I(\xi)$ holds for all countable ordinals ξ. Indeed, this is so because if $f^{-1}(y)$ is countable then $\alpha(f^{-1}y)$ exists and is a countable ordinal.

Of course, the proof will proceed by transfinite induction. Since $\alpha(f^{-1}(y))$ is always a successor, the limit steps of the induction are trivial. So assume now that $\xi = \eta + 1$ and $I(\eta)$ is valid. We want to show that then so is $I(\xi)$. Assume, indirectly, that this is not the case, hence

$$\big|\big\{y \in Y\colon \alpha\big(f^{-1}(y)\big) = \xi\big\}\big| > \operatorname{dis}(X).$$

Let Z be the set that appears on the left-hand side of this inequality. For each $y \in Z$ we have $\alpha(f^{-1}(y)) = \xi = \eta + 1$ and hence we may pick by the above remark a point $x_y \in f^{-1}(y)$ such that

$$\alpha\big(f^{-1}(y) \setminus \{x_y\}\big) \leqslant \eta.$$

Since $|Z| > \operatorname{dis}(X)$ we may clearly find a subset $Z_0 \subset Z$ with $|Z_0| > \operatorname{dis}(X)$ such that $D = \{x_y\colon y \in Z_0\}$ is a discrete subspace of X. Recall that we have $|D \setminus f^{-1}(f[D'])| \leqslant \omega$, hence if we set $Z_1 = f[D] \cap f[D']$ then $Z_1 \subset Z_0$ and $|Z_0 \setminus Z_1| \leqslant \omega$, consequently we have $|Z_1| = |Z_0| > \operatorname{dis}(X)$.

Let us now consider the restriction g of the map f to the closed subspace D' of X. Then g is a continuous surjection from D' onto $f[D']$, hence the inductive hypothesis $I(\eta)$ may be applied to it to conclude that

$$\big|\big\{y \in f[D']\colon \alpha\big(g^{-1}(y)\big) \leqslant \eta\big\}\big| \leqslant \operatorname{dis}(D') \leqslant \operatorname{dis}(X).$$

On the other hand, we have $f[D'] \supset Z_1$ and for each $y \in Z_1$,

$$g^{-1}(y) \subset f^{-1}(y) \setminus \{x_y\}$$

holds because $D \cap D' = \emptyset$. But then, by the choice of x_y, for all $y \in Z_1$ we have $\alpha(g^{-1}(y)) \leqslant \eta$. This is a contradiction since $|Z_1| > \operatorname{dis}(X)$.

This contradiction completes the proof of the transfinite induction and with it the proof of our theorem. □

Assume now that X is a crowded first countable compactum with $\operatorname{dis}(X) < \mathfrak{c}$, i.e. a counterexample to Problem 4. Then any uncountable closed subspace of X is of cardinality $\mathfrak{c}$ and non-scattered. Hence it is an immediate corollary of Theorem 6 that if e.g. f is any continuous surjection of X onto the interval $[0, 1]$ (and such maps always exist) then for almost all (more precisely: for all but $\operatorname{dis}(X)$ many) points $r \in [0, 1]$ we have $|f^{-1}(r)| = \mathfrak{c}$. In some, admittedly non-precise, sense this means that a counterexample to Problem 4 must be "complicated".

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