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EXTREMAL PSEUDOCOMPACT ABELIAN GROUPS ARE COMPACT METRIZABLE

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ABSTRACT. Every pseudocompact Abelian group of uncountable weight has both a proper dense pseudocompact subgroup and a strictly finer pseudocompact group topology.

1. INTRODUCTION

All topological groups here are assumed to satisfy the Hausdorff separation axiom. A pseudocompact group G is said to be *r-extremal* [resp. *s-extremal*] if G admits no strictly finer pseudocompact group topology [resp. G has no proper dense pseudocompact subgroup]. Early formulations of these notions appeared in [6], [7]. From the fact that a pseudocompact space of countable weight is compact and metrizable it follows readily (as in [7, 2.3]) that every pseudocompact group of countable weight is both r-extremal and s-extremal. It is natural to ask whether there are extremal pseudocompact groups of uncountable weight. This question has generated much attention during the last two decades. See [1], [12] and [10] for more information. An affirmative answer was given in [7] for zero-dimensional Abelian groups. In [2] it was shown that no pseudocompact Abelian group of cardinality greater than $\mathfrak{c}$ is s-extremal. For partial answers in the class of connected groups, see for example [3], [12] and [1].

The aim of this paper is to answer the question for Abelian groups.

Theorem 1.1. *A pseudocompact Abelian group of uncountable weight is neither r-extremal nor s-extremal.*

We keep this presentation short by invoking several essential results established in the literature. We plan in [5] to present a polished, complete and self-contained proof of Theorem 1.1.

We announced our results at the annual meeting of the American Mathematical Society in January, 2006 [4].

2. PRELIMINARIES

In this section we fix notation and we cite the results we need from the literature.

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The symbol wX denotes the weight of a topological space X. A subspace of a space X is *G_δ-dense* in X if it meets every nonempty G_δ-subset of X. If X is a set and κ a cardinal number, then $[X]^{\leq\kappa}$ denotes $\{A \subseteq X : |A| \leq \kappa\}$.

For Abelian groups we use additive notation. Let G be an Abelian group. If $A \subseteq G$, then $\langle\langle A\rangle\rangle$ denotes the subgroup of G generated by A. A subset X of G is called *independent* if for every $x \in X$ we have $\langle\langle\{x\}\rangle\rangle \cap \langle\langle X \setminus \{x\}\rangle\rangle = \{0\}$. If A is a subgroup of G, then a subset X of G is said to be *independent over* A if it is independent and $\langle\langle X\rangle\rangle \cap A = \{0\}$. The cardinality of a maximal independent set of elements of infinite order is called the *torsion-free rank* of G, here denoted $r_0(G)$. It is known that $r_0(G)$ is an invariant of G, i.e., all such maximal independent subsets of G have the same cardinality. It is clear that if $h\colon G \to H$ is a surjective homomorphism, then $r_0(H) \leq r_0(G)$. See [11, pp. 85-86] for additional details. The torsion subgroup of an Abelian group G is denoted by tG.

If G is a (not necessarily Abelian) totally bounded group, then $\overline{G}$ denotes its (compact) Weil completion. It was shown in [8] that a topological group G is pseudocompact if and only if it is G_δ-dense in $\overline{G}$. Hence a dense subgroup of a pseudocompact group G is pseudocompact if and only if it is G_δ-dense in G.

Let G be a topological group. Then

$$\Lambda(G) = \{N \subseteq G : N \text{ is a closed, normal, } G_\delta\text{-subgroup of } G\}.$$

Now we collect some information needed later in our proof of the main result.

Theorem 2.1. *Let G be a pseudocompact group such that $wG > \omega$, and let $N \in \Lambda(G)$. Then*

(a) [9, 3.3] *G/N is compact and metrizable,*
(b) [7, 6.2] *N is pseudocompact, and*
(c) [3, 2.7] *$wN = wG$.*

Lemma 2.2 ([3, 2.13(b),(c)]). *Let G be a pseudocompact group and let $G = \bigcup_{n<\omega} A_n$, where each A_n is a subgroup of G. Then there exist $N \in \Lambda(G)$ and $n < \omega$ such that $A_n \cap N$ is G_δ-dense in N.*

Theorem 2.3 ([1, 4.4], [12, 3.7.1]). *Let G be a pseudocompact Abelian group. If G contains a proper, dense pseudocompact subgroup H such that G/H can be mapped homomorphically onto some nondegenerate compact group, then G is not r-extremal.*

Theorem 2.4 ([1, 5.7], [12, 6.4.2]). *Let G be a pseudocompact Abelian group of uncountable weight. If there exists $N \in \Lambda(G)$ such that no connected $M \in \Lambda(G)$ is contained in N, then G is neither r-extremal nor s-extremal.*

Theorem 2.5 ([2, 4.5], [1, 5.10], [12, 7.3]). *Let G be a pseudocompact Abelian group of uncountable weight such that $r_0(G) > \mathfrak{c}$. Then G is neither r-extremal nor s-extremal.*

3. Lemmas

In this section we collect some simple results to be used later. The technique used in the proof of Lemma 3.1 is well-known, and was used in many earlier results. See e.g., [3, 2, 12, 1]. For the benefit of the reader we provide the (simple) details. (Note added September 15, 2006. The referee has pointed out that a proof of Lemma 3.1 is also available in the preprint [10].)

Lemma 3.1. *Let G be a pseudocompact Abelian group, and let A be a G_δ-dense subgroup of some $N \in \Lambda(G)$ such that $r_0(N/A) \geq \mathfrak{c}$. Then G contains a G_δ-dense subgroup H such that $r_0(G/H) \geq \mathfrak{c}$.*

Proof. The conditions imply that there is a subset X of $N \setminus A$ of elements of infinite order such that $|X| = \mathfrak{c}$ and X is independent over A. Split X into two disjoint sets X_0 and X_1, each of cardinality $\mathfrak{c}$.

By Theorem 2.1(a), the number of cosets of N in G is at most $\mathfrak{c}$. (In fact, either $|G/N| < \omega$ or $|G/N| = \mathfrak{c}$.) Let $\{a_\alpha + N : \alpha < \lambda\}$ be a faithful enumeration of G/N. We assume without loss of generality that $a_0 = 0$. By recursion on $\alpha < \lambda$ we will choose $x_\alpha \in X_0 \cup \{0\}$ such that

$$\langle\!\langle X_1 \rangle\!\rangle \cap (\langle\!\langle \{a_\beta + x_\beta : \beta \leq \alpha\} \rangle\!\rangle + A) = \{0\}.$$

Let $x_0 = 0$. Let $\alpha < \lambda$ and suppose that x_β has been defined for all $\beta < \alpha$. Put $B_\alpha = \langle\!\langle \{a_\beta + x_\beta : \beta < \alpha\} \rangle\!\rangle$. Then $|B_\alpha| < \mathfrak{c}$ and $\langle\!\langle X_1 \rangle\!\rangle \cap (B_\alpha + A) = \{0\}$. Suppose that for every $x \in X_0$ we have that

$$\langle\!\langle X_1 \rangle\!\rangle \cap (\langle\!\langle B_\alpha \cup \{a_\alpha + x\} \rangle\!\rangle + A) \neq \{0\}.$$

Then for every $x \in X_0$ there exist $b_x \in B_\alpha$, $n_x \in \mathbb{Z}$, $p_x \in A$ and $q_x \in \langle\!\langle X_1 \rangle\!\rangle \setminus \{0\}$ such that

$$q_x = b_x + n_x(a_\alpha + x) + p_x. \tag{$\dagger$}$$

Note (since $q_x \notin B_\alpha + A$) that no n_x is equal 0 . Since $|X_0| = \mathfrak{c}$, there are distinct $x, y \in X_0$, $n \in \mathbb{Z} \setminus \{0\}$ and $b \in B_\alpha$ such that $n = n_x = n_y$ and $b = b_x = b_y$. But then by subtracting the equation (†) for x and y, we get

$$n(x - y) = q_x - q_y + p_y - p_x \in \langle\!\langle X_1 \rangle\!\rangle + A,$$

which contradicts the independence of X over A. This completes the transfinite recursion.

Now put $B = \bigcup_{\alpha<\lambda} B_\alpha$. Then $\langle\!\langle X_1 \rangle\!\rangle \cap (B + A) = \{0\}$, hence $r_0(G/(B + A)) \geq |X_1| = \mathfrak{c}$. It is clear that $B + A$ is G_δ-dense in G. □

Lemma 3.2. *Let κ be an infinite cardinal. Suppose that $\mathcal{A}$ is a family of subsets of 2^κ with the following properties:*

1. *if $\mathcal{B} \in [\mathcal{A}]^{\leq\kappa}$, then $\bigcap \mathcal{B} \in \mathcal{A}$, and*
2. *each element of $\mathcal{A}$ has cardinality 2^κ.*

Then there is a countably infinite family $\mathcal{B}$ of subsets of 2^κ such that

- (i) *$\mathcal{B}$ is pairwise disjoint, and*
- (ii) *if $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|A \cap B| = 2^\kappa$.*

Proof. We give 2^κ the standard Tychonov product topology. Let $\mathcal{V}$ be the collection of all nonempty clopen subsets V of 2^κ for which there is an element $A(V) \in \mathcal{A}$ such that $|V \cap A(V)| < 2^\kappa$. Clearly, $|\mathcal{V}| \leq \kappa$. Let $\mathcal{D} = \{A(V) : V \in \mathcal{V}\}$, $Y = \bigcap \mathcal{D}$, and $\widetilde{V} = \bigcup \mathcal{V}$. Now $|V \cap Y| \leq |V \cap A(V)| < 2^\kappa$ for every $V \in \mathcal{V}$, so

$$\left|\widetilde{V} \cap Y\right| < 2^\kappa$$

since 2^κ has cofinality at least κ^+. Then $|Y| = 2^\kappa$ by (1) and (2), hence $|Y \setminus \widetilde{V}| = 2^\kappa$. There is consequently a countably infinite pairwise disjoint family $\mathcal{B}$ of clopen subsets of 2^κ such that $B \cap (Y \setminus \widetilde{V}) \neq \emptyset$ for every $B \in \mathcal{B}$. To see that $\mathcal{B}$ is as required, pick arbitrary $B \in \mathcal{B}$ and $A \in \mathcal{A}$. If $|B \cap A| < 2^\kappa$, then $B \in \mathcal{V}$ and hence $B \subseteq \widetilde{V}$, which contradicts the fact that $B \cap (Y \setminus \widetilde{V}) \neq \emptyset$. $\square$

Remark 3.3. The inclusion $\bigcup \mathcal{B} \subseteq 2^\kappa$ is necessarily proper (since 2^κ is compact). Replacing any one element of $\mathcal{B}$ by the complement in 2^κ of the union of the remaining elements, we may hence assume without loss of generality that $\mathcal{B}$ is a partition.

4. Proof of Theorem 1.1

We now present the proof of our main result. By Theorem 2.5, it suffices to consider groups G of torsion-free rank at most $\mathfrak{c}$. Furthermore, by Theorem 2.4 we may assume that every $N \in \Lambda(G)$ contains a connected $M \in \Lambda(G)$. Henceforth, let G be a pseudocompact Abelian group of uncountable weight satisfying those two conditions.

Lemma 4.1. *If H is a nontrivial connected subgroup of G, then $r_0(H) = \mathfrak{c}$.*

Proof. It is clear that $r_0(H) \leq \mathfrak{c}$. Let $0 \neq x \in H$ and let h be a continuous homomorphism from H to $\mathbb{T}$ such that $h(x) \neq h(0)$. Then $h(H) = \mathbb{T}$ since H (and hence $h(H)$) is connected. It follows that $\mathfrak{c} \geq r_0(H) \geq r_0(\mathbb{T}) = \mathfrak{c}$, as asserted. $\square$

Since $G \in \Lambda(G)$, there is a connected $C \in \Lambda(G)$. Hence $r_0(C) = \mathfrak{c}$ by Theorem 2.1(b),(c) and Lemma 4.1.

Let $\overline{C}$ be the closure of C in $\overline{G}$. Then $\overline{C}$ is a compact, connected group, hence is divisible [13, Theorem 24.25]. By [13, Theorem A.14] or [11, Theorem 23.1], there is a cardinal number λ such that $\overline{C}$ is (algebraically) isomorphic to

$$\bigoplus_{\alpha<\lambda} \mathbb{Q}_\alpha \oplus t\overline{C},$$

where each $\mathbb{Q}_\alpha$ is a copy of the group of rational numbers $\mathbb{Q}$. Then $\overline{C}/t\overline{C} = \bigoplus_{\alpha<\lambda} \mathbb{Q}_\alpha$. Let $\pi\colon \overline{C} \to \bigoplus_{\alpha<\lambda} \mathbb{Q}_\alpha$ be the natural homomorphism. For $x \in \overline{C}$ let $S(x) = \{\alpha < \lambda : \pi(x)_\alpha \neq 0_\alpha\}$, and for $E \subseteq \overline{C}$ let $S(E) = \bigcup_{x\in E} S(x)$.

Lemma 4.2. *If $N \in \Lambda(C)$, then $|S(\pi(N))| = \mathfrak{c}$.*

Proof. Since $N \in \Lambda(G)$, we may assume without loss of generality that N is connected. Moreover, N is nontrivial by Theorem 2.1(c). So $r_0(N) = \mathfrak{c}$ by Lemma 4.1. That $|S(\pi(N))| = \mathfrak{c}$ is then clear. $\square$

Writing $S = S(\pi(C))$, we have $|S| = \mathfrak{c}$. Hence $\lambda \geq \mathfrak{c}$, and

$$C \subseteq \bigoplus_{\alpha\in S} \mathbb{Q}_\alpha \oplus \bigoplus_{\alpha\notin S} \{0_\alpha\} \oplus t\overline{C}.$$

For every $\beta \in S$, let $\rho_\beta\colon \bigoplus_{\alpha\in S} \mathbb{Q}_\alpha \to \mathbb{Q}_\beta$ be the projection.

For every nonempty $A \subseteq S$, put

$$G(A) = C \cap \Big(\bigoplus_{\alpha\in A} \mathbb{Q}_\alpha \oplus \bigoplus_{\alpha\notin A} \{0_\alpha\} \oplus t\overline{C} \Big),$$

and let

$$\mathcal{A} = \{A \subseteq S : \text{there is } N \in \Lambda(C) \text{ such that } N \subseteq G(A)\}.$$

Lemma 4.3. *$\mathcal{A}$ is closed under countable intersections, and every $A \in \mathcal{A}$ has size $\mathfrak{c}$.*

Proof. That $\mathcal{A}$ is closed under countable intersections is clear, since if $\mathcal{B}$ is any family of subsets of S, then

$$\bigcap_{B\in\mathcal{B}} G(B) = G(\bigcap \mathcal{B})$$

and $\Lambda(G)$ is closed under countable intersections.

Now take an arbitrary $A \in \mathcal{A}$. We want to prove that $|A| = \mathfrak{c}$. Take $N \in \Lambda(C)$ such that $N \subseteq G(A)$. Then $\pi(N) \subseteq A$, so $\mathfrak{c} = |S| \geq |A| \geq |\pi(N)| = \mathfrak{c}$ by Lemma 4.2. □

By Lemma 3.2 and Remark 3.3, there consequently is a (faithfully indexed) partition $\mathcal{B} = \{B_n : n < \omega\}$ of S such that $|B_n \cap A| = \mathfrak{c}$ for each $B_n \in \mathcal{B}$, $A \in \mathcal{A}$. For every $n < \omega$, let

$$V_n = G\Big(\bigcup_{i\leq n} B_i\Big).$$

Then $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq \cdots$, and $C = \bigcup_{n<\omega} V_n$. By Theorem 2.1(b) and Lemma 2.2, there exist $N \in \Lambda(C)$ and $m < \omega$ such that $H := V_m \cap N$ is G_δ-dense in N. We may assume without loss of generality that $m = 0$, i.e., that $V_m = V_0 = G(B_0)$.

Lemma 4.4. $r_0(N/H) \geq \mathfrak{c}$.

Proof. We will prove that there is a subset X of N of cardinality $\mathfrak{c}$ such that

(1) each $x \in X$ has infinite order,
(2) X is independent,
(3) $\langle\langle X\rangle\rangle \cap H = \{0\}$

(hence $\langle\langle X\rangle\rangle$ is isomorphic to $\bigoplus_{\alpha<\mathfrak{c}} \mathbb{Z}_\alpha$, where each $\mathbb{Z}_\alpha$ is a copy of the group of integers $\mathbb{Z}$). Choose $x_0 \in N \setminus G(B_0)$ and define $W_0 = B_0$. Let $0 < \alpha < \mathfrak{c}$ and suppose that x_β and W_β have been defined for all $\beta < \alpha$. Then, set

$$W_\alpha = B_0 \cup \bigcup_{\beta<\alpha} S(x_\beta),$$

and observe that

$$|B_1 \cap W_\alpha| = \Big|B_1 \cap \bigcup_{\beta<\alpha} S(x_\beta)\Big| < \mathfrak{c}.$$

Hence $W_\alpha \notin \mathcal{A}$, since B_1 meets every element of $\mathcal{A}$ in a set of size $\mathfrak{c}$, which means that $N \not\subseteq G(W_\alpha)$; let x_α be any point in $N \setminus G(W_\alpha)$. This completes the transfinite construction.

We claim that $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ satisfies (1), (2) and (3). To prove this, let $\alpha < \mathfrak{c}$, and let $n \in \mathbb{Z} \setminus \{0\}$ be arbitrary. By construction we have

$$x_\alpha \notin G\Big(B_0 \cup \bigcup_{\beta<\alpha} S(x_\beta)\Big),$$

so $S(x_\alpha) \not\subseteq B_0 \cup \bigcup_{\beta<\alpha} S(x_\beta)$; let $\gamma \in S(x_\alpha)$ witness that relation. Then $\rho_\gamma(\pi(x_\alpha)) \neq 0$ and $\rho_\gamma(\pi(x_\beta + h)) = 0$ for every $\beta < \alpha$ and $h \in H$. Then clearly x_α has infinite order, and

$$nx_\alpha \notin \langle\langle\{x_\beta : \beta < \alpha\}\rangle\rangle + H,$$

as required. □

Now we complete the proof of Theorem 1.1. From Lemmas 3.1 and 4.4 we conclude that G has a proper dense pseudocompact subgroup H such that $r_0(G/H) \geq \mathfrak{c}$. This proves that G is not s-extremal. To see that G is not r-extremal, note first that $r_0(G/H) \geq \mathfrak{c}$, so G/H contains a subgroup isomorphic to $\bigoplus_{\alpha<\mathfrak{c}} \mathbb{Z}_\alpha$. The latter can be mapped homomorphically onto $\mathbb{T}$, and since homomorphisms into a divisible group always extend by [13, Theorem A.7], we are done by Theorem 2.3.

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