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HOMOGENEOUS SPACES AND TRANSITIVE ACTIONS BY ANALYTIC GROUPS

JAN VAN MILL

Abstract

If X is homogeneous, analytic, and strongly locally homogeneous, then there is an analytic group acting transitively on X. There is an example of an analytic space on which some separable metrizable group acts transitively, but on which no analytic group acts transitively.

1. *Introduction*

Note: all spaces under discussion are separable and metrizable. Let X be a homogeneous space. It was asked in [**8**] whether there is a topological group G admitting a transitive action on X. This question was recently answered in the negative in [**9**]. For X locally compact, the question has an affirmative answer. If $\alpha X = X \cup \{\infty\}$ denotes its Alexandrov one-point compactification, then for G one can take the subgroup of the homeomorphism group of αX consisting of those homeomorphisms that fix ∞ (the topology is the compact-open topology).

A space X is *strongly locally homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ which is supported on B (that is, f is the identity outside B) and moves x to y. This notion is due to Ford [**3**]. Most of the well-known homogeneous spaces are SLH. An SLH-space does not need to be either connected or homogeneous: the topological sum of the 1-sphere and the 2-sphere is an easy counterexample. A *connected* SLH-space, however, is homogeneous. The pseudo-arc is an example of a homogeneous continuum that is not SLH. It was shown in [**8**] that if X is homogeneous and SLH, then there is a topological group G admitting a transitive (and micro-transitive [**1**]) action on X.

A space is *analytic* if it is a continuous image of the space of irrational numbers. It is known that every Polish space is analytic. The aim of this paper is, among other things, to show that the results in [**8**] can be used quite easily to show that if X is homogeneous, SLH, and Borel/analytic, then there is a Borel/analytic group acting transitively on X. Our main result is an example of an analytic space on which some (separable metrizable) group acts transitively, but on which no *analytic* group acts transitively.

2. *Preliminaries*

The collection of analytic subsets of a space X is denoted by $\mathbf{\Sigma}^1_1(X)$. A space is X *coanalytic* if there is a Polish space $\hat{X}$ which contains X such that $\hat{X} \setminus X \in \mathbf{\Sigma}^1_1(X)$. The collection of coanalytic subsets of a space X is denoted by $\mathbf{\Pi}^1_1(X)$. It is well known, and easy to prove, that both $\mathbf{\Sigma}^1_1(X)$ and $\mathbf{\Pi}^1_1(X)$ are closed under countable unions and countable intersections. For details about these concepts, see, for example, [**5**].

If X is a space, then $\mathcal{H}(X)$ denotes its group of homeomorphisms. The identity function on a set X is denoted by 1_X. If $A \subseteq X$, then $\mathcal{H}(X|A) = \{h \in \mathcal{H}(X) : h{\restriction}A = 1_A\}$. If X is compact, then the compact-open topology on $\mathcal{H}(X)$ is Polish and compatible with its group structure.

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It is easy to see that the compact-open topology on $\mathcal{H}(X)$ coincides with the topology of uniform convergence on $\mathcal{H}(X)$.

An *action* α of a topological group G on a space X is a continuous function

$$(g,x) \mapsto gx \colon G \times X \xrightarrow{\alpha} X$$

such that $ex = x$ for every $x \in X$ and $g(hx) = (gh)x$ for $g, h \in G$ and $x \in X$ (here, e denotes the neutral element of G). It is easily seen that for each $g \in G$ the function $x \mapsto gx$ is a homeomorphism of X whose inverse is the function $x \mapsto g^{-1}x$. The action of G on X is *transitive* if for all $x, y \in X$ there is an element $g \in G$ such that $gx = y$ (and hence X is homogeneous).

LEMMA 2.1. *Let G be a topological group, acting on a space X by the action α. If Y is a space containing X, and K is compact in X, then $\alpha^{-1}(K)$ is closed in $G \times Y$.*

Proof. Observe that

$$\alpha^{-1}(K) = \{(g, g^{-1}x) : g \in G,\ x \in K\}.$$

Now let $(g_n, g_n^{-1}x_n)_n$ be a sequence in $\alpha^{-1}(K)$ converging to some element $(g, y) \in G \times Y$. Since K is compact, we may assume without loss of generality that $x_n \to x$ for some $x \in K$. Since $g_n^{-1} \to g^{-1}$, we get $g_n^{-1}x_n \to g^{-1}x$; that is, $y = g^{-1}x$, as required. □

Let (X, ϱ) be a metric space. For every $x \in X$ and $\varepsilon > 0$, we put

$$B_\varepsilon(x) = \{y \in X : \varrho(x,y) < \varepsilon\}.$$

A zero-dimensional space X is *strongly homogeneous* if all of its nonempty clopen subsets are homeomorphic. It is not difficult to see that every strongly homogeneous space is homogeneous.

LEMMA 2.2. *Let X be a strongly homogeneous space containing more than one point. Then X admits a zero-dimensional compactification γX having the following property: for all nonempty clopen subsets B and B' of γX, there exists a homeomorphism $f\colon B \to B'$ such that $f(B \cap X) = B' \cap X$.*

Proof. Since X is strongly homogeneous, there are a countable subalgebra $\mathcal{B}$ of the Boolean algebra of clopen subsets of X and a countable subgroup F of $\mathcal{H}(X)$ such that $\mathcal{B}$ is a base and for all nonempty $B, B' \in \mathcal{B}$ with $X \setminus B \neq \emptyset \neq X \setminus B'$ there is an element $f \in F$ such that $f(B) = B'$. Let γX be the Stone space of $\mathcal{B}$. Observe that every $f \in F$ extends to a homeomorphism $\bar{f}\colon \gamma X \to \gamma X$. By applying the fact that X has no isolated points, it now easily follows that γX is as required. □

If X is a space, then $\mathcal{K}(X)$ denotes the space of nonempty compact subsets of X with the Vietoris topology. The standard example of a coanalytic space that is not analytic is $\mathcal{K}(\mathbb{Q})$, where $\mathbb{Q}$ denotes the space of rational numbers. This is due to Hurewicz [**4**] (see also [**5**, Exercise 33.5]). It was shown by Michalewski [**6**, Theorem 7] that $\mathcal{K}(\mathbb{Q})$ is strongly homogeneous (and hence homogeneous). By Lemma 2.2, we may consequently think of $\mathcal{K}(\mathbb{Q})$ as a dense subset X of the Cantor set C, such that for all nonempty clopen subsets B and B' of C, there is a homeomorphism $f\colon B \to B'$ such that $f(B \cap X) = B' \cap X$.

Throughout the paper, $\mathbb{N}$ stands for the set of all positive integers. Let X and Y be spaces, and let $f\colon X \to Y$ be continuous. Inverse images of one-point sets under f are called *fibers* of f. We say that f is *monotone* if all fibers $f^{-1}(y)$ are connected.

Let X be a space, and let A and B be two disjoint closed subsets of X. A *partition* between A and B is a closed subset $S \subseteq X$ such that $X \setminus S$ can be written as the disjoint union of open sets U and V with $A \subseteq U$ and $B \subseteq V$.

3. *Positive results*

We will now sketch the proofs of our positive results. Since they follow rather easily from the results in [**8**], we will be brief.

We adopt the terminology in [**8**, Corollary 3.2]. Let X be a space. For $f, g \in \mathcal{H}(X)$, we put

$$S(f,g) = \{x \in X : f(x) = g(x)\}.$$

Then $S(f,g)$ is evidently a closed subset of X.

Proposition 3.1. *Let X be homogeneous and* SLH. *Then there are a compactification γX of X and a subset A of $\mathcal{H}(\gamma X|X)$ such that*

(i) *the 'natural action' of A on X is transitive*;
(ii) *A is a continuous image of a closed subspace of $\mathbb{N}^\infty \times \mathcal{H}(\gamma X) \times X \times X$.*

Proof. Let γX, $\mathcal{G}$ and $\mathcal{W}$ be as in [**8**, Corollary 3.2]. If X is finite, then there is nothing to prove, so we assume without loss of generality that X is infinite. This implies that $\mathcal{G}$ is infinite as well. We endow $\mathcal{G}$ with the discrete topology, and put

$$\begin{aligned} Z = \{(g_i)_i, (h_i)_i, f, x, y) \in \mathcal{G}^\infty \times \mathcal{G}^\infty \times \mathcal{H}(\gamma X) \times X^2 :& \\ (\forall\, i \in \mathbb{N})\ (\gamma X \setminus B_{1/i}(x) \subseteq S(g_i, f) \text{ and } \gamma X \setminus B_{1/i}(y) \subseteq S(h_i, f^{-1})\}.& \end{aligned}$$

It is left as an exercise to the reader to prove that Z is closed in

$$\mathcal{G}^\infty \times \mathcal{G}^\infty \times \mathcal{H}(\gamma X) \times X^2 \approx \mathbb{N}^\infty \times \mathcal{H}(\gamma X) \times X^2.$$

Claim 1. *If $((g_i)_i, (h_i)_i, f, x, y) \in Z$, then $f \in \mathcal{H}(\gamma X|X)$.*

Take an arbitrary $p \in \gamma X \setminus X$. There is an i such that $p \notin B_{1/i}(x)$. This means that $p \in S(g_i, f)$; that is, $f(p) = g_i(p) \in \gamma X \setminus X$. It follows similarly that $f^{-1}(p) \in \gamma X \setminus X$.

Take arbitrary $x, y \in X$. By applying [**8**, Lemma 3.4], we see that there are a sequence $(g_i)_i$ in $\mathcal{G}$ and a decreasing neighborhood base $(U_i)_i$ of x in γX such that

(i) the infinite left-product $f = \lim_{i\to\infty} g_i \circ \ldots \circ g_1$ is a homeomorphism of γX such that $f(x) = y$;
(ii) $f(X) = X$;
(iii) if $p \notin U_i$, then $f(p) = g_i \circ \ldots \circ g_1(p)$.

For every i, let $\alpha_i = g_i \circ \ldots \circ g_1$, and $\beta_i = g_1^{-1} \circ \ldots \circ g_i^{-1}$.

Claim 2. *There are functions $\xi, \eta\colon \mathbb{N} \to \mathbb{N}$ such that $((\alpha_{\xi(n)})_n, (\beta_{\eta(n)})_n, f, x, y) \in Z$.*

Pick an arbitrary $n \in \mathbb{N}$. There is an $i \in \mathbb{N}$ such that $U_i \subseteq B_{1/n}(x)$; hence by (3),

$$\gamma X \setminus B_{1/n}(x) \subseteq \gamma X \setminus U_i \subseteq S(\alpha_i, f).$$

So put $\xi(n) = i$. This defines the function ξ, and η can be found in a similar way.

Let A be the image of Z under the projection $\mathcal{G}^\infty \times \mathcal{G}^\infty \times \mathcal{H}(\gamma X) \times X^2 \to \mathcal{H}(\gamma X)$. Then by Claims 1 and 2, A is as required. $\square$

If G is a topological group and $A \in \mathbf{\Sigma}_1^1(G)$, then the subgroup $\tilde{A}$ of G generated by A belongs to $\mathbf{\Sigma}_1^1(G)$ as well. This is clear since $\tilde{A}$ is the union of countably many continuous images of finite powers of A, and $\mathbf{\Sigma}_1^1(G)$ is closed under countable unions.

THEOREM 3.2. *Let X be homogeneous and* SLH*. If X is absolutely Borel/analytic, then there is an absolutely Borel/analytic group G that acts transitively on X.*

Proof. Let γX and A be as in Proposition 3.1 for X. Since A is analytic, so is the subgroup A_0 of $\mathcal{H}(\gamma X|X)$ generated by A. Observe that A_0 acts transitively on X since it contains A. So we are done if the group that we are after *must* be analytic. For the remaining part of the proof, assume that X is absolutely Borel. It is clear from the definition that $\mathcal{H}(\gamma X|X) \in \mathbf{\Pi}_1^1(\mathcal{H}(\gamma X))$; hence by the Lusin separation theorem [**5**, Theorem 14.7] there is a Borel set A_1 in $\mathcal{H}(\gamma X|X)$ that contains A_0. The subgroup A_1' of $\mathcal{H}(\gamma X|X)$ generated by A_1 is analytic, and is contained in $\mathcal{H}(\gamma X|X)$. So by another application of the Lusin separation theorem there is a Borel subset A_2 of $\mathcal{H}(\gamma X|X)$ containing A_1', and so on. At the end of this process, $H = \bigcup_{n<\omega} A_n$ is a Borel subgroup of $\mathcal{H}(\gamma X|X)$. (A similar trick is used in the proof of [**2**, Theorem 2.2.7].) $\square$

Since a zero-dimensional homogeneous space is evidently SLH, we get the following corollary.

COROLLARY 3.3. *Let X be a homogeneous zero-dimensional absolute Borel set. Then there is a* (*zero-dimensional*) *Borel group G that admits a transitive action on X.*

(It is not completely obvious that the group G can be chosen to be zero-dimensional. However, it is easily seen that the compactification γX in [**8**, Section 3] can be chosen to be zero-dimensional, provided that X is. Then the group that we get from it is zero-dimensional. The details of checking this are left to the reader.)

REMARK 3.4. Let X be a homogeneous coanalytic SLH-space. We do not know whether there is a coanalytic group which admits a transitive action on X.

REMARK 3.5. Let X be a homogeneous Polish SLH-space. The question naturally arises as to whether there is a Polish group acting transitively on X. This is indeed the case; see [**9**]. The group that we get from [**8**] is not Polish, but a suitable modification of it turns out to be *Polishable*; that is, it has a stronger Polish group topology (see, for example, [**10**] for more information on such groups).

4. *Negative results*

We will now construct an analytic space on which some topological group acts transitively, but on which no analytic group acts transitively.

A space X is *countable dense homogeneous* if for all countable dense subsets D and E in X there is a homeomorphism $h\colon X \to X$ with $h(D) = E$. By a standard back-and-forth technique, it is easy to show that a Polish SLH-space is countable dense homogeneous. For details, see, for example, [**7**, Theorem 1.6.9]. By noting that a connected countable dense homogeneous space is homogeneous [**7**, Corollary 1.6.8], it is easy to verify that the same proof yields the following stronger result.

LEMMA 4.1. *Let X be connected, Polish and* SLH*. If D and E are disjoint countable dense subsets of X, then for all $x, y \in X \setminus (D \cup E)$ there is a homeomorphism $f\colon X \to X$ such that $f(D) = D$, $f(E) = E$, and $f(x) = y$.*

Let $\mathbb{S}$ denote the 2-sphere $\mathbb{S}^2$. It is clear that $\mathbb{S}$ is SLH, and has the property that a nonempty connected open subset remains connected after the removal of an arbitrary countable set.

PROPOSITION 4.2. *There are disjoint countable dense subsets D and E of $\mathbb{S}$, and elements $f, f_n \in \mathcal{H}(\mathbb{S})$, $n \in \mathbb{N}$, such that*

(i) *for every $n \in \mathbb{N}$, $f_n(D) = D$, $f_n(E) = E$;*
(ii) $f(D) = D \cup E$;
(iii) $\lim_{n\to\infty} f_n = f$.

Proof. To begin with, we define similar sets and homeomorphisms in $\mathbb{R}$. To this end, put

$$D' = \mathbb{N}\pi + \mathbb{Q}, \qquad E' = \mathbb{Q}.$$

Let $(d_n)_n$ be a sequence of rational numbers converging to π. Define $f', f'_n \in \mathcal{H}(\mathbb{R})$, $n \in \mathbb{N}$, by

$$f'_n(x) = x - d_n, \qquad f'(x) = x - \pi.$$

We extend these homeomorphisms over the two-point compactification $[\infty, \infty]$ of $\mathbb{R}$ in the trivial way. Observe that $f'_n(\pm\infty) = \pm\infty$, $n \in \mathbb{N}$, and $f'(\pm\infty) = \pm\infty$. Consider the square $\mathbb{D} = [-\infty, +\infty]^2$, and let

$$D = D' \times D', \qquad E = E' \times E'.$$

Finally, define $f, f_n \in \mathcal{H}(\mathbb{D})$, $n \in \mathbb{N}$, by

$$f(x) = \big(f'(x_1), f'(x_2)\big), \qquad f_n(x) = \big(f'_n(x_1), f'_n(x_2)\big).$$

An easy check shows that D, E, f and $(f_n)_n$ are as required, except for the fact that they are subsets or, respectively, homeomorphisms of $\mathbb{D}$ and not of $\mathbb{S}$. But this can easily be fixed by collapsing the boundary of $\mathbb{D}$ to a single point. $\square$

THEOREM 4.3. *There is an analytic space on which some topological group acts transitively, but on which no analytic group acts transitively.*

Proof. Let $X \in \mathbf{\Pi}^1_1(\mathsf{C}) \setminus \mathbf{\Sigma}^1_1(\mathsf{C})$ be such that for all nonempty clopen subsets B and B' of C there exists a homeomorphism $h\colon B \to B'$ such that $h(B \cap X) = B' \cap X$; see Section 2. In addition, let D, E, $(f_n)_n$ and f be as in Proposition 4.2. Put $Z = \mathsf{C} \times \mathbb{S}$, and

$$Y = Z \setminus ((\mathsf{C} \times D) \cup (X \times E)).$$

Since $\mathbf{\Sigma}^1_1(Z)$ is closed under countable intersections, we have $Y \in \mathbf{\Sigma}^1_1(Z)$. It is clear that $Y \notin \mathbf{\Pi}^1_1(Z)$, since $Z \setminus Y$ contains a closed copy of X. We claim that Y is as required.

For every $x \in \mathsf{C}$ and nonempty open *connected* set V in $\mathbb{S}$, put $S_V(x) = (\{x\} \times V) \cap Y$. Observe that $S_V(x)$ is connected, being homeomorphic to V minus a countable (dense) set.

Let $U \subseteq \mathsf{C}$ be clopen and nonempty. In addition, let $V \subseteq \mathbb{S}$ be nonempty open and connected. Then

$$\pi_{U,V}\colon (U \times V) \cap Y \to U$$

denotes the restriction to $(U \times V) \cap Y$ of the projection $U \times V \to U$.

CLAIM 1. *$\pi_{U,V}$ is open, surjective and monotone.*

This is clear, since $\pi^{-1}_{U,V}(x) = S_V(x)$ is a dense connected subset of $\{x\} \times V$ for every $x \in U$.

CLAIM 2. *If $x \in \mathsf{C} \setminus X$ and $y \in X$, then there is a homeomorphism $g \in \mathcal{H}(Z|Y)$ such that $g(S_{\mathbb{S}}(x)) = S_{\mathbb{S}}(y)$.*

Partition $\mathsf{C} \setminus \{x\}$ into nonempty clopen sets $\{R_n : n \in \mathbb{N}\}$. Similarly, partition $\mathsf{C} \setminus \{y\}$ into nonempty clopen sets $\{S_n : n \in \mathbb{N}\}$. We assume that for all $n \neq m$ we have $R_n \neq R_m$ and $S_n \neq S_m$. For every n, let $\xi_n \colon R_n \to S_n$ be a homeomorphism such that $\xi_n(R_n \cap X) = S_n \cap X$. Now define $g \colon Z \to Z$ by

$$g(p,q) = \begin{cases} (\xi_n(p), f_n(q)) & (p \in R_n,\ n \in \mathbb{N}), \\ (y, f(q)) & (p = x). \end{cases}$$

An easy check shows that g is as required.

Claim 3. *If $(a,b),(c,d) \in Y$, then there is a homeomorphism $g \in \mathcal{H}(Z|Y)$ such that $g(a,b) = (c,d)$.*

By Claim 2, it suffices to consider the case when $a = c \in X$. By Lemma 4.1 there is a homeomorphism $\eta \colon \mathbb{S} \to \mathbb{S}$ such that $\eta(D) = D$, $\eta(E) = E$, and $\eta(b) = d$. Now define $g \colon Z \to Z$ by

$$g(x,y) = (x, \eta(y)).$$

Then g is clearly as required.

So we conclude that $\mathcal{H}(Z|Y)$ acts transitively on Y. We will now show that such a group cannot be chosen to be analytic. Striving for a contradiction, assume that G is an analytic group acting transitively on Y by the action $\alpha \colon G \times Y \to Y$.

Let $z \in Z$ and $g \in G$ be arbitrary. If U is a neighborhood of (g,z) in $G \times Z$, then U intersects $G \times Y$ since Y is dense in Z. This implies that the set

$$\Theta_{g,z} = \bigcap\{\overline{\alpha(U)} : U \text{ is a neighborhood of } (g,z) \text{ in } G \times Z\}$$

is nonempty by compactness (here, 'closure' means closure in Z). We will prove that $\Theta_{g,z}$ is in fact a singleton. By the continuity of α, this is certainly the case if $z \in Y$, for then $\alpha(g,z) = gz$ is the unique element in $\Theta_{g,z}$. So we assume that $z \in Z \setminus Y$ and, striving for a contradiction, we assume that $\Theta_{g,z}$ contains at least two distinct points, say z_0 and z_1.

Since $Z \setminus Y$ is zero-dimensional, being the union of countably many closed zero-dimensional subspaces [**7**, Theorem 3.2.8], there is a partition K in Z between $\{z_0\}$ and $\{z_1\}$ such that $K \subseteq Y$; see [**7**, Corollary 3.1.6]. Write $Z \setminus K$ as $L \cup M$, where L and M are disjoint open subsets such that $z_0 \in L$ and $z_1 \in M$.

Claim 4. $\overline{\alpha^{-1}(L)} \cap \overline{\alpha^{-1}(M)} \subseteq \alpha^{-1}(K)$ *(here, 'closure' means closure in $G \times Z$).*

To prove this, assume that there is an element

$$(h,(p,q)) \in \overline{\alpha^{-1}(L)} \cap \overline{\alpha^{-1}(M)} \setminus \alpha^{-1}(K). \tag{1}$$

By Lemma 2.1, $\alpha^{-1}(K)$ is closed in $G \times Z$; hence there are open neighborhoods U of h in G, V of p in C, and W of q in $\mathbb{S}$ such that

$$(U \times (V \times W)) \cap \alpha^{-1}(K) = \emptyset. \tag{2}$$

We may assume without loss of generality that V is clopen, and that W is connected. Consider the map

$$\pi_{V,W} \colon (V \times W) \cap Y \to V.$$

The function

$$\phi = 1_U \times \pi_{V,W} \colon U \times ((V \times W) \cap Y) \to U \times V$$

is open, surjective and monotone (Claim 1). Since α is continuous, equation (2) consequently implies that each fiber of ϕ is contained either in $\alpha^{-1}(L)$ or in $\alpha^{-1}(M)$. Since ϕ is open, there are complementary open subsets S and T of $U \times V$ such that

$$\phi^{-1}(S) \subseteq \alpha^{-1}(L), \qquad \phi^{-1}(T) \subseteq \alpha^{-1}(M).$$

Hence $\{S \times W, T \times W\}$ is an open partition of $U \times (V \times W)$, and we may assume without loss of generality that

$$(h, (p, q)) \in S \times W.$$

Observe that

$$\alpha(S \times W) = \alpha(\phi^{-1}(S)) \subseteq L,$$

and hence $S \times W$ is a neighborhood of $(h, (p, q))$ that misses $\alpha^{-1}(M)$. But this contradicts inclusion (1).

Now put

$$L' = \overline{\alpha^{-1}(L)} \setminus \alpha^{-1}(K), \qquad M' = \overline{\alpha^{-1}(M)} \setminus \alpha^{-1}(K).$$

Then by Claim 4 and Lemma 2.1, L' and M' are disjoint open subsets of $G \times Z$ that cover $(G \times Z) \setminus \alpha^{-1}(K)$. In addition, $L' \cap (G \times Y) = \alpha^{-1}(L)$ and $M' \cap (G \times Y) = \alpha^{-1}(M)$. Since $(g, z) \in G \times (Z \setminus Y)$, we may assume without loss of generality that $(g, z) \in L'$. But then L' is a neighborhood of (g, z) such that

$$\alpha(L') = \alpha(\alpha^{-1}(L)) = L.$$

Hence

$$\Theta_{g,z} \subseteq \overline{L} \subseteq L \cup K,$$

but this contradicts the fact that $z_1 \in \Theta_{g,z}$.

Claim 5. *The action $\alpha\colon G \times Y \to Y$ can be extended to an action $\beta\colon G \times Z \to Z$.*

Indeed, for $(g, z) \in G \times Z$, let $\beta(g, z)$ be the unique point in $\Theta_{g,z}$. That β is continuous is trivial, and that it is an action is easy to verify. (See also the proof of [**2**, Theorem 2.2.7].)

We are now ready for the final contradiction. Since β extends α and $\mathsf{C} \times D$ is σ-compact, we have

$$Z \setminus Y \supseteq \beta(G \times (\mathsf{C} \times D)) \in \mathbf{\Sigma}^1_1(Z)$$

since G is analytic. But $Z \setminus Y$ is in $\mathbf{\Pi}^1_1(Z) \setminus \mathbf{\Sigma}^1_1(Z)$, and thus there are $x \in X$ and $p \in E$ such that

$$(x, p) \notin \beta\big(G \times (\mathsf{C} \times D)\big). \tag{3}$$

Now consider the set $\beta(G \times \{(x, p)\})$. It clearly also belongs to $\mathbf{\Sigma}^1_1(Z)$. Observe that by (3), we have

$$\beta(G \times \{(x, p)\}) \cap (\mathsf{C} \times D) \subseteq \beta(G \times \{(x, p)\}) \cap \beta(G \times (\mathsf{C} \times D)) = \emptyset. \tag{4}$$

Hence $\beta(G \times \{(x, p)\}) \subseteq X \times E$. The projection $X \times E \to X$ maps $\beta(G \times \{(x, p)\})$ onto an element of $\mathbf{\Sigma}^1_1(Z)$. Since $X \notin \mathbf{\Sigma}^1_1(Z)$, there is an element $x' \in X$ not contained in that projection. As a consequence, by (4), we have

$$(\beta(G \times \{(x, p)\}) \cap (\{x'\} \times (D \cup E)) = \emptyset. \tag{5}$$

Now take an arbitrary element $q \in \mathbb{S} \setminus (D \cup E)$. Since G acts transitively on Y, there is an element $g \in G$ such that $g(x, q) = (x', q)$. By connectivity of $\mathbb{S}$ and zero-dimensionality of C, the homeomorphism of Z that is associated to g maps $\{x\} \times \mathbb{S}$ onto $\{x'\} \times \mathbb{S}$. This means that (x, p) must be mapped onto an element of $\{x'\} \times (D \cup E)$ — but we have just seen in (5) that this is impossible. $\square$

Remark 4.4. In the light of Theorem 4.3, it is quite natural to ask whether there is a Polish space on which some topological group acts transitively, but on which no Polish group acts transitively. We do not know the answer to this question. However, we recently constructed an example of a homogeneous Polish space on which no (separable metrizable) group acts transitively; see [**9**].

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Jan van Mill
Faculteit Exacte Wetenschappen
Afdeling Wiskunde
Vrije Universiteit
De Boelelaan 1081A
1081 HV Amsterdam
The Netherlands

vanmill@cs.vu.nl