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A COUNTEREXAMPLE CONCERNING LINE-FREE GROUPS AND COMPLETE ERDŐS SPACE

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ABSTRACT. We present a weakly closed, one-dimensional, line-free subgroup of the separable Banach space c that is not homeomorphic to complete Erdős space. The existence of this example disproves a conjecture of Dobrowolski, Grabowski, and Kawamura.

Complete Erdős space was first featured by Erdős in [8], who proved that it is totally disconnected and one-dimensional. It can be represented by, for instance,

$$\mathfrak{E}_{\mathrm{c}} = \{z \in \ell^2 : z_i \in \mathbb{R} \setminus \mathbb{Q} \text{ for each } i \in \mathbb{N}\},$$

where ℓ^2 is the Hilbert space of square summable real sequences. $\mathfrak{E}_{\mathrm{c}}$ is a universal element of the class of almost zero-dimensional spaces; for background information see [11, 9, 3, 4, 5]. A subset of a topological space is called a *C-set* if it can be written as an intersection of clopen subsets of the space. A topological space is called *almost zero-dimensional* if every point has a neighbourhood basis consisting of C-sets. Every almost zero-dimensional space is at most one-dimensional; see [11, 10, 1].

An additive subgroup of a vector space is called *line-free* if it does not contain nontrivial linear subspaces. It is remarked in [2] that a topological classification of the line-free closed subgroups of Banach spaces produces a classification of all closed subgroups of Banach spaces. Let G be an arbitrary nondiscrete, weakly closed, line-free, additive subgroup of a separable Banach space E. Dobrowolski, Grabowski, and Kawamura [7] proved that G is homeomorphic to complete Erdős space whenever E is reflexive. In addition, Ancel, Dobrowolski, and Grabowski [2] showed that E contains zero-dimensional examples of such groups G precisely if E contains an isomorphic copy of c_0. These results prompted Dobrowolski, Grabowski, and Kawamura [7] to formulate the following

Conjecture. *Every separable, nondiscrete, weakly closed, one-dimensional, line-free subgroup of a Banach space is homeomorphic to* $\mathfrak{E}_{\mathrm{c}}$.

We present a counterexample to this conjecture, thereby finding a new topological type that closed subgroups of Banach spaces can have. We shall distinguish our example from $\mathfrak{E}_{\mathrm{c}}$ by the following property of $\mathfrak{E}_{\mathrm{c}}$. A topological space is called *somewhere zero-dimensional* if it contains a point at which the space is

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zero-dimensional, that is, the point has a clopen neighbourhood basis. Dijkstra, van Mill, and Steprāns [6] have shown that $\mathfrak{E}_c$ has the property that every point $x \in \mathfrak{E}_c$ has a neighbourhood U such that every closed subset of U is either empty or somewhere zero-dimensional.

Counterexample. We construct our counterexample in the Banach space c. We find it convenient to represent c as the space of all continuous real-valued functions f on the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. The norm $\|f\| = \sup\{|f(1/n)| : n \in \mathbb{N}\}$ makes c a separable Banach space. For $n \in \mathbb{N}$ let φ_n be the element of the dual of c that is given by $\varphi_n(f) = 2^n f(1/n)$. Since $\{\varphi_n : n \in \mathbb{N}\}$ is clearly a total sequence of functionals, we have that

$$G = \{f \in c : \varphi_n(f) \in \mathbb{Z} \text{ for each } n \in \mathbb{N}\}$$

is a line-free, weakly closed, additive subgroup of c. We first verify that G is almost zero-dimensional and hence that $\dim G \leq 1$. Consider an arbitrary closed ε-ball $B_\varepsilon(f) = \{g \in c : \|g - f\| \leq \varepsilon\}$ in c. Let $g \in G \setminus B_\varepsilon(f)$ and note that $|g(1/n) - f(1/n)| > \varepsilon$ for some $n \in \mathbb{N}$. Then $\{h \in G : h(1/n) = g(1/n)\}$ is an obviously clopen subset of G that is disjoint from $B_\varepsilon(f)$. Thus $G \cap B_\varepsilon(f)$ is a C-set in G, proving the almost zero-dimensionality of G. The fact $\dim G \leq 1$ also follows from [2, Theorem 3.1], when we note that the φ_n's form a norming sequence; see Dijkstra and van Mill [5, Remark 30].

We shall now show with the method of Erdős [8] that for each $\varepsilon > 0$ the set $A = G \cap B_\varepsilon(\mathbf{0})$, where $\mathbf{0}$ stands for the zero function, is not zero-dimensional at each of its points. We may then conclude that $\dim G \geq 1$ and that G is not homeomorphic to $\mathfrak{E}_c$. (However, according to [4, Propositions 6.3 and 6.10] G is homeomorphic to a dense subset of $\mathfrak{E}_c$.) Let $f \in A$ be arbitrary. Since $A = -A$ we may assume that $f(0) \leq 0$. Let U be a subset of $A \cap B_{\varepsilon/3}(f)$ such that $f \in U$. We show that U has boundary points in A. For each $n \in \mathbb{N}$ we let $\alpha_n \in G$ be defined by $\alpha_n(x) = 2^{-n}$ for $x \leq 1/n$ and $\alpha_n = 0$ for $x > 1/n$. Note that $\|\alpha_n\| = \alpha_n(0) = 2^{-n}$. We construct by recursion a sequence $g_1, g_2, g_3, \ldots$ in U as follows. We put $g_1 = f$. Assume that g_{n-1} has been found. Since U is bounded and $g_{n-1} \in U$, there is a $k \in \{0\} \cup \mathbb{N}$ such that

$$g_{n-1} + k\alpha_n \in U \quad \text{and} \quad g_{n-1} + (k+1)\alpha_n \notin U.$$

Defining $g_n = g_{n-1} + k\alpha_n$ we trivially have the following properties:

(1) $g_n \geq g_{n-1}$,
(2) $g_n + \alpha_n \in G \setminus U$, and
(3) $\|g_n - g_{n-1}\| = g_n(0) - g_{n-1}(0)$.

Since the sequence $g_1(0), g_2(0), \ldots$ is nondecreasing and bounded by ε, we have that it converges, say, to L. By property (3) we have $\sum_{n=1}^{\infty} \|g_{n+1} - g_n\| = L - g_1(0)$, thus $g_1, g_2, \ldots$ is a Cauchy sequence. Put $g = \lim_{n\to\infty} g_n$ and note that g is in the closure of U in A because A is closed. Since the closure of U is contained in $B_{\varepsilon/3}(f)$, we have $g(0) \leq f(0) + \varepsilon/3 \leq \varepsilon/3$. Select an $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/3$ and $g(1/n) < 2\varepsilon/3$ for each $n > N$. Let $n > N$. If $i < n$, then $(g_n + \alpha_n)(1/i) = g_n(1/i) \in [-\varepsilon, \varepsilon]$. If $i \geq n$, then

$$-\varepsilon \leq g_n(1/i) \leq (g_n + \alpha_n)(1/i) = g_n(1/i) + 2^{-n} \leq g(1/i) + \varepsilon/3 \leq \varepsilon.$$

Thus $g_n + \alpha_n \in A$ for every $n > N$. With property (2) we have that $g = \lim_{n\to\infty}(g_n + \alpha_n)$ is also in the closure of $A \setminus U$, thus U is not clopen in A and A is not zero-dimensional at f.

Since c is isomorphic to c_0, our construction also applies to that space and, by the Hahn-Banach Theorem, to every locally convex space that contains an isomorphic copy of c_0.

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