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STRONG LOCAL HOMOGENEITY AND COSET SPACES

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ABSTRACT. We prove that for every homogeneous and strongly locally homogeneous separable metrizable space X there is a metrizable compactification γX of X such that, among other things, for all $x, y \in X$ there is a homeomorphism $f\colon \gamma X \to \gamma X$ such that $f(x) = y$. This implies that X is a coset space of some separable metrizable topological group G.

1. INTRODUCTION

All spaces under discussion are Tychonoff. If G is a topological group acting on a space X, then for every $x \in X$ we let $\gamma_x\colon X \to G$ be defined by $\gamma_x(g) = gx$. We also let $G_x = \{g \in G : gx = x\}$ denote the *stabilizer* of $x \in X$. Then G_x is evidently a closed subgroup of G.

A space X is a *coset space* provided that there is a topological group G with *closed* subgroup H such that X and $G/H = \{xH : x \in G\}$ are homeomorphic. Observe that G acts transitively on G/H. It is well known, and easy to prove, that G/G_x is homeomorphic to X if γ_x is open. So for a space X to be a coset space it is necessary and sufficient that there is a topological group G acting transitively on X such that for some $x \in X$ (equivalently: for all $x \in X$) the function $\gamma_x\colon G \to X$ is open.

A space X is *strongly locally homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ which is supported on B (that is, f is the identity outside B) and moves x to y. This notion is due to Ford [2]. The topological sum of the spheres $\mathbb{S}^1$ and $\mathbb{S}^2$ is SLH, but not homogeneous. It is not hard to prove that a *connected* SLH-space is homogeneous, and that every Polish SLH-space is countably dense homogeneous. Most of the well-known homogeneous continua are strongly locally homogeneous: the Hilbert cube, the universal Menger continua and manifolds without boundaries. The pseudo-arc is an example of a homogeneous continuum which is not SLH. Observe that a zero-dimensional homogeneous space is evidently SLH (the clopen sets do the job). Ford [2] essentially proved that every Tychonoff homogeneous and SLH-space X is a coset space (see also Mostert [6, Theorem 3.2]). The proof goes as follows. One thinks of X as a subspace of its Čech-Stone compactification βX. The subgroup

$$G = \{g \in \mathcal{H}(\beta X) : g(X) = X\}$$

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of the homeomorphism group $\mathcal{H}(\beta X)$ of βX endowed with the compact-open topology acts transitively on X, and by strong local homogeneity, $\gamma_x \colon G \to X$ is open for every $x \in X$ (see the proof of Theorem 1.1 below).

In this note we are, among other things, interested in the question whether Ford's theorem holds within the class of all separable and metrizable spaces.

It is known that many homogeneous spaces are coset spaces. Every topological group is evidently a coset space. Ungar [7] proved that if X is separable metrizable, homogeneous and locally compact, then X is a coset space. This is a consequence of the Effros Theorem on transitive actions of Polish groups on Polish spaces (Effros [1]). Not all homogeneous spaces are coset spaces; see Ford [2] and van Mill [5].

Here is our main result.

Theorem 1.1. *Let a separable metrizable X be both homogeneous and* SLH. *Then X is a coset space of some separable metrizable topological group G.*

We prove that a homogeneous and SLH-space X has a (metrizable) compactification γX such that, among other things, for all $x, y \in X$ there is a homeomorphism $h \colon \gamma X \to \gamma X$ such that $h(x) = y$ and $h(X) = X$. Theorem 1.1 is a corollary of this. Such compactifications surface at several places in the literature. If G is a topological group acting on a space X, then X admits a compactification γX such that the action of G on X can be extended to an action of G on γX if and only if the motion-continuous functions on X separate the points and the closed subsets of X (such a compactification is called *equivariant*). Here a continuous real-valued function f on X is *motion-continuous* if for every $\varepsilon > 0$ there exists a neighborhood U of the neutral element of G such that for all $g \in U$ and all $x \in X$ we have $|f(gx) - f(x)| < \varepsilon$. Observe that for an equivariant compactification γX we have that for every $g \in G$ the homeomorphism $x \mapsto gx$ of X can be extended to the homeomorphism $y \mapsto gy$ of γX. For a locally compact G acting on X an equivariant compactification of X exists (see de Vries [9] for details) and similarly if the action is transitive and the space is of the second category. See Uspenskiĭ [8] for details (I am indebted to Michael Megrelishvili for informing me about this result). As was shown by Megrelishvili [3], not all actions can be 'equivariantly compactified', even if the group and the space under consideration are both Polish.

From now on, *all topological spaces under discussion are separable and metrizable.*

2. Homogeneous spaces

For a space X we let $\mathcal{H}(X)$ denote the homeomorphism group of X. If $A \subseteq X$, then $\mathcal{H}(X|A) = \{h \in \mathcal{H}(X) : h(A) = A\}$.

If X is a space, then $\mathcal{O}(X)$ denotes the collection of open subsets O in X such that for all $x, y \in O$ there is an element $f \in \mathcal{H}(X)$ which is supported on O and sends x to y. Observe that $\mathcal{O}(X)$ is invariant under $\mathcal{H}(X)$. It is obvious that X is homogeneous if and only if $X \in \mathcal{O}(X)$. Finally, observe that X is SLH if and only if $\mathcal{O}(X)$ is a base for X.

Let G be a topological group acting on a space X. The action is *transitive* if for all $a, b \in X$ there is an element $g \in G$ such that the homeomorphism $x \mapsto gx$ of X takes a onto b. Hence if G acts transitively on X, then X is homogeneous. The *natural action* of $\mathcal{H}(X)$ on X is defined by the formula

$$(h, x) \mapsto h(x) : \mathcal{H}(X) \times X \to X.$$

A topology on $\mathcal{H}(X)$ is called *admissible* if it makes $\mathcal{H}(X)$ a topological group and makes the natural action of $\mathcal{H}(X)$ on X continuous. Since all topologies that we consider are separable and metrizable, it is not clear whether $\mathcal{H}(X)$ has such a topology. Observe that if $\mathcal{H}(X)$ admits an admissible topology, then its natural action on X is transitive if and only if X is homogeneous. If X is compact, then the *compact-open* topology on $\mathcal{H}(X)$ is admissible. If ϱ is an admissible metric on X, then the formula

$$\hat{\varrho}(f,g) = \max\bigl\{\varrho\bigl(f(x),g(x)\bigr) : x \in X\bigr\}$$

defines a metric on $\mathcal{H}(X)$ that generates the compact-open topology.

3. Strongly locally homogeneous spaces

In this section we will prove that every homogeneous and SLH-space X has a compactification γX with certain homogeneity properties. We first prove that the homogeneity of X can in some sense be captured in a certain countable family of open subsets of X. Then we use these countably many open sets to define a (countable) Wallman base for X of which γX is the corresponding Wallman compactification. For background information on Wallman compactifications, see [4, §A.9].

Proposition 3.1. *Let X be SLH and homogeneous. There are a countable subcollection $\mathcal{V}$ of $\mathcal{O}(X)$ and a countable subgroup $\mathcal{G}$ of $\mathcal{H}(X)$ such that*

(1) *$X \in \mathcal{V}$ and $\mathcal{V}$ is a base of X,*
(2) *$\mathcal{V}$ is invariant under $\mathcal{G}$,*
(3) *for all $V \in \mathcal{V}$, $x, y \in V$ and $\varepsilon > 0$, there exist $A, B \in \mathcal{V}$ and $g \in \mathcal{G}$ such that*
  (i) $\overline{A \cup B} \subseteq V$, $x \in A$, $y \in B$,
  (ii) $\operatorname{diam} A < \varepsilon$, $\operatorname{diam} B < \varepsilon$,
  (iii) *g is supported on V, and $g(A) = B$.*

Proof. Let $\mathcal{U}$ be a countable subcollection of $\mathcal{O}(X)$ which is a base of X. By homogeneity we may assume that $X \in \mathcal{U}$. We will construct $\mathcal{V}$ in such a way that it contains $\mathcal{U}$. So then (1) is automatically satisfied.

Fix $O \in \mathcal{O}(X)$ for a moment and let $\mathcal{A}$ denote the collection of all homeomorphisms of X which are supported on O. Put $\mathcal{U}' = \{U \in \mathcal{U} : \overline{U} \subseteq O\}$. For every $U \in \mathcal{U}'$ and $n \in \mathbb{N}$, put

$$\mathcal{E}(U,n) = \{\alpha(U) : \alpha \in \mathcal{A}, \operatorname{diam} \alpha(U) < \tfrac{1}{n}\}.$$

There is a countable subcollection $\mathcal{A}(U,n)$ of $\mathcal{A}$ such that

$$\bigcup \mathcal{E}(U,n) = \bigcup \{\alpha(U) : \alpha \in \mathcal{A}(U,n)\}.$$

Now let $x, y \in O$ and $n \in \mathbb{N}$ be arbitrary. Pick $\beta \in \mathcal{A}$ such that $\beta(x) = y$. Since $\mathcal{U}$ is a base, there clearly is an element $U \in \mathcal{U}'$ such that $\operatorname{diam} U < \frac{1}{n}$ and $\operatorname{diam} \beta(U) < \frac{1}{n}$. This implies that $y \in \beta(U) \subseteq \bigcup \mathcal{E}(U,n)$. There consequently is an element $\alpha \in \mathcal{A}(U,n)$ such that $y \in \alpha(U)$.

Put

$$\mathcal{H} = \{1_X\} \cup \bigcup_{n=1}^{\infty} \bigcup_{U \in \mathcal{U}'} \mathcal{A}(U,n), \qquad \mathcal{W} = \{\alpha(U) : U \in \mathcal{U}', \alpha \in \mathcal{H}\}.$$

Then both $\mathcal{H}$ and $\mathcal{W}$ are countable. Observe that by the above for all $x, y \in O$ and $n \in \mathbb{N}$ there are $W_0, W_1 \in \mathcal{W}$ and $\alpha \in \mathcal{H}$ such that $x \in W_0 \subseteq \overline{W}_0 \subseteq O$, $y \in W_1 \subseteq \overline{W}_1 \subseteq O$, $\operatorname{diam} W_0 < \frac{1}{n}$, $\operatorname{diam} W_1 < \frac{1}{n}$ and $\alpha(W_0) = W_1$.

It is now clear how to proceed. For each element of the base $\mathcal{U}$ we construct as above countable subcollections of $\mathcal{O}(X)$ and $\mathcal{H}(X)$, respectively, which 'deal' with that element. This is the first step of an inductive process. There are countably many sets and homeomorphisms only at this stage. We may clearly assume that these sets are invariant under these homeomorphisms, and that the homeomorphisms form a subgroup of $\mathcal{H}(X)$. Then we proceed with each of these countably many sets in precisely the same way, etc. At each step of our countable process we have countable collections. At the end of our process we consequently obtain the countable base $\mathcal{V}$ and the countable subgroup $\mathcal{G}$ we are looking for. $\square$

Let X be homogeneous and strongly locally homogeneous. To X we associate the countable base $\mathcal{V}$ and countable group $\mathcal{G}$ of Proposition 3.1.

The closed collection

$$\{X \setminus V : V \in \mathcal{V}\} \cup \{\overline{V} : V \in \mathcal{V}\}$$

can be enlarged to a (countable) Wallman base $\mathcal{F}$ of X ([4, Lemma A.9.1]). We may assume without loss of generality that $\mathcal{F}$ is invariant under $\mathcal{G}$. The Wallman compactification $\gamma X = \omega(X, \mathcal{F})$ of X consequently has the property that each homeomorphism $g \in \mathcal{G}$ can be extended to a homeomorphism $\hat{g} \colon \gamma X \to \gamma X$.

For every open subset $U \subseteq X$, put $\hat{U} = \gamma X \setminus \overline{X \setminus U}$ (here closure means closure in γX). The open collection $\widehat{\mathcal{V}} = \{\hat{V} : V \in \mathcal{V}\}$ is clearly a local base in γX at every point of X.

Take $V \in \mathcal{V}$ and $g \in \mathcal{G}$. Then,

$$\hat{g}(\hat{V}) = \gamma X \setminus \overline{\hat{g}(X \setminus V)} = \gamma X \setminus \overline{X \setminus g(V)} = \widehat{g(V)}.$$

So the collection $\widehat{\mathcal{V}}$ is invariant under the subgroup $\widehat{\mathcal{G}} = \{\hat{g} : g \in \mathcal{G}\}$ of $\mathcal{H}(\gamma X)$. Moreover, if g is supported on V, then $\hat{g}$ is supported on $\hat{V}$. In addition, if $V, W \in \mathcal{V}$ are such that the closure $\overline{V}$ of V in X is contained in W, then $\overline{V}$ and $X \setminus W$ are disjoint members from $\mathcal{F}$ and hence have disjoint closures in γX. This implies that the closure of V in γX is contained in $\hat{W}$. These observations complete the proof of the following:

Corollary 3.2. *Let X be* SLH *and homogeneous. Then there are a compactification γX of X and a countable collection $\mathcal{W}$ of open subsets of γX and a countable subgroup $\mathcal{G}$ of $\mathcal{H}(\gamma X | X)$ such that*

(1) *$\gamma X \in \mathcal{W}$ and $\mathcal{W} \restriction X$ is a base of X,*
(2) *$\mathcal{W}$ is invariant under $\mathcal{G}$,*
(3) *for all $W \in \mathcal{W}$, $x, y \in W \cap X$ and $\varepsilon > 0$, there exist $A, B \in \mathcal{W}$ and $g \in \mathcal{G}$ with*
   (i) *$\overline{A \cup B} \subseteq W$, $x \in A$, $y \in B$,*
   (ii) *$\operatorname{diam} A < \varepsilon$, $\operatorname{diam} B < \varepsilon$,*
   (iii) *g is supported on W, and $g(A) = B$.*

In the remaining part of this section, we adopt the notation in Corollary 3.2. Let $x, y \in W$ for certain $W \in \mathcal{W}$. Suppose that there are $A_i, B_i \in \mathcal{W}$ and $g_i \in \mathcal{G}$

for every i such that

(A1) if $i \geq 2$, then g_i is supported on B_{i-1},
(A2) $\operatorname{diam} B_i \leq 2^{-i}$, $\overline{B}_{i+1} \subseteq B_i$,
(A3) $y \in B_i \subseteq W$,
(A4) $A_i = (g_i \circ \cdots \circ g_1)^{-1}(B_i)$,
(A5) $\operatorname{diam} A_i \leq 2^{-i}$, $\overline{A}_{i+1} \subseteq A_i$,
(A6) $x \in A_i \subseteq W$.

We say that the sequences $(A_i)_i$, $(B_i)_i$ and $(g_i)_i$ are *admissible* for x and y.

If both x and y belong to X, then there are such sequences.

Lemma 3.3. *If $x, y \in X$, then there are admissible sequences $(A_i)_i, (B_i)_i$ in $\mathcal{W}$ and $(g_i)_i$ in $\mathcal{G}$.*

Proof. Put $A_0 = B_0 = W$ and $g_0 = 1_{\gamma X}$. Suppose that

$$A_0, \ldots, A_i, \quad B_0, \ldots, B_i, \quad g_0, \ldots, g_i$$

have been constructed for some i. Observe that

$$a = g_i \circ \cdots \circ g_0(x) \in B_i$$

by (A4). Let $U \subseteq A_i$ be a closed neighborhood of x and $V \subseteq B_i$ a closed neighborhood of y such that

(1) $\operatorname{diam} U \leq 2^{-(i+1)}$, $\operatorname{diam} V \leq 2^{-(i+1)}$,
(2) $g_i \circ \cdots \circ g_0(U) \subseteq B_i$.

By the properties of $\mathcal{W}$ and $\mathcal{G}$ there are $E, F \in \mathcal{W}$ and $g_{i+1} \in \mathcal{G}$ such that

(3) $a \in E \subseteq g_i \circ \cdots \circ g_0(U)$, $y \in F \subseteq V$,
(4) $g_{i+1}(E) = F$,
(5) g_{i+1} is supported on B_i.

Define $A_{i+1} = (g_i \circ \cdots \circ g_0)^{-1}(E)$ and $B_{i+1} = F$. Then our choices are easily seen to be as required. $\square$

Let Z be a compact space and let $(h_n)_n$ be a sequence in $\mathcal{H}(Z)$. It is clear that for each $n \in \mathbb{N}$ we have

$$f_n = h_n \circ \cdots \circ h_1 \in \mathcal{H}(Z).$$

If $f = \lim_{n\to\infty} f_n$ exists in $\mathcal{H}(Z)$, then it will be denoted by

$$\lim_{n\to\infty} h_n \circ \cdots \circ h_1$$

and is called the *infinite left product* of the sequence $(h_n)_n$.

Lemma 3.4. *Suppose that the sequences $(A_i)_i$, $(B_i)_i$ and $(g_i)_i$ are admissible for $x, y \in W$. Then*

(1) *$g = \lim_{i\to\infty} g_i \circ \cdots \circ g_1$ is a homeomorphism of γX that is supported on W,*
(2) *$\hat{\varrho}(g_i \circ \cdots \circ g_1, g) \leq 2^{-i}$,*
(3) *$g(X \setminus \{x\}) = X \setminus \{y\}$,*
(4) *$g(x) = y$,*
(5) *if $p \notin A_i$, then $g(p) = g_i \circ \cdots \circ g_1(p)$.*

Proof. For every i put $f_i = g_i \circ \cdots \circ g_1$. We will first prove that $g(p) = \lim_{i\to\infty} f_i(p)$ exists for every $p \in \gamma X$. Indeed, if $p = x$, then $f_i(p) \in B_i$ for every i by (A6) and (A4). Hence $g(x) = y$ by (A2) and (A3). If $p \notin A_i$, then $q = f_i(p) \notin B_i$ by (A4). Hence (A1) gives us that $g_j(q) = q$ for every $j \geq i+1$. This shows that $\lim_{i\to\infty} f_i(p)$

exists and is equal to q. So we proved that g is well defined, and that (4) and (5) hold.

Observe that by (A1) and (A2) we have that $\hat{\varrho}(f_{i+1}, f_i) = \hat{\varrho}(g_{i+1}, 1_X) \leq 2^{-i-1}$ for every i. This easily implies that g is a continuous surjection ([4, Proposition 1.3.8 and Lemma A.3.1]). That g is one-to-one follows by similar considerations. Hence g is a homeomorphism. Observe that (5) implies (3) since X is invariant under $\mathcal{G}$. It remains to prove (2). But this is trivial since g_j is supported on B_i for every $j > i$ and $\operatorname{diam} B_i \leq 2^{-i}$. □

So we completed the proof of the following:

Theorem 3.5. *For every homogeneous and* SLH*-space X there are a compactification γX of X and a collection of open subsets $\mathcal{W}$ of γX having the following properties:*

1. *$\gamma X \in \mathcal{W}$ and $\mathcal{W} \restriction X$ is a base of X,*
2. *for all $W \in \mathcal{W}$ and $x, y \in W \cap X$ there is a homeomorphism f of γX such that $f(X) = X$, $f(x) = y$ and f is supported on W.*

Remark 3.6. It would be interesting to know whether a similar result holds for Tychonoff spaces. Let X be a Tychonoff space of weight α. If X is homogeneous and SLH, does there exist a Hausdorff compactification γX of X of weight α such that for all $x, y \in X$ there is an element $g \in \mathcal{H}(\gamma X)$ such that $g(X) = X$ and $g(x) = y$?

4. Coset spaces

Many homogeneous spaces are coset spaces. In the introduction we mentioned the following classes: locally compact homogeneous spaces and topological groups. Motivated by Ford [2], we add the homogeneous SLH-spaces to this.

Proof of Theorem 1.1. Let γX and $\mathcal{W}$ be as in Theorem 3.5. It is clear that the subgroup
$$\mathcal{G} = \{g \in \mathcal{H}(\gamma X) : g(X) = X\}$$
acts transitively on X. So it remains to prove that for some fixed $x \in X$ the continuous surjection $\gamma_x \colon \mathcal{G} \to X$ is open. To prove this, let $g \in \mathcal{G}$ and $\varepsilon > 0$ be arbitrary. Put
$$B = \{h \in \mathcal{G} : \hat{\varrho}(h, g) < \varepsilon\}.$$
We claim that Bx is open. To prove this, take an arbitrary $h \in B$, let $\delta = \hat{\varrho}(h, g)$, and put $\gamma = \varepsilon - \delta$. Let $W \in \mathcal{W}$ be such that
$$hx \in W \cap X \subseteq \{y \in X : \varrho(y, hx) < \tfrac{1}{2}\gamma\}.$$
We claim that $W \cap X \subseteq Bx$, which is clearly as required. To prove this, take an arbitrary $p \in W \cap X$. There is $\xi \in \mathcal{G}$ which is supported on W and has the property that $(\xi h)x = p$. Observe that $\hat{\varrho}(\xi h, h) = \hat{\varrho}(\xi, 1_{\gamma X}) < \gamma$, i.e., $\hat{\varrho}(\xi h, g) < \varepsilon$. So $p \in Bx$, and this is what we had to prove. □

Since a zero-dimensional homogeneous space is SLH, we obtain:

Corollary 4.1. *Let X be zero-dimensional and homogeneous. Then X is a coset space.*

Without too much difficulty it can be shown that a homogeneous zero-dimensional space is a coset space of some zero-dimensional topological group. The details of checking this are left to the reader.

The results in this note suggest the following problem that seems to be non-trivial (recall that all spaces are separable and metrizable).

Question 4.2. Let X be a homogeneous space. Does X admit a transitive action of a topological group G? What if X is Polish?

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