BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 53, No. 4, 2005

GENERAL TOPOLOGY

On Countable Dense and Strong Local Homogeneity

by

Jan VAN MILL

Presented by Czesław BESSAGA

Summary. We present an example of a connected, Polish, countable dense homogeneous space X that is not strongly locally homogeneous. In fact, a nontrivial homeomorphism of X is the identity on no nonempty open subset of X.

1. Introduction. *All spaces under discussion are separable and metrizable.* A space X is *strongly locally homogeneous* (abbreviated SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f\colon X \to X$ which is supported on B (that is, f is the identity outside B) and moves x to y. This notion is due to Ford [7]. Most of the well-known homogeneous continua are SLH, but not all: the pseudo-arc is an example of a homogeneous continuum that is not SLH.

A space X is *countable dense homogeneous* (abbreviated CDH) provided that for all countable dense subsets D and E of X there is a homeomorphism f of X such that $f(D) = E$. Bennett [2] showed that a connected CDH-space is homogeneous.

Bennett [2] also showed that every locally compact SLH-space is CDH. This was generalized to Polish spaces by Fletcher and McCoy [6], and independently, but later, by Anderson, Curtis and van Mill [1]. That this cannot be generalized to Baire spaces was shown by van Mill [10] (a similar example with better properties was constructed by Saltsman [12]).

The question whether every *connected* CDH-space is SLH is due to the author [10], and was repeated as Problem 382 in the *Open Problems in Topology* book by Fitzpatrick and Zhou [5]. It was solved negatively

2000 *Mathematics Subject Classification*: 57N20, 54G20.
Key words and phrases: strongly locally homogeneous, countable dense homogeneous, connected space, Polish space, Hilbert space, infinite-dimensional topology.

under the Continuum Hypothesis by Saltsman [13] (nonmetric counterexamples were earlier constructed in [4] and [15]). His example is based on a careful transfinite induction procedure in the plane, and inevitably has bad completeness properties. The aim of this note is to present a solution to the question without using additional set-theoretical assumptions.

EXAMPLE 1.1. There is a convex subspace X of Hilbert space ℓ^2 with the following properties:

(1) X is CDH (hence X is homogeneous),
(2) a nontrivial homeomorphism of X is the identity on no nonempty open subset of X,
(3) $X \times X \approx \ell^2$.

Observe that (3) implies that X is Polish (among other things). This comes as no surprise, since Hrušák and Zamora Avilés [8] recently proved that all Borel CDH-spaces are Polish, and that this extends to all spaces under the Axiom of Determinacy (see however [3]).

Our space is complete, but not compact. The following question, which appeared as Problem 383 in [5], remains open:

QUESTION 1.2. Is every CDH-continuum SLH?

Kennedy [9] obtained an interesting partial answer to this question: she proved that if a continuum is 2-homogeneous, and has a nontrivial homeomorphism that is the identity on some nonempty open set, then it is SLH.

I am indebted to Jim Rogers for some useful information. I am also indebted to the referee whose observations simplified and clarified my original construction substantially.

2. Preliminaries. If X and (Y, ϱ) are spaces, then $C(X, Y)$ denotes the collection of all continuous functions from X to Y. If $f, g \in C(X, Y)$, then

$$\widehat{\varrho}(f, g) = \sup\{\varrho(f(x), g(x)) : x \in X\}.$$

If X is compact, then $\widehat{\varrho}$ is a metric and the topology induced by $\widehat{\varrho}$ is separable. See, e.g., [11, § 1.3] for details.

Let Q denote the Hilbert cube $\prod_{n=1}^{\infty}[-1, 1]_n$ with its admissible metric

$$\varrho(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|.$$

We assume that the reader is familiar with the basic notions in infinite-dimensional topology (see [11]).

The identity function on a set X will be denoted by 1_X. We say that an indexing $\{X_n : n \in I\}$ is *faithful* provided that $X_n \neq X_m$ if $n \neq m$. We let $\mathbb{I}$ denote the closed unit interval $[0,1]$.

3. Dense collections of compacta. Let X be a nonempty compact space. We say that a countable collection $\mathcal{X}$ of Z-sets in Q is *X-dense* if

(1) $\mathcal{X}$ is pairwise disjoint and every $X' \in \mathcal{X}$ is homeomorphic to X,
(2) for every $f \in C(X,Q)$ and $\varepsilon > 0$ there are an $X' \in \mathcal{X}$ and a homeomorphism $\alpha\colon X \to X'$ such that $\widehat{\varrho}(\alpha, f) < \varepsilon$.

If $X = \{\mathrm{pt}\}$, then the singleton subsets of any countable dense subset of Q form an X-dense collection. Since Q is CDH, all such $\{\mathrm{pt}\}$-dense collections are "topologically unique". We will show that this holds for any nonempty compact space.

LEMMA 3.1. *Let X be a nonempty compact space. Then there exists an X-dense collection $\mathcal{X}$ of Z-sets in Q.*

Proof. Let $F = \{f_1, f_2, \dots\}$ be a countable dense subset of $C(X,Q)$. We assume that every element of F is listed infinitely often. For every n we will construct a Z-imbedding $\alpha_n\colon X \to Q$ such that

$$\widehat{\varrho}(\alpha_n, f_n) < 2^{-n}, \quad \alpha_n(X) \subseteq Q \setminus \bigcup_{i=1}^{n-1} \alpha_i(X).$$

Then $\mathcal{X} = \{\alpha_n(X) : n \in \mathbb{N}\}$ is the collection we are after. Assume that $\alpha_1, \dots, \alpha_{n-1}$ have been constructed. Since $\bigcup_{i=1}^{n-1} \alpha_i(X)$ is a Z-set in Q, there is a map $f\colon X \to Q \setminus \bigcup_{i=1}^{n-1} \alpha_i(X)$ such that $\widehat{\varrho}(f, f_n) < 2^{-n}$. By the Mapping Replacement Theorem [11, Theorem 5.3.11], we may assume without loss of generality that f is a Z-imbedding. Hence $\alpha_n = f$ is as required. ∎

LEMMA 3.2. *Let X be a nonempty compact space. If $h\colon Q \to Q$ is a homeomorphism, and $\mathcal{X}$ is an X-dense collection of Z-sets in Q, then $\{h(A) : A \in \mathcal{X}\}$ is X-dense as well.*

Proof. Use the fact that h is uniformly continuous. ∎

We will now prove that X-dense collections are "topologically unique". The proof is basically identical to the standard proof that Q is CDH. For the convenience of the reader, we will present the details.

PROPOSITION 3.3. *Let X be a nonempty compact space. In addition, let $\mathcal{S}$ and $\mathcal{T}$ be X-dense collections of Z-sets in Q. Then there is a homeomorphism $h\colon Q \to Q$ arbitrarily close to the identity such that $h(\bigcup \mathcal{S}) = \bigcup \mathcal{T}$.*

Proof. Let $\{S_1, S_2, \dots\}$ and $\{T_1, T_2, \dots\}$ be faithful enumerations of $\mathcal{S}$ respectively $\mathcal{T}$. Using the Inductive Convergence Criterion [11, Theorem 1.6.2], we construct a sequence $(h_n)_n$ of homeomorphisms of Q such that its infinite left product h is a homeomorphism and the following conditions are satisfied:

(1) $h_n \circ \cdots \circ h_1(S_i) = h_{2i} \circ \cdots \circ h_1(S_i) \in \mathcal{T}$ for each i and $n \geq 2i$,
(2) $(h_n \circ \cdots \circ h_1)^{-1}(T_i) = (h_{2i+1} \circ \cdots \circ h_1)^{-1}(T_i) \in \mathcal{S}$ for each i and each $n \geq 2i+1$.

Let $h_1 = 1_Q$, assume $h_1, \ldots, h_{2i-1}$ have been defined for a certain i, and put $\alpha = h_{2i-1} \circ \cdots \circ h_1$.

We first claim that if $\alpha(S_i) \cap (T_1 \cup \cdots \cup T_{i-1}) \neq \emptyset$ then $\alpha(S_i) \in \{T_1, \ldots, T_{i-1}\}$. For assume that for some $j \leq i-1$ we have $\alpha(S_i) \cap T_j \neq \emptyset$. Observe that by (2),

$$\alpha^{-1}(T_j) = (h_{2i-1} \circ \cdots \circ h_1)^{-1}(T_j) = (h_{2j+1} \circ \cdots \circ h_1)^{-1}(T_j) \in \mathcal{S},$$

hence $\alpha^{-1}(T_j) = S_i$ since $\mathcal{S}$ is pairwise disjoint.

If $\alpha(S_i) \in \{T_1, \ldots, T_{i-1}\}$, take $h_{2i} = 1_Q$. Otherwise, by the above the complement U_{2i} of the Z-set

$$K = (T_1 \cup \cdots \cup T_{i-1}) \cup \alpha(S_1 \cup \cdots \cup S_{i-1})$$

is a neighborhood of $\alpha(S_i)$. Let $\kappa\colon X \to \alpha(S_i)$ be any homeomorphism. Since $\mathcal{T}$ is X-dense, there are an index $k \geq i$ and a homeomorphism $\lambda\colon X \to T_k$ such that $T_k \subseteq U_{2i}$, and κ and λ are as close as we please. Hence there exists a "small" homeomorphism $\xi\colon \alpha(S_i) \to T_k$. By the Z-set Homeomorphism Extension Theorem [11, Theorem 5.3.7] this homeomorphism can be extended to a "small" homeomorphism h_{2i} of Q which restricts to the identity on K. Then h_{2i} clearly satisfies the required inductive hypotheses.

Put $\beta = h_{2i} \circ \cdots \circ h_1$. If $T_i \cap \beta(S_1 \cup \cdots \cup S_i) \neq \emptyset$ then $T_i \in \{\beta(S_1), \ldots, \beta(S_i)\}$. For assume that $T_i \cap \beta(S_j) \neq \emptyset$ for some $j \leq i$. Observe that by (1),

$$\beta(T_j) = h_{2i} \circ \cdots \circ h_1(T_j) = h_{2j} \circ \cdots \circ h_1(T_j) \in \mathcal{S},$$

hence $\beta(T_j) = S_i$ since $\mathcal{S}$ is pairwise disjoint.

If $T_i \in \{\beta(S_1), \ldots, \beta(S_i)\}$, take $h_{2i+1} = 1_Q$. Otherwise, by the above the complement U_{2i+1} of the Z-set

$$L = (T_1 \cup \cdots \cup T_{i-1}) \cup \beta(S_1 \cup \cdots \cup S_i)$$

is a neighborhood of T_i. Observe that $\beta(\mathcal{T})$ is X-dense by Lemma 3.2. Hence there is an index $\ell \geq i+1$ such that $\beta(T_\ell)$ is contained in U_{2i+1} and "closely approximates" S_i. In fact, we may choose $\beta(T_\ell)$ so close to S_i that there exists a "small" homeomorphism $\eta\colon \beta(T_\ell) \to S_i$. By the Z-set Homeomorphism Extension Theorem [11, Theorem 5.3.7] this homeomorphism can be extended to a "small" homeomorphism h_{2i+1} of Q which restricts to the identity on L. Then h_{2i+1} clearly satisfies the required inductive hypotheses.

If the approximations are chosen small enough, the conditions of the Inductive Convergence Criterion [11, Theorem 1.6.2] are satisfied so that $h = \lim_{i\to\infty} h_i \circ \cdots \circ h_1$ exists and is a homeomorphism of Q arbitrarily close to the identity. In addition, (1) and (2) easily imply that $h(\bigcup \mathcal{S}) = \bigcup \mathcal{T}$. ∎

REMARK 3.4. Let X and Y be nonempty compact spaces. If $\mathcal{S}$ is X-dense and $\mathcal{T}$ is Y-dense, and $\bigcup \mathcal{S} \cap \bigcup \mathcal{T} = \emptyset$, then $\mathcal{S} \cup \mathcal{T}$ is Z-dense, where Z is the topological sum of X and Y. Hence by Proposition 3.3 we can simultaneously push two dense collections in place provided their unions are disjoint. By repeating the proof of Proposition 3.3, this can easily be generalized to countably many pairwise disjoint "dense" collections.

4. The example. Now let $\mathcal{Q}$ be a Q-dense collection of Z-sets in Q (Lemma 3.1). Put $Y = Q \setminus \bigcup \mathcal{Q}$. Then Y is a G_δ-subset of Q and hence it is Polish. In addition, Y is connected, being the complement of a σ-Z-set in Q. By Remark 3.4 we get:

COROLLARY 4.1. *Y is* CDH.

Since Y is connected, this implies that Y is homogeneous. Much more is true. Observe that the identity $Q \to Q$ can be approximated arbitrarily closely by maps $Q \to Q \setminus Y$. Hence every compact subset of Y is a Z-set in Q. By the proof of Proposition 3.3, this shows that homeomorphisms between compact subsets of Y extend to homeomorphisms of Y (with control).

We will now analyze the homeomorphisms of Y. To this end, let $h\colon Y \to Y$ be an arbitrary homeomorphism. Our goal is to prove that if $h \neq 1_Y$ then h is not the identity on some closed non-Z-set of Y. We originally proved this for open sets, the strengthening to closed non-Z-sets was suggested by the referee. Let $\{Q_n : n \in \mathbb{N}\}$ be a faithful enumeration of $\mathcal{Q}$.

PROPOSITION 4.2. *There is a permutation $\varrho\colon \mathbb{N} \to \mathbb{N}$ such that for every $A \subseteq Y$, if $\overline{A} \cap Q_n \neq \emptyset$ then $\overline{h(A)} \cap Q_{\varrho(n)} \neq \emptyset$ (here closure means closure in Q).*

Proof. Let $\Gamma \subseteq Q \times Q$ be the graph of h, let $M = \overline{\Gamma}$, and let $\pi_i\colon M \to Q$ be the restrictions to M of the projection maps $Q \times Q \to Q$, $i = 1, 2$. It was shown in the proof of [1, Lemma 3.6] that the maps π_1, π_2 are monotone surjections such that $\pi_1^{-1}(\bigcup \mathcal{Q}) = \pi_2^{-1}(\bigcup \mathcal{Q})$.

Now take an arbitrary $n \in \mathbb{N}$. Then $\pi_1^{-1}(Q_n)$ is a continuum since π_1 is monotone. Since by Sierpiński's theorem from [14] no continuum is the countable infinite union of disjoint nonempty compacta, there is a unique element $m \in \mathbb{N}$ such that $\pi_1^{-1}(Q_n) \subseteq \pi_2^{-1}(Q_m)$. Since π_2 is monotone and π_1 is continuous, $\pi_1(\pi_2^{-1}(Q_m))$ is consequently a continuum that contains Q_n and is contained in $Q \setminus Y$. Hence again by Sierpiński's theorem, $\pi_1(\pi_2^{-1}(Q_m)) = Q_n$. So we conclude that $\pi_1^{-1}(Q_n) = \pi_2^{-1}(Q_m)$. Now define $\varrho(n) = m$. Since n was arbitrary, this defines a function $\varrho\colon \mathbb{N} \to \mathbb{N}$. It is clear from the construction that ϱ is a bijection.

Let $A \subseteq Y$ be such that $\overline{A} \cap Q_n \neq \emptyset$. Pick an arbitrary open neighborhood U of $Q_{\varrho(n)}$. Then $\pi_2^{-1}(U)$ is an open neighborhood of $\pi_2^{-1}(Q_{\varrho(n)}) = \pi_1^{-1}(Q_n)$. Since π_1 is a closed map, there is an open neighborhood V of Q_n such that $\pi_1^{-1}(V) \subseteq \pi_2^{-1}(U)$. Pick an element $a \in A \cap V$. Then $(a, h(a)) \in \pi_1^{-1}(V)$, hence $h(a) = \pi_2(a, h(a)) \in \pi_2(\pi_1^{-1}(V)) \subseteq U$, as required. ∎

This leads us to the result we are after.

THEOREM 4.3. *Let $h\colon Y \to Y$ be a homeomorphism. If h is the identity on some closed subset of Y that is not a Z-set in Y, then h is the identity.*

Proof. Let V be the closed non-Z-set on which h is the identity. Striving for a contradiction, assume that there is an element $x \in Y$ such that $h(x) \neq x$. Since $Q \setminus Y$ is a σ-Z-set, the closure $\overline{V}$ of V in Q is not a Z-set in Q. Let $\varepsilon > 0$ be such that if $\beta\colon Q \to Q$ is continuous and $\widehat{\varrho}(\beta, 1_Q) < \varepsilon$, then $\beta(Q) \cap \overline{V} \neq \emptyset$. Since singleton subsets of Q are Z-sets, there exist maps $Q \to Q \setminus \{h(x)\}$ arbitrarily close to the identity. Hence we may pick a continuous function $\alpha\colon Q \to Q$ such that $x \in \alpha(Q)$, $h(x) \notin \alpha(Q)$, and $\widehat{\varrho}(\alpha, 1_Q) < \varepsilon/2$.

There is a neighborhood W of x in Y such that $\overline{h(W)} \cap (W \cup \alpha(Q)) = \emptyset$ (here closure means closure in Q). Observe that $V \cap W = \emptyset$. Let W' be an open subset of Q such that $W' \cap Y = W$. By construction, there are an $n \in \mathbb{N}$ and a homeomorphism $f_n\colon Q \to Q_n$ such that $\widehat{\varrho}(\alpha, f_n) < \varepsilon/2$. Since $x \in \alpha(Q)$, we may assume that $\widehat{\varrho}(\alpha, f_n)$ is so small that $f_n(Q) = Q_n$ intersects W'. Since $\widehat{\varrho}(f_n, 1_Q) < \varepsilon$ we get $Q_n \cap \overline{V} \neq \emptyset$, hence there is a sequence $(x_n)_n$ in V converging to an element of Q_n. By Proposition 4.2, the sequence $(h(x_n))_n$ has a limit point in $Q_{\varrho(n)}$. Since $x_n = h(x_n)$ for every n, this means that $\varrho(n) = n$. Now let $(y_n)_n$ be a sequence in W converging to an element of Q_n. Then $(h(y_n))_n$ has a limit point in $Q_{\varrho(n)} = Q_n$, again by Proposition 4.2. But this is impossible since $h(y_n) \in h(W)$ for every n, and $\overline{h(W)} \cap Q_n = \emptyset$. ∎

COROLLARY 4.4. *Let $h\colon Y \to Y$ be a homeomorphism. If h is the identity on some nonempty open subset of Y, then h is the identity.*

Proof. It suffices to observe that every nonempty open set contains a closed set with nonempty interior and that Z-sets have empty interior. ∎

REMARK 4.5. By Wong [16], there is a Cantor set K in Q which is not a Z-set. Hence if $L = K \cap Y$, then L is a zero-dimensional closed subset of the strongly infinite-dimensional space Y with the following curious property: if h is any homeomorphism of Y that restricts to the identity on L, then h is the identity on Y. It can be shown however that there are involutions on Y with a unique fixed point.

REMARK 4.6. Obviously there are similar examples that are finite-dimensional. For example, replace Q by $\mathbb{I}^3$, and consider an "$\mathbb{I}$-dense" sequence of Z_1-arcs that are contained in the interior of $\mathbb{I}^3$.

We will now show that Y has the additional properties promised in Example 1.1. That Y is homeomorphic to a convex subset of ℓ^2 follows from [1, Theorem 3.1] since $Q \setminus Y$ is a σ-Z-set in Q. By observing that the identity function $Q \to Q$ can be approximated arbitrarily closely by maps $Q \to Q \setminus Y$, $Y \times Y \approx \ell^2$ follows from [1, Theorem 3.5].

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Jan van Mill
Faculteit Exacte Wetenschappen
Afdeling Wiskunde
Vrije Universiteit
De Boelelaan 1081A
1081 HV Amsterdam, The Netherlands
E-mail: vanmill@cs.vu.nl

Received September 29, 2005;
received in final form February 5, 2006 (7479)