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ON THE CHARACTER AND π-WEIGHT OF HOMOGENEOUS COMPACTA

BY

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ABSTRACT

Under GCH, $\chi(X) \leq \pi(X)$ for every homogeneous compactum X. CH implies that a homogeneous compactum of countable π-weight is first countable. There is a compact space of countable π-weight and uncountable character which is homogeneous under MA+$\neg$CH, but not under CH.

1. Introduction

For all undefined notions, see Engelking [11], Kunen [19] and Juhász [16]. Recall that $\chi(X)$ and $\pi(X)$ denote the **character** and **π-weight** of X. *All spaces under discussion are Tychonoff.*

A space X is **homogeneous** if for all $x, y \in X$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$. Every topological group is clearly homogeneous. In this paper we are interested in the class of all homogeneous *compacta.* There are many known examples of second countable homogeneous compacta (compact groups, Hilbert cube, Cantor set, Menger compacta, spheres, etc.). The list of known homogeneous compacta that are not second countable is rather limited, however. See §6 for details.

Our first contribution is to point out that known results easily imply the following curious inequality:

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THEOREM 1.1: *Let X be a homogeneous compactum. Then $2^{\chi(X)} \leq 2^{\pi(X)}$.*

(Observe that homogeneity is essential. If $X = \beta\omega$, then $\chi(X) = \mathfrak{c}$ and $\pi(X) = \omega$.)

There are two ingredients in the proof. The first one is the result of van Douwen [7] (see also [16, 2.38]) that $|X| \leq 2^{\pi(X)}$ for every homogeneous space X. The second ingredient is the classical Čech–Pospišil Theorem, see [16, 3.16], that if X is compact and if for some κ, $\chi(x, X) \geq \kappa$ for every $x \in X$, then $|X| \geq 2^{\kappa}$. So to complete the proof, all one needs to observe is that the homogeneity of X clearly implies that all points in X have the same character, hence $|X| \geq 2^{\chi(X)}$. (A similar result was proved by Hart and Kunen in [14, 2.5.1(2)]. They applied the Čech–Pospišil Theorem and Arhangel′skiĭ's Theorem from [2] to conclude that if X is an infinite homogeneous compactum then $|X| = 2^{\chi(X)}$.)

Theorem 1.1 has some interesting consequences.

COROLLARY 1.2: *Let X be a homogeneous compactum. Then $\chi(X) < 2^{\pi(X)}$.*

Simply apply Cantor's Theorem that $2^{\kappa} > \kappa$ for every cardinal κ.

COROLLARY 1.3 (GCH): *If X is a homogeneous compactum then $\chi(X) \leq \pi(X)$.*

This inequality is much more appealing than the one in Theorem 1.1.

COROLLARY 1.4 ($\mathfrak{c} < 2^{\omega_1}$): *Every homogeneous compactum of countable π-weight is first countable.*

The question naturally arises whether Corollaries 1.3 and 1.4 can be proved in ZFC alone. In the light of Corollary 1.4, our first result on this question is not very surprising.

THEOREM 1.5 (MA): *Let X be a homogeneous compactum of countable π-weight. If X has weight less than $\mathfrak{c}$, then X is first countable.*

This again follows easily from known results. Juhász [17, Theorem 5] proved that under MA+$\neg$CH, if X is a countably compact space of weight less than $\mathfrak{c}$ having a dense set of points of countable π-character, then X is somewhere first countable. So if X is moreover homogeneous then it must be first countable.

In the light of this result, the following question is quite natural. Let X be a compact homogeneous space of countable π-weight. Assume that X has weight less than $\mathfrak{c}$. Does X have countable weight under MA? The answer to this question is in the negative. Let G be a dense subgroup of $\mathbb{R}$ of cardinality ω_1 such that $1 \in G$. In the unit interval $\mathbb{I}$, split every point $g \in G \cap (0,1)$ in two distinct points g^- and g^+. Order the set obtained in this way in the natural way, where g^- precedes g^+ if g is split. The ordered compact space that we obtain by this procedure has weight ω_1, has countable π-weight, and is homogeneous by the method of van Douwen [8]. (This example is also in Hart and Kunen [14] (next-to-last paragraph of §2.5).)

Theorem 1.5 raises the question what can be said about spaces of weight $\mathfrak{c}$. This is answered in our next result, which is our principle contribution.

THEOREM 1.6: *There is in* ZFC *a compact space with countable π-weight and character ω_1 which is homogeneous under* MA+$\neg$CH, *but not under* CH.

So we conclude that the existence of a non-first countable homogeneous compactum of countable π-weight is independent of ZFC.

In the proof of Theorem 1.6 we make good use of recent results of Matveev [24]. I am indebted to Michael Hrušák for bringing Matveev's paper to my attention.

2. Preliminaries

If X is a space then τX denotes its topology.

Let X be a space. If $x \in X$ then $\chi(x, X)$, **the character of x in X**, is the minimum cardinality of a neighborhood base at x; the **character of X** is

$$\chi(X) = \sup\{\chi(x, X) : x \in X\}.$$

A π-basis of X is a collection $\mathcal{U} \subseteq \tau X \setminus \{\emptyset\}$ such that for every nonempty open set $V \subseteq X$ there is a $U \in \mathcal{U}$ with $U \subseteq V$. The minimum cardinality of a π-basis of X is called the **π-weight**, $\pi(X)$, of X.

A continuous closed surjection $f\colon X \to Y$ is **irreducible** if $f[A] \neq Y$ for every proper closed subset of Y. Observe that f has the following property: for every nonempty open subset U of X there is a nonempty open subset V of Y such that $f^{-1}[V] \subseteq U$.

Let X and Y be spaces, and for certain $x \in X$, let $f\colon X \setminus \{x\} \to Y$ be a continuous function. The **f-boundary** $\partial_f Y$ of Y is the set of all elements $y \in Y$ having the following property: for every neighborhood U of x in X and for every neighborhood V of y in Y we have $U \cap f^{-1}[V] \neq \emptyset$.

If A and B are sets then $A \subseteq^* B$ means that $|A \setminus B| < \omega$. We say that A is a **pseudo-intersection** for a family of sets $\mathcal{F}$ if $A \subseteq^* F$ for every $F \in \mathcal{F}$. We also say that a family of sets $\mathcal{F}$ has the **sfip** (strong finite intersection property) if every nonempty finite subfamily has infinite intersection. As usual, we let $\mathfrak{p}$ denote the minimum cardinality of a subfamily of infinite subsets of ω with the *sfip* which has no infinite pseudo-intersection.

If $f, g \in \omega^\omega$ then $f \leq^* g$ abbreviates the following statement:

$$\exists N < \omega \forall n \geq N[f(n) \leq g(n)].$$

The cardinal $\mathfrak{b}$ is the minumum cardinality of an unbounded (with respect to $\leq^*$) subset of ω^ω. It is known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{c}$. It is also known by Bell [5] that $\mathsf{MA}_{\sigma\text{-centered}}(\kappa)$ holds if and only if $\kappa < \mathfrak{p}$. See van Douwen [9] and Vaughan [29] for proofs and more information on these and various other 'small' uncountable cardinals.

We let C denote *the* Cantor set 2^ω. We think of a point of C as a countably infinite sequence consisting of 0's and 1's. If $t = i_0 i_2 \cdots i_n$ is a finite sequence consisting of 0's and 1's and $s = s_0 s_1 \cdots \in \mathsf{C}$, then $t^\frown s$ denotes the sequence

$$i_0 i_1 \cdots i_n s_0 s_1 \cdots$$

in C. *A* Cantor set is a space homeomorphic to C. For example, C^ω is a Cantor set. We use the standard Boolean group structure '+' on C. The neutral element of C will be denoted by e.

For every $i \in I$ let X_i be a space. Consider the product $X = \prod_{i \in I} X_i$. It is convenient to introduce the following notation. If $E \subseteq I$ then

$$\pi_E \colon \prod_{i \in I} X_i \to \prod_{i \in E} X_i$$

denotes the projection.

3. Proof of Theorem 1.6: part 1

In this section we will construct in ZFC a certain compact space X of countable π-weight and character ω_1. Observe that by Corollary 1.4, X is not homogeneous under CH. But X *is* homogeneous under MA+$\neg$CH, as will be shown in §5.

For the description of the example, it is convenient to use the resolution method of Fedorchuk [12] (see also Watson [30]). This method allows for a more or less 'concrete' description of the example, which turns out to be helpful in verifying the continuity of certain maps.

Suppose that X is a topological space and that $\{Y_x : x \in X\}$ are topological spaces and, for each $x \in X$, $f_x \colon X \setminus \{x\} \to Y_x$ is a continuous function. We topologize

$$Z = \bigcup \{\{x\} \times Y_x : x \in X\}$$

as follows. If $x \in X$, U_x is an open neighborhood of x in X, and $W \subseteq Y_x$ is open, then

$$U_x \otimes W = (\{x\} \times W) \cup \bigcup \{\{x'\} \times Y_{x'} : x' \in U_x \cap f_x^{-1}[W]\}.$$

The collection

$$\{U_x \otimes W : (U_x \in \tau X, x \in U_x) \ \& \ (W \in \tau Y_x)\}$$

is an open basis for Z. Topologized in this way, Z is called the **resolution** of X at each point $x \in X$ into Y_x by the mapping f_x. Let π_0: $Z \to X$ be the 'projection'. Then π_0 is continuous by Watson [30, 3.1.35]. Observe that for every basic open subset $U_x \otimes W$ of X we have $\pi_0[U_x \otimes W] \subseteq U_x$. Also observe that for every $x \in X$ the set $\{x\} \times Y_x$ is a closed topological copy of Y_x in Z.

It is known and easy to prove that if X is compact, and all Y_x are compact, then so is the resolution. In addition, if $\partial_{f_x} Y_x = Y_x$ for every $x \in X$ then π_0: $Z \to X$ is irreducible. See Fedochuk [12] and Watson [30] for details.

It is convenient to describe the topology of a resolution in terms of neighborhood bases.

LEMMA 3.1: *Suppose that Z is the resolution of X at each point $x \in X$ into Y_x by the mapping f_x. Suppose that $\langle x, y\rangle \in Z$, that $\mathcal{U}$ is an open neighborhood base at x in X and $\mathcal{W}$ is an open neighborhood base at y in Y_x with $Y_x \in \mathcal{W}$. Then for every neighborhood O of $\langle x, y\rangle$ in Z there are elements $U_x \in \mathcal{U}$ and $W \in \mathcal{W}$ such that $\langle x, y\rangle \in U_x \otimes W \subseteq O$.*

Proof: Suppose that $a \in X$, E_a is an open neighborhood of a in X, and $V \subseteq Y_a$ is open such that

$$\langle x, y\rangle \in E_a \otimes V \subseteq O.$$

If $x = a$ then there is nothing to prove. Simply observe that if $U_x \in \mathcal{U}$ is such that $U_x \subseteq E_a$ and $W \in \mathcal{W}$ is such that $y \in W \subseteq V$, then $\langle x, y\rangle \in U_x \otimes W \subseteq O$. So assume that $x \neq a$. Then $x \in E_a \cap f_a^{-1}[V]$. By continuity of f_a, the set $F_x = E_a \cap f_a^{-1}[V]$ is an open neighborhood of x. Let $U_x \in \mathcal{U}$ be such that $U_x \subseteq F_x$. We claim that $\langle x, y\rangle \in U_x \otimes Y_x \subseteq E_a \otimes V$. To this end, take an arbitrary element $\langle p, q\rangle \in U_x \otimes Y_x$. Since $p \in U_x \subseteq F_x = E_a \cap f_a^{-1}[V]$ it follows that $p \neq a$, and hence $\langle p, q\rangle \in E_a \otimes V$, as required. ∎

Let K denote the 'uncountable' torus T^{ω_1}. Here T denotes the circle group. Since T is Abelian, we use additive notation. This group is monothetic by Hewitt and Ross [15, Corollary 25.15], i.e., there is an element $d \in K$ such that the subgroup generated by $\{d\}$ is dense in K. Put $d_n = (n+1) \cdot d$ for $n < \omega$. It is easy to see that

$$D = \{d_n : n < \omega\}$$

is dense in K.

Our example X is C with each point replaced by a copy of K. To ensure homogeneity under additional axioms, this has to be done in a uniform way.

For every $n < \omega$ let

$$C_n = \{x \in \mathsf{C} : (x_i = 0 \text{ if } i < n) \ \& \ (x_n = 1)\}.$$

Hence $C_0 = \{x \in \mathsf{C} : x_0 = 1\}$. It is clear that every C_n is clopen, that $C_n \cap C_m = \emptyset$ if $n \neq m$ and that $\bigcup_{n<\omega} C_n = \mathsf{C} \setminus \{e\}$. Define $f\colon \mathsf{C} \setminus \{e\} \to K$ as follows:

$$f(x) = d_n \quad \Longleftrightarrow \quad x \in C_n.$$

It is clear that f is continuous, and that $\partial_f K = K$. To see this, simply observe that every neighborhood of e contains all but finitely many C_n's, and every nonempty open subset of K contains infinitely many d's. Now for every $x \in \mathsf{C}$, let $K_x = K$, and define $f_x\colon \mathsf{C} \setminus \{x\} \to K_x$ by

$$f_x(y) = f(x+y).$$

Let X be the resolution of C at each point $x \in \mathsf{C}$ into K_x by the mapping f_x. By the above, X is a compact space and the standard mapping $\pi\colon X \to \mathsf{C}$ is a continuous, irreducible surjection. Hence $\pi(X) = \omega$. Since T^{ω_1} embeds into X, Lemma 3.1 implies that $\chi(X) = \omega_1$. It therefore suffices to prove that X is homogeneous under $\mathsf{MA}+\neg\mathsf{CH}$.

It will be convenient to identify X and $\mathsf{C} \times K$ as sets. One should be a little careful with this identification since the resolution topology on X does not coincide with the product topology on $\mathsf{C} \times K$. Observe that π_0 is nothing but the projection $\mathsf{C} \times K \to C$. So in the resolution topology, the projection onto the first coordinate is continuous. The projection onto the second coordinate is definitely not continuous.

To prove homogeneity, we consider two types of actions on X. The first type will be described here. We will postpone the description of the second type to §5. Our aim here is to construct for every translation $x \mapsto a+x$ of C a homeomorphism $T_a\colon X \to X$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "T_a"] \arrow[d, "\pi_0"'] & X \arrow[d, "\pi_0"] \\ \mathsf{C} \arrow[r, "x\mapsto a+x"'] & \mathsf{C} \end{tikzcd}$$

commutes. To this end, for every $a \in \mathsf{C}$ define $T_a\colon X \to X$ in the obvious way by:

$$T_a(\langle x, y\rangle) = \langle a+x, y\rangle.$$

It is clear that T_a is a well-defined bijection, and that it makes our diagram commutative. Observe that T_a is nothing but the product of the translation $x \mapsto a+x$ and the identity on the second factor of X.

Since all our groups are Abelian, we use additive notation. This is slightly confusing in the proof of the next result, since the group operation on C is Boolean and so for a given x the notation does not tell us whether we think of x or its inverse $-x$. The proof definitely looks more natural, when written down in multiplicative notation. Since the proof is simple anyway, we will not bother doing that.

LEMMA 3.2: *The function T_a is a homeomorphism for every $a \in \mathsf{C}$.*

Proof: Since $T_a \circ T_a$ is the identity on X, it suffices to verify continuity. To this end, let $p \in \mathsf{C}$, U_p an open neighborhood of p in C, and W an open subset of K. We claim that $T_a^{-1}[U_p \otimes W]$ is open in X. Put $q = a+p$ and $E_q = a+U_p$. Then E_q is an open neighborhood of q. We claim that $T_a^{-1}[U_p \otimes W] = E_q \otimes W$, which clearly suffices.

To prove this, take an arbitrary element $\langle u, v\rangle \in T_a^{-1}[U_p \otimes W]$. Then $\langle a+u, v\rangle \in U_p \otimes W$ and so $a+u \in U_p$, i.e., $u \in a+U_p = E_q$. This means that E_q is also a neighborhood of u. First assume that $a+u = p$. Then $u = q$ and $\langle p, v\rangle = \langle a+u, v\rangle \in U_p \otimes W$ gives us that $v \in W$. Hence indeed $\langle u, v\rangle \in E_q \otimes W$. So assume that $a+u \neq p$. Then $\langle a+u, v\rangle \in U_p \otimes W$ implies $a+u \in U_p \cap f_p^{-1}[W]$. Notice that

$$W \ni f_p(a+u) = f(p + a+u) = f(q+u) = f_q(u).$$

Since $u \in E_q$ and $u \neq q$ this yields $\langle u, v\rangle \in E_q \otimes W$, as required.

So we proved that $T_a^{-1}[U_p \otimes W] \subseteq E_q \otimes W$. Since there was nothing specific about a and p, we also implicitly proved that $T_a^{-1}[E_q \otimes W] \subseteq U_p \otimes W$, i.e.,

$$E_q \otimes W \subseteq T_a[U_p \otimes W] = T_a^{-1}[U_p \otimes W].$$

So we are done. ∎

Let η: $K \to K$ be the translation $\eta(x) = d+x$.

For the second type of action on X, which will be described in §5, we need that 'many' clopen subsets of X are homeomorphic. All we need is formulated and proved in the next result.

LEMMA 3.3: *For every $n < \omega$, $\pi_0^{-1}[C_n] \approx X$.*

Proof: Define a homeomorphism $\xi\colon \mathsf{C} \to C_n$ by

$$\xi(x) = \underbrace{000\cdots0}_{n\times}1^\frown x.$$

We claim that the function $\varphi\colon X \to \pi_0^{-1}[C_n]$ defined by

$$\varphi(\langle x,y\rangle) = \langle \xi(x), \eta^{n+1}(y)\rangle$$

is a homeomorphism. It is clearly a bijection, so by compactness it suffices to verify continuity. To this end, let $\langle a,b\rangle \in \pi_0^{-1}[C_n]$ be an arbitrarily chosen point. Consider an open neighborhood U_a of a in C_n, and a nonempty open set W in K. By Lemma 3.1, it suffices to prove that $\varphi^{-1}[U_a \otimes W]$ is open. Let $x = \xi^{-1}(a)$ and $V_x = \xi^{-1}[U_a]$. Then $a = 00\cdots01^\frown x$ and V_x is an open neighborhood of x. We claim that

$$V_x \otimes \eta^{-(n+1)}[W] = \varphi^{-1}[U_a \otimes W],$$

which clearly suffices.

To prove this, pick an arbitrary element $\langle u,v\rangle \in X$. Suppose first that $u = x$. Then $\xi(u) = a$, hence on the one hand

$$\langle u,v\rangle \in V_x \otimes \eta^{-(n+1)}[W] \;\Leftrightarrow\; v \in \eta^{-(n+1)}[W] \;\Leftrightarrow\; \eta^{n+1}(v) \in W,$$

while on the other hand

$$\langle u,v\rangle \in \varphi^{-1}[U_a \otimes W] \;\Leftrightarrow\; \langle a, \eta^{n+1}(v)\rangle \in U_a \otimes W \;\Leftrightarrow\; \eta^{n+1}(v) \in W.$$

So if $u = x$ then there is nothing to prove. We may therefore assume without loss of generality that $u \neq x$, hence $u+x \neq e$. Let $m < \omega$ be such that $u+x \in C_m$. Now observe that

$$\begin{aligned}
\langle u,v\rangle \in V_x \otimes \eta^{-(n+1)}[W] \;&\Leftrightarrow\; u \in V_x \cap f_x^{-1}[\eta^{-(n+1)}[W]]\\
&\Leftrightarrow\; u \in V_x \;\&\; f_x(u) \in \eta^{-(n+1)}[W]\\
&\Leftrightarrow\; u \in V_x \;\&\; f(x+u) \in \eta^{-(n+1)}[W]\\
&\Leftrightarrow\; u \in V_x \;\&\; d_m \in \eta^{-(n+1)}[W]\\
&\Leftrightarrow\; u \in V_x \;\&\; d_{m+n+1} \in W.
\end{aligned}$$

Notice that since $u \neq x$ we have $\xi(u) \neq a$. So

$$\begin{aligned}\langle u, v\rangle \in \varphi^{-1}[U_a \otimes W] &\Leftrightarrow \langle \xi(u), \eta^{n+1}(v)\rangle \in U_a \otimes W.\\ &\Leftrightarrow \xi(u) \in U_a \cap f_a^{-1}[W]\\ &\Leftrightarrow u \in V_x \ \&\ f_a(00\cdots01^\frown u) \in W\\ &\Leftrightarrow u \in V_x \ \&\ f(00\cdots01^\frown x + 00\cdots01^\frown u) \in W\\ &\Leftrightarrow u \in V_x \ \&\ f(\underbrace{00\cdots00}_{(n+1)\times}{}^\frown(x+u)) \in W.\end{aligned}$$

Since

$$x+u \in C_m \Leftrightarrow \underbrace{00\cdots00}_{(n+1)\times}{}^\frown(x+u) \in C_{m+n+1}$$

we conclude that

$$\langle u, v\rangle \in \varphi^{-1}[U_a \otimes W] \Leftrightarrow u \in V_x \ \&\ d_{m+n+1} \in W,$$

which means that we are done. ∎

Consider two 'blown-up points' $\{y\} \times K$ and $\{z\} \times K$ of X. If $x = z+y$ then clearly

$$T_x[\{y\} \times K] = \{z\} \times K.$$

So since T_x is a homeomorphism, for proving homogeneity it suffices to prove that elements of X with the same first coordinates can be homeomorphed onto each other. This would be simple if the product of a homeomorphism on the second factor of X and the identity on the first factor of X would be a homeomorphism of X. But it is easy to see that this need not be the case. So we have to think of something else and set theory enters the picture.

4. Matveev's Theorem extended a bit

The following interesting result was recently proved by Matveev [24]:

THEOREM 4.1: *Let $\kappa < \mathfrak{p}$. Then all compactifications of ω with remainder homeomorphic to 2^κ are homeomorphic.*

We need the permutations on ω that are guaranteed by this result for the second type of action on X. But in order to do that, we first have to generalize the theorem a bit. This will be done in our next result. The proof is based on the main idea of Matveev in [24].

THEOREM 4.2: *Let $a\omega$ and $b\omega$ be compactifications of ω. Assume that*

(1) *there is a retraction $r\colon a\omega \to a\omega \setminus \omega$,*

(2) *there is a retraction $s\colon b\omega \to b\omega \setminus \omega$,*

(3) *$f\colon a\omega \setminus \omega \to b\omega \setminus \omega$ is a homeomorphism.*

If the weight of $a\omega \setminus \omega$ is less than $\mathfrak{p}$, then f can be extended to a homeomorphism $\bar{f}\colon a\omega \to b\omega$.

So the theorem simply says that there is a permutation $\pi\colon \omega \to \omega$ such that $\bar{f} = f \cup \pi$ is a homeomorphism.

Proof: Let $\kappa < \mathfrak{p}$ be the weight of $a\omega \setminus \omega$. The sets $A = r[\omega]$ and $B = s[\omega]$ are countable. So we may pick an open base $\mathcal{U} = \{U_{\alpha,0} : \alpha < \kappa\}$ for $a\omega \setminus \omega$ such that for every $\alpha < \kappa$,

$$(\overline{U}_{\alpha,0} \setminus U_{\alpha,0}) \cap (A \cup f^{-1}[B]) = \emptyset.$$

For $\alpha < \kappa$, put $U_{\alpha,1} = (a\omega \setminus \omega) \setminus \overline{U}_{\alpha,0}$, $V_{\alpha,0} = f[U_{\alpha,0}]$ and $V_{\alpha,1} = f[U_{\alpha,1}]$. Observe that

$$(r^{-1}[U_{\alpha,0}] \cap \omega) \cup (r^{-1}[U_{\alpha,1}] \cap \omega) = \omega, \quad (r^{-1}[U_{\alpha,0}] \cap \omega) \cap (r^{-1}[U_{\alpha,1}] \cap \omega) = \emptyset,$$

and

$$(s^{-1}[V_{\alpha,0}] \cap \omega) \cup (s^{-1}[V_{\alpha,1}] \cap \omega) = \omega, \quad (s^{-1}[V_{\alpha,0}] \cap \omega) \cap (s^{-1}[V_{\alpha,1}] \cap \omega) = \emptyset.$$

For $n < \omega$, let $f_n\colon \kappa \to 2$ be defined by:

$$f_n(\alpha) = 0 \iff r(n) \in U_{\alpha,0}.$$

Observe that if $f_n(\alpha) = 1$ then $r(n) \in U_{\alpha,1}$. Consider the collection

$$\{V_{\alpha,f_n(\alpha)} : \alpha < \kappa\}.$$

This is a collection of open subsets of $b\omega \setminus \omega$, each element of which contains $f(r(n))$. As a consequence,

$$\{s^{-1}[V_{\alpha,f_n(\alpha)}] \cap \omega : \alpha < \kappa\}$$

has the *sfip*, and hence has an infinite pseudo-intersection H_n by $\kappa < \mathfrak{p}$. For each $\alpha < \kappa$, define $h_\alpha\colon \omega \to \omega$, as follows:

$$h_\alpha(n) = 1 + \max(H_n \setminus s^{-1}[V_{\alpha,f_n(\alpha)}]).$$

So each element $m \in H_n$ with $m \geq h_\alpha(n)$ belongs to $s^{-1}[V_{\alpha,f_n(\alpha)}] \cap \omega$. Since $\kappa < \mathfrak{p} \leq \mathfrak{b}$, there is a function $g_0\colon \omega \to \omega$ such that $h_\alpha \leq^* g_0$ for every $\alpha < \kappa$. Since each H_n is infinite, there is a strictly increasing function $\pi_0\colon \omega \to \omega$ such that $g_0 \leq \pi_0$ and $\pi_0(n) \in H_n$ for every n.

CLAIM 1: *For every $\alpha < \kappa$ and $i < 2$, $\pi_0[r^{-1}[U_{\alpha,i}] \cap \omega] \subseteq^* s^{-1}[V_{\alpha,i}] \cap \omega$.*

Let $\alpha < \kappa$ and $i < 2$ be arbitrary. Let $N < \omega$ be such that $h_\alpha(n) \leq g_0(n)$ for every $n \geq N$. Take an arbitrary $n \geq N$ with $n \in r^{-1}[U_{\alpha,i}] \cap \omega$. Then $\pi_0(n) \in H_n$ and $\pi_0(n) \geq h_\alpha(n)$. Hence, by construction, $\pi_0(n) \in s^{-1}[V_{\alpha,i}] \cap \omega$.

By a completely analogous reasoning, one obtains an injection $\pi_1\colon \omega \to \omega$ such that:

CLAIM 2: *For every $\alpha < \kappa$ and $i < 2$, $\pi_1[s^{-1}[V_{\alpha,i}] \cap \omega] \subseteq^* r^{-1}[U_{\alpha,i}] \cap \omega$.*

By the proof of the Cantor–Bernstein Theorem, there are partitions of ω into the sets M_1 and M_2, and into the sets N_1 and N_2, such that $\pi_0[M_1] = N_1$ and $\pi_1[N_2] = M_2$. Define the permutation $\pi\colon \omega \to \omega$ by

$$\pi(n) = \begin{cases} \pi_0(n) & (n \in M_1), \\ \pi_1^{-1}(n) & (n \in M_2). \end{cases}$$

By combining Claims 1 and 2, we get:

CLAIM 3: *For every $\alpha < \kappa$ and $i < 2$, $\pi[r^{-1}[U_{\alpha,i}] \cap \omega] =^* s^{-1}[V_{\alpha,i}] \cap \omega$.*

It now suffices to prove the following:

CLAIM 4: *$\bar{f} = f \cup \pi\colon a\omega \to b\omega$ is a homeomorphism.*

By compactness it suffices to prove that $\bar{f}$ is continuous. To this end, let V be a nonempty open subset of $b\omega$. We may assume without loss of generality that $V \cap (b\omega \setminus \omega) \neq \emptyset$. Consider $\bar{f}^{-1}[V]$. We need to prove that it is a neighborhood of an arbitrarily chosen point $x \in \bar{f}^{-1}[V] \setminus \omega$. Since f is a homeomorphism, $W = f^{-1}[V \cap (b\omega \setminus \omega)]$ is a neighborhood of x in $a\omega \setminus \omega$. Pick $\alpha < \kappa$ such that $x \in U_{\alpha,0} \subseteq \overline{U}_{\alpha,0} \subseteq W$, and consider the set $V_{\alpha,0} = f[U_{\alpha,0}]$. Observe that $f(x) \in f[U_{\alpha,0}] \subseteq \overline{f[U_{\alpha,0}]} \subseteq V$. We claim that $E = s^{-1}[V_{\alpha,0}] \setminus V$ is finite. Striving for a contradiction, assume otherwise, and let $p \in b\omega \setminus \omega$ be a limit point of E. Since $s^{-1}[V_{\alpha,1}]$ is open and misses $s^{-1}[V_{\alpha,0}]$, it follows that $p \notin V_{\alpha,1}$, hence $p \in \overline{V}_{\alpha,0} \subseteq V$. Hence V is a neighborhood of p, and consequently intersects E. This is a contradiction. By Claim 3, we therefore conclude that $r^{-1}[U_{\alpha,0}] \setminus \bar{f}^{-1}[V]$ is a finite subset of ω, and so is a clopen subset of $a\omega$. Hence $\bar{f}^{-1}[V]$ is indeed a neighborhood of x. ∎

Remark 4.3: The existence of the retractions r and s is essential in Theorem 4.2. From a Hausdorff gap, it is possible to construct a compactification $\gamma\omega$ of ω containing a point $p \in \gamma\omega \setminus \omega$ such that $\gamma\omega \setminus \{p\}$ is not normal, while $\gamma\omega \setminus (\omega \cup \{p\})$ is the union of two disjoint copies of ω_1 with the order topology. See van Douwen [6] for details. It is not difficult to prove that there is a compactification $a\omega$ of ω such that $a\omega \setminus \omega$ contains a point q such that $a\omega \setminus (\omega \cup \{q\})$ is a copy of ω_1 with the order topology. It is easy to see that $a\omega \setminus \{q\}$ is normal. Hence, if we consider the topological sum of two copies of $a\omega$ with the points corresponding to q identified to a single point, then one obtains a compactification $b\omega$ of ω such that $\gamma\omega \setminus \omega \approx b\omega \setminus \omega$, while $\gamma\omega$ and $b\omega$ are not homeomorphic.

There are quite a few nontrivial compactifications of ω to which Theorem 4.2 can be applied. We are particularly interested in the case of compactifications whose remainders are products of second countable compacta. A result of Arhangel'skiĭ, Chandler, Faulkner and Vipera [4, Theorem 3.3] implies that all such compactifications are 'good' for us. For completeness sake, we will present a short and elementary proof of a special case of their result (all of Theorem 3.2 in [4] can be proved by a slightly more complicated argument).

PROPOSITION 4.4: *Let $\gamma\omega$ be a compactification of ω. If K is a second countable compactum, and $f\colon \gamma\omega \setminus \omega \to K$ is continuous, then f can be extended to a continuous function $\bar{f}\colon \gamma\omega \to K$.*

Proof: We may assume that K is a subspace of the Hilbert cube Q. Let ϱ be an admissible metric for Q. The function f can be extended to a continuous function $g\colon \gamma\omega \to Q$. For each $n < \omega$, let $\xi(n) \in K$ be such that

$$\varrho(g(n), K) = \varrho(g(n), \xi(n)).$$

We claim that $\bar{f} = f \cup \xi\colon \gamma\omega \to K$ is continuous. To this end, let $U \subseteq K$ be open. We have to prove that $\bar{f}^{-1}[U]$ is open in $\gamma\omega$. To this end, pick an arbitrary element $x \in (\gamma\omega \setminus \omega) \cap \bar{f}^{-1}[U]$, and pick $\varepsilon > 0$ such that the open ball B with center $f(x)$ and radius 2ε in Q has the property that $B \cap K \subseteq U$. Let B' be the open ball with center $f(x)$ and radius ε in Q. Then $V = g^{-1}[B']$ is an open neighborhood of x in $\gamma\omega$. We claim that $V \subseteq \bar{f}^{-1}[U]$. To prove this, take an arbitrary element $n \in V \cap \omega$. Then $g(n) \in B'$, hence $\varrho(g(n), K) \leq \varrho(g(n), f(x)) < \varepsilon$. As a consequence, $\varrho(g(n), \xi(n)) < \varepsilon$, so $\varrho(f(x), \xi(n)) < 2\varepsilon$, i.e., $\xi(n) \in B \cap K \subseteq U$. ∎

So we clearly have:

COROLLARY 4.5: *Let $\gamma\omega$ be a compactification of ω such that $\gamma\omega \setminus \omega$ is a product of second countable compacta. Then $\gamma\omega \setminus \omega$ is a retract of $\gamma\omega$.*

5. Proof of Theorem 1.6: part 2

Let $\mathcal{U}$ be the collection of all clopen neighborhoods U_e of $e \in \mathsf{C}$ having the property that for every $n < \omega$, either $C_n \subseteq U_e$ or $C_n \cap U_e = \emptyset$. It is clear that $\mathcal{U}$ is a neighborhood base at e.

Now take any two elements $x, y \in K$ and consider the points $\langle e, x\rangle$ and $\langle e, y\rangle$ of X. As observed in §3, all we need to do is to prove that these points can be homeomorphed onto each other. We aim at applying Theorem 4.2. From now on, assume that $\omega_1 < \mathfrak{p}$. This is true, for example, under $\mathsf{MA}+\neg\mathsf{CH}$.

For every $n < \omega$, pick an arbitrary element $x_n \in C_n$, and consider the point

$$p_n = \langle x_n, f(x_n)\rangle = \langle x_n, d_n\rangle \in X.$$

Clearly, $p_n \in \pi_0^{-1}[C_n]$. Put $E = \{p_n : n < \omega\}$ and notice that E is a discrete subset of X.

LEMMA 5.1: $\overline{E} = E \cup (\{e\} \times K)$.

Proof: If $\langle u, v\rangle \in \overline{E} \setminus E$ then clearly $u = e$. Conversely, consider a point of the form $\langle e, v\rangle$, where $v \in K$. We aim at proving that every neighborhood of $\langle e, v\rangle$ meets E. By Lemma 3.1, it suffices to consider a neighborhood of the form $U_e \otimes W$, where $U_e \in \mathcal{U}$ and W is an open neighborhood of v in K. Since D is dense in K, there are infinitely many m with $d_m \in W \cap D$. Since $(x_n)_{n<\omega}$ converges to e, we may pick one of those m's with $x_m \in U_e$. Since $f(x_m) = d_m$, we have $x_m \in f^{-1}[W]$, i.e.,

$$p_m = \langle x_m, f(x_m)\rangle = \langle x_m, d_m\rangle \in U_e \otimes W.$$

So $U_e \otimes W$ indeed meets E. ∎

Let f: $\{e\} \times K \to \{e\} \times K$ be a homeomorphism with $f(\langle e, x\rangle) = \langle e, y\rangle$. Since K is a product of second countable compacta, $\{e\} \times K$ is a retract of $\overline{E}$ by Corollary 4.5. By Theorem 4.2 there consequently is a homeomorphism $\bar{f}$: $\overline{E} \to \overline{E}$ which extends f. So there is a permutation τ: $E \to E$ such that $\bar{f} = f \cup \tau$. If $m, m' < \omega$, let $\xi_{m,m'}$: $\pi_0^{-1}[C_m] \to \pi_0^{-1}[C_{m'}]$ be any homeomorphism (Lemma 3.3). Define F: $X \to X$ by the following formula:

$$F(\langle u, v\rangle) = \begin{cases} f(\langle e, v\rangle) & (u = e), \\ \xi_{m,m'}(\langle u, v\rangle) & (u \in C_m, \tau(p_m) = p_{m'}). \end{cases}$$

It is clear that F is a bijection. We claim that F is a homeomorphism. For this, it suffices to check the continuity of F at the points of $\{e\} \times K$.

LEMMA 5.2: *Let $U_e \in \mathcal{U}$, $W \subseteq K$ open, and $m < \omega$.*
(1) *If $\pi_0^{-1}[C_m] \cap (U_e \otimes W) \neq \emptyset$ then $\pi_0^{-1}[C_m] \subseteq U_e \otimes W$.*
(2) *If $\pi_0^{-1}[C_m] \cap F^{-1}[U_e \otimes W] \neq \emptyset$ then $\pi_0^{-1}[C_m] \subseteq F^{-1}[U_e \otimes W]$.*

Proof: For (1), take an arbitrary element $\langle u, v\rangle \in \pi_0^{-1}[C_m] \cap (U_e \otimes W)$. Then

$$u \in C_m \cap U_e \cap f_e^{-1}[W] = C_m \cap U_e \cap f^{-1}[W].$$

So $C_m \subseteq U_e$ and $f(u) \in W$. Since f is constant on C_m, we even have $f[C_m] \subseteq W$. Hence if $\langle u', v'\rangle$ is an arbitrary element in $\pi_0^{-1}[C_m]$ then $u' \in U_e \cap f^{-1}[W]$, i.e., $\langle u', v'\rangle \in U_e \otimes W$.

For (2), let $\tau(p_m) = p_{m'}$, and pick an arbitrary element

$$\langle u, v\rangle \in \pi_0^{-1}[C_m] \cap F^{-1}[U_e \otimes W].$$

Then $F(\langle u, v\rangle) \in \pi_0^{-1}[C_{m'}] \cap (U_e \otimes W)$. Hence by (1),

$$F[\pi_0^{-1}[C_m]] = \pi_0^{-1}[C_{m'}] \subseteq U_e \otimes W. \qquad \blacksquare$$

Take an arbitrary element $\langle e, v\rangle \in \{e\} \times K$, let $U_e \in \mathcal{U}$ be arbitrary, and let $W \subseteq K$ be an open neighborhood of v in K. We claim that $F^{-1}[U_e \otimes W]$ is open.

To prove this, pick an arbitrary element $\langle a, b\rangle \in F^{-1}[U_e \otimes W]$. Assume first that $a \neq e$, say $a \in C_m$ for some $m < \omega$. Then by Lemma 5.2(2), $\pi_0^{-1}[C_m]$ is a neighborhood of $\langle a, b\rangle$ which is contained in $F^{-1}[U_e \otimes W]$. So we are done in this case. Assume next that $a = e$. Hence $\langle e, b\rangle \in F^{-1}[U_e \otimes W]$. Since F and $\bar{f}$ both extend f, we have

$$\bar{f}(\langle e, b\rangle) \in U_e \otimes W.$$

Since $\bar{f}$ is a homeomorphism, by Lemma 3.1 we may pick $V_e \in \mathcal{U}$ and a nonempty open set H in K such that

$$(*) \qquad \langle e, b\rangle \in (V_e \otimes H) \cap \overline{E} \subseteq \bar{f}^{-1}[U_e \otimes W].$$

We claim that

$$V_e \otimes H \subseteq F^{-1}[U_e \otimes W],$$

which clearly suffices. To this end, take an arbitrary element $\langle u, v\rangle \in V_e \otimes H$. We may assume that $u \neq e$, say $u \in C_m$ for $m < \omega$. By Lemma 5.2(1), $\pi_0^{-1}[C_m] \subseteq$

$V_e \otimes H$. This implies that $p_m \in V_e \otimes H$. Let $\tau(p_m) = p_{m'}$. Observe that $p_{m'} \in U_e \otimes W$ by $(*)$; hence

$$F[\pi_0^{-1}[C_m]] \cap (U_e \otimes W) = \pi_0^{-1}[C_{m'}] \cap (U_e \odot W) \neq \emptyset.$$

So $\langle u, v\rangle \in \pi_0^{-1}[C_m] \subseteq F^{-1}[U_e \otimes W]$ by Lemma 5.2(2).

6. Examples of homogeneous compacta

There is a variety of homogeneous compacta. They seem to fall into two subclasses. The first subclass is the class of homogeneous compacta which admit an algebraic structure of some sort. The second subclass consists of products of first countable compacta which are homogeneous. An important example is the Alexandroff double arrow line, [1, Ex. A_7], and its generalizations by van Douwen [8]. These spaces do not admit the structure of a topological group, have countable π-weight, are first countable but not second countable. Other examples include the ordered compacta of Maurice [25, 26]. Some of his spaces have cellularity $\mathfrak{c}$. Yet another example is the non-metrizable homogeneous Eberlein compact space constructed in [27].

Jensen pointed out that it is easy to construct homogeneous compact Souslin lines from $\diamondsuit$. The square of such a space is an example of a compact homogeneous space with uncountable cellularity. A related result is Kunen's recent construction under CH of a compact L-space which is even a right topological group, [21] (he asks whether there can be a homogeneous Souslin line which admits the structure of a right topological group). The square of his space satisfies the countable chain condition. Another result which generates homogeneous compacta is due to Dow and Pearl [10]: they proved, extending a result of Lawrence [23], that the infinite power of any zero-dimensional first countable space is homogeneous.

There are many unsolved questions on homogeneous compacta. To illustrate our ignorance, let us point out that it is not known whether every compact homogeneous space has cellularity at most $\mathfrak{c}$ (this is van Douwen's Problem), and whether every homogeneous compactum contains a nontrivial convergent sequence (this is W. Rudin's Problem). Observe that a counterexample to one of these questions is not first countable and is not a topological group. To see this, simply notice that an infinite compact group satisfies the countable chain condition and contains a nontrivial convergent sequence. Both statements follow easily from Kuz′minov's Theorem in [22] that every compact group is dyadic. Also, every first countable compactum has cardinality at most $\mathfrak{c}$ by Arhangel′skiĭ [2] and hence has cellularity at most $\mathfrak{c}$. It is easy to see that an arbitrary product of compact topological groups and first countable compacta does not yield a counterexample for basically the same reasons, cf., [16, p. 107]. So a counterexample, if it exists, must be something completely different.

For more information and questions on homogeneous compacta, see, e.g., Arhangel′skiĭ [3] and Kunen [20].

The proof that our space X is not homogeneous under CH (or even under $\mathfrak{c} < 2^{\omega_1}$) is based on a very simple cardinality argument (see §1). There are many other nonhomogeneity results in the literature which in essence boil down to cardinality considerations. Frolík's Theorem in [13] that $\mathbb{N}^*$ is not homogeneous is such an example. The proofs of these results were sometimes replaced by better proofs, presenting explicit topological properties shared by some but not all points of the spaces under consideration. In the case of Frolík's Theorem this was done by Kunen in [18]: he showed that some but not all points in $\mathbb{N}^*$ are weak P-points. Van Douwen called such arguments 'honest' nonhomogeneity proofs. For the space X constructed in this paper, it seems impossible to present an 'honest' proof of its nonhomogeneity in some model of set theory. Simply observe that it is homogeneous under $\mathsf{MA}+\neg\mathsf{CH}$. This is a rather curious phenomenon which deserves further study.

In private conversation, Kunen remarked that it is not clear whether compact right topological group implies anything interesting which does not follow from just compact homogeneous. As he observed, not every compact homogeneous space is a right topological group. The Hilbert cube is homogeneous by Keller's Theorem but is not a right topological group since it has the fixed point property (see [28] for details). In addition, there is an example of a compact right topological group under $\diamondsuit$ which is first countable and fails to have the countable chain condition (the square of the space in Kunen [21, Theorem 6.2]). So compact right topological groups need not be dyadic and first countable compact right topological groups need not be metrizable.

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