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ω-far points in large spaces

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To the memory of Murray Bell

Abstract

We prove that every normal non-compact space which is nowhere of cardinality at most $\mathfrak{c}$ has an ω-far point. This provides a partial answer to a question of van Douwen.

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1. Introduction

All spaces under discussion are Tychonoff, and if X is a space then X^* denotes $\beta X \setminus X$. Let X be a crowded (i.e., no isolated points) space. A point $p \in X^*$ is called an *ω-far point* of X provided that $p \notin \mathrm{cl}_{\beta X} D$ for any countable closed discrete set $D \subseteq X$. The concept of ω-far point was introduced by van Douwen [1], who proved that normal non-Lindelöf spaces and non-compact metrizable spaces have ω-far points. He used the concept of an ω-far point to present what he called "honest" proofs of the non-homogeneity of certain Čech–Stone remainders. The question raised in [1] of whether all non-pseudocompact crowded spaces have ω-far points is still open (although it has an affirmative answer under MA by [7]).

The concept of an ω-far point is strongly related to that of a remote point. A point $p \in X^*$ is called a *remote point* of X provided that $p \notin \mathrm{cl}_{\beta X} D$ for any nowhere dense

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$D \subseteq X$. If X is crowded then a remote point is certainly ω-far, but the converse need not be true.

Let X be a space. An *ω-filter* on X is a closed filter $\mathcal{F}$ on X such that for every countable subset D of X there exists an element $F \in \mathcal{F}$ with $F \cap D = \emptyset$. So, roughly speaking, an ω-filter is a closed filter "avoiding" all countable sets. Observe that an ω-filter consists of uncountable sets. If $\mathcal{F}$ is an ω-filter on the normal space X then any point $x \in \bigcap_{F \in \mathcal{F}} \mathrm{cl}_{\beta X} F$ is ω-far. To see this, note that if $D \subseteq X$ is countable closed and discrete, then there exists $F \in \mathcal{F}$ such that $F \cap D$ is empty. Now it suffices to observe that F and D have disjoint closures in βX since X is normal.

Observe that no compact space has an ω-filter. For if $\mathcal{F}$ is an ω-filter on a compact space X then $\bigcap \mathcal{F} \neq \emptyset$. Since $\mathcal{F}$ consists of closed sets, and "avoids" all singleton subsets of $\bigcap \mathcal{F}$, this is a contradiction.

The known ZFC proofs of the existence of ω-far points in certain spaces either prove the stronger result that ω-filters exist or prove the stronger result that remote points exist. Let us demonstrate this by a simple example in presenting van Douwen's proof that every non-Lindelöf space has an ω-filter. Indeed, let X be non-Lindelöf, and let $\mathcal{U}$ be an open cover of X having no countable subcover. The collection

$$\mathcal{F} = \left\{X \setminus \bigcup \mathcal{V}\colon\ \mathcal{V} \in [\mathcal{U}]^{\leqslant \omega}\right\}$$

generates an ω-filter on X. Hence by the above, every normal non-Lindelöf space has an ω-far point.

In this note we are interested in proving the existence of ω-far points by constructing ω-filters. By what we just observed, we need to consider non-compact Lindelöf spaces only. Hence all spaces we are interested in are normal. This simplifies things a bit. Our main result is that every non-compact Lindelöf space which is nowhere of cardinality at most $\mathfrak{c}$ has an ω-filter (as usual, if $\mathcal{P}$ is any topological property, we say that a space is *nowhere* $\mathcal{P}$ provided that no non-empty open subset of X has $\mathcal{P}$). This yields a partial answer to van Douwen's problem.

We finish this introduction by making some remarks on spaces with or without an ω-filter. It is clear that no countable space has an ω-filter. For uncountable spaces the situation is unclear. Let L be the one-point Lindelöfication of an uncountable discrete space, and let X be the product $L \times \mathbb{Q}$, here $\mathbb{Q}$ denotes the space of rational numbers. Then X is a non-compact crowded Lindelöf space having no ω-filter. (But X has a non-compact clopen subset of countable weight and therefore has a remote point and, since X is crowded, an ω-far point.) It is even consistent to have examples of uncountable subsets of the real line $\mathbb{R}$ having no ω-filter. Let $X \subset \mathbb{R}$ be uncountable and *concentrated* on a countable set (i.e., X contains a countable dense subset D such that each neighborhood of D is co-countable in X). Such sets exist under CH, see [8] for more details. Now X does not have an ω-filter since any closed set F which is disjoint from D will be countable. The question of which spaces do have an ω-filter is open. It is even unknown which subsets of $\mathbb{R}$ have an ω-filter.

We are indebted to the referee for finding and correcting some inaccuracies in an earlier version of this note.

2. A tool for constructing ω-filters

Let X be a space. We say that X has property $(*)_0$ if it contains a point countable uncountable family of closed sets. Every uncountable space has an uncountable family consisting of singletons and as a consequence has property $(*)_0$. So this concept is not very interesting. We say that X has property $(*)_1$ if it contains a point countable uncountable family of closed sets, each of which has property $(*)_0$. So this property simply says that X contains a point countable uncountable family of uncountable closed sets. Not every uncountable space has property $(*)_1$, as the one point Lindelöfication of an uncountable discrete space shows. We say that X has property $(*)_{n+1}$ if it contains a point countable uncountable family of closed sets, each of which has property $(*)_n$. One may note that if "point-countable" is strengthened to "pairwise disjoint", then a space X has property $(*)_n$ exactly when it contains a tree of closed sets $\{F_t\colon\ t \in \omega_1^{\leqslant n+1}\}$. That is, $F_\emptyset = X$, and for each $s \in \omega_1^{\leqslant i}$ for $i \leqslant n$, the $F_{s\frown\alpha}$, for $\alpha < \omega_1$, form a disjoint family of closed non-empty subsets of F_s.

We now present our main tool for constructing ω-filters in "large" spaces. The idea of the proof goes back to Kunen [6] and Dow and van Mill [5].

Theorem 2.1. *Let X be the topological sum of the spaces X_n, $n < \omega$. If for every n, X_n has property $(*)_n$ then X has an ω-filter.*

Proof. For each $n < \omega$ we fix a family of closed subsets of X_n indexed by $\leqslant n$-sized subsets of ω_1, denoted $\{B_n(F)\colon\ F \subset \omega_1,\ |F| \leqslant n\}$. We set $B_n(\emptyset) = X_n$, and, for each finite subset $F \subseteq \omega_1$ with $|F| < n$, we select $B_n(F \cup \{\alpha\}) \subseteq B_n(F)$ for all $\max F < \alpha < \omega_1$ to be a point-countable family of closed sets each with property $(*)_{n-|F|-1}$.

For each $\beta < \omega_1$ let $\{a(\beta, n)\colon\ n < \omega\}$ be an increasing chain of finite subsets of β so that $\beta = \bigcup_{n<\omega} a(\beta, n)$. Define $G_\beta \subseteq X$ so that $G_\beta \cap X_n$ is the union of all $B_n(\{\alpha_1, \ldots, \alpha_k, \beta\})$ such that $k < n$ and $\{\alpha_1, \ldots, \alpha_k\} \subseteq a(\beta, n)$. This is a finite union, so G_β is closed. We claim that the filter generated by $\mathcal{G} = \{G_\beta\colon\ \beta < \omega_1\}$ is as required.

Claim 1. $\mathcal{G}$ has the finite intersection property.

Take arbitrary $\beta_1 < \beta_2 < \cdots < \beta_k < \omega_1$ and fix n large enough so that for each $i < j \leqslant k$, $\beta_i \in a(\beta_j, n)$. It follows that $B_n(\{\beta_1, \ldots, \beta_k\}) \subseteq G_{\beta_1} \cap \cdots \cap G_{\beta_k}$.

Claim 2. For every countable $D \subseteq X$ there exists $\beta < \omega_1$ such that $G_\beta \cap D = \emptyset$.

For convenience of notation, let $B_n(F) = \emptyset$ for each $F \subset \omega_1$ with $n < |F|$. Note that the family $\{B_n(\{\alpha\})\colon\ n < \omega,\ \alpha < \omega_1\}$ is point-countable, hence there is a β_0 such that $B_n(\{\alpha\})$ is disjoint from D for each $\alpha \geqslant \beta_0$. Similarly, the family $\{B_n(\{\gamma\} \cup \{\alpha\})\colon\ n < \omega,\ \gamma < \beta_0$ and $\alpha < \omega_1\}$ is point-countable, hence there is a $\beta_1 \geqslant \beta_0$ in ω_1 so that $B_n(\{\gamma\} \cup \{\alpha\})$ is disjoint from D for each $n \in \omega$, $\gamma < \omega_1$ and $\beta_1 \leqslant \alpha < \omega_1$ (note $\min(\gamma, \alpha) \geqslant \beta_0$ is handled because $B_n(\{\gamma, \alpha\}) \subset B_n(\{\min(\gamma, \alpha)\})$). Proceeding by induction on k, there is a $\beta_k \in \omega_1$ so that for each $F \subset \omega_1$ with $|F| \leqslant k$, each $n \in \omega$ and $\alpha \geqslant \beta_k$, D is disjoint from $B_n(F \cup \{\alpha\})$. Therefore, for each $\beta \geqslant \sup\{\beta_k\colon\ k \in \omega\}$, G_β is disjoint from D. $\square$

3. ω-filters in second countable spaces

The aim of this section is to consider the second countable spaces which have an ω-filter. Recall that a space which is concentrated on a countable set will not have an ω-filter.

Lemma 3.1. *Let X be a space of cardinality ω_1 which is not concentrated on a countable set. Then X has property $(*)_1$.*

Proof. It is clear that X has property $(*)_0$. Let $X = \{x_\alpha\colon \alpha < \omega_1\}$ and choose F_α any uncountable closed set which is disjoint from $\{x_\beta\colon \beta < \alpha\}$. It is clear that the family $\{F_\alpha\colon \alpha < \omega_1\}$ is point-countable. $\square$

Corollary 3.2. *If X has cardinality ω_1 and no uncountable closed subset of X is concentrated on a countable set, then X has property $(*)_n$ for all $n \in \omega$.*

Theorem 3.3. *Let X be a second countable space which has an ω-filter. Then X contains a non-compact closed subspace which is nowhere concentrated on a countable set.*

Proof. Let $\mathcal{F}$ be an ω-filter on X, an fix an arbitrary $F \in \mathcal{F}$. Let $\mathcal{U}$ be the family of all relatively open subsets of F which are concentrated on a countable set. There is a countable $\mathcal{U}' \subseteq \mathcal{U}$ with $\bigcup\mathcal{U}' = \bigcup\mathcal{U}$. As a consequence, $\bigcup\mathcal{U}$ is concentrated on a countable set, say D. It also follows that $A = F \setminus \bigcup\mathcal{U}$ is nowhere concentrated on a countable set. It is enough to show that A is a member of $\mathcal{F}$ and that A is not compact. There exists $F' \in \mathcal{F}$ with $F' \cap D = \emptyset$. Then $F' \cap \bigcup\mathcal{U}$ is countable. So there also exists $F'' \in \mathcal{F}$ with $F'' \cap (F' \cap \bigcup\mathcal{U}) = \emptyset$, hence A contains $F \cap F' \cap F''$ a member of $\mathcal{F}$. Also, A is not compact because otherwise $\bigcap\mathcal{F} \neq \emptyset$, which contradicts the fact that $\mathcal{F}$ "avoids" all countable sets, in particular, all singleton subsets of X. $\square$

Question 1. If a second countable non-compact space has the property that no uncountable closed set is concentrated on a countable set, does this space have an ω-filter?

4. ω-filters in large spaces

We have already observed that a non-Lindelöf space has an ω-filter so the aim of this section is to prove the following result.

Theorem 4.1. *Let X be a non-compact, Lindelöf space. If X is nowhere of cardinality $\leqslant \mathfrak{c}$ then X has an ω-filter (in particular, X has an ω-far point).*

We begin by making some reductions that will be useful in separating the proof into cases. Since X is Lindelöf but not compact, there is a discrete family $\{X_n\colon n \in \omega\}$ of non-empty regular closed subsets of X. Since X is nowhere of cardinality $\leqslant \mathfrak{c}$, it follows that each X_n is nowhere of cardinality $\leqslant \mathfrak{c}$. We will assume that X is equal to the union of these X_n's.

Our proof will split into cases according to the weight of the X_n's. The following two cases are not exclusive but are easily seen to be inclusive.

In the first case we may assume that no relatively open subset of any of the X_n's has weight at most $\mathfrak{c}$. We will prove that such a space has an ω-filter (not requiring that X be nowhere of cardinality $\leqslant \mathfrak{c}$). In the case that infinitely many of the X_n has a nowhere ccc open subset, it follows immediately from Theorem 2.1 that we have the stronger result, namely that each X_n has property $(*)_n$. Therefore, in this case we may assume that X is ccc.

In the second case, we may assume that each X_n has weight at most $\mathfrak{c}$. In this case we are able to prove a stronger result, namely that each X_n has the property $(*)_n$.

The following result is a consequence of the proof of van Mill [7, Theorem 7.2], however in the interest of completeness we prove a pair of results that are improvements of the corresponding ideas in [7].

Theorem 4.2. *Let X be the topological sum of the spaces X_n, $n < \omega$, where each X_n is ccc and nowhere of weight $\leqslant \mathfrak{c}$. Then X has has an ω-filter.*

The main step is an improvement (basically a new proof) of the key construction in [7].

Theorem 4.3. *Let X be a ccc space which is nowhere of weight less than $\kappa = \mathfrak{c}^+$. Then X contains an independent family of disjoint* (*even completely separated*) *regular closed sets.*

Proof. Fix any chain $\{M_\alpha\colon\ \alpha < \kappa\}$ of elementary submodels each of cardinality $\mathfrak{c}$ and each closed under ω-sequences (i.e., $M_\alpha^\omega \subset M_\alpha$ for each $\alpha < \kappa$). Also arrange it so that $M_\alpha \in M_{\alpha+1}$ for each $\alpha \in \kappa$. The reader can find basic details about elementary submodels and chains in [3] and [4].

Since the weight of a regular space is bounded by its π-weight raised to its cellularity we have that no open subset of X has π-weight less than κ. Let B denote the Boolean algebra $RO(X)$. Therefore for each α we can choose (using the notation of [7]) a regular open set $\emptyset \neq U_\alpha \in M_{\alpha+1}$ such that no non-empty member of $M_\alpha \cap B$ is contained in U_α. In addition, we can (obviously) choose any $C_\alpha \in B \cap M_{\alpha+1}$ which is completely separated from the complement of U_α.

Now we use that M_α is closed under ω-sequences to note that each member of B has a projection into $M_\alpha \cap B$. Indeed, for any $U \in B$, we define $\mathrm{pr}_\alpha(U)$ to be the meet in B of those members of $M_\alpha \cap B$ which contain U. Since B is a complete ccc Boolean algebra and since M_α is closed under ω-sequences, $\mathrm{pr}_\alpha(U)$ is actually a member of $B \cap M_\alpha$. For each α, let A_α be the complement (in $RO(X)$) of U_α, and we will show that $\{\langle A_\alpha, C_\alpha\rangle\colon\ \alpha \in \kappa\}$ contains a κ-sized independent family.

For each α, let A'_α be the projection in M_α of A_α and let C'_α be the projection in M_α of C_α. We show that $A'_\alpha \cap C'_\alpha$ is not empty. First of all,

$$A'_\alpha \cap C'_\alpha \supset A_\alpha \cap C'_\alpha \supset C'_\alpha \setminus U_\alpha.$$

This latter set is not empty since C'_α is a non-empty member of M_α and U_α does not contain any such element.

By the pressing down lemma there is a stationary subset S of κ and a pair A', C' in $RO(X)$ such that for all $\alpha \in S$, $A'_\alpha = A'$ and $C'_\alpha = C'$. We finish by checking that $\{\langle A_\alpha, C_\alpha\rangle\colon \alpha \in S\}$ has the appropriate finite intersection property.

Suppose that $n \in \omega$ and $\beta_0 < \cdots < \beta_{n-1} < \kappa$ and $f \in 2^n$. For each β_i, let B_{β_i} denote A_{β_i} if $f(i) = 0$ and denote C_{β_i} otherwise.

Claim. By induction on $j \leqslant n$, $A' \cap C' \cap \bigcap_{i<j} B_{\beta_i}$ is not empty.

For $j = 0$, we just have $A' \cap C'$ which we have proven is not empty. Now suppose that $H = A' \cap C' \cap \bigcap_{i<j} B_{\beta_i}$ and note that H is a non-empty member of M_{β_j} which is contained in $A' \cap C'$. It suffices to show that $H \cap A_{\beta_j}$ and $H \cap C_{\beta_j}$ are both non-empty. The proof for C_{β_j} also works for A_{β_j} by symmetry. By the definition of $\mathrm{pr}_{\beta_j}(C_{\beta_j}) = C'$, it follows that if H is disjoint from C_{β_j} it must also be disjoint from C', which, of course, it is not. □

Proof of Theorem 4.2. Let $\kappa = \mathfrak{c}^+$. We can easily deduce from Theorem 4.3 that each of our spaces X_n maps densely into $\mathbb{I}^\kappa$. Indeed, Theorem 4.3 implies that each of the X_n map into $\mathbb{I}^\kappa$ by a mapping, f_n, which has the property that 2^κ is contained in the closure of the range. Since there is a mapping from $\mathbb{I}^\kappa$ to itself which sends 2^κ onto $\mathbb{I}^\kappa$, it follows that there is a mapping from X_n to a dense subset of $\mathbb{I}^\kappa$.

Now $\mathbb{I}^\kappa$ has a remote filter (see [2]) and countable sets are nowhere dense. This implies that every dense subset of $\mathbb{I}^\kappa$ has an ω-filter (simply trace the remote filter on the dense set; we do not run into problems here since every element of a remote filter has nonempty interior and so intersects the dense set).

So we are done since if a space S can be mapped onto a space T with an ω-filter then S has an ω-filter. □

We are able to prove, from CH, the stronger result that spaces such as in Theorem 4.3 have property $(*)_n$ for all n (see Theorem 4.8). However we have been unable to decide this in ZFC, even for $n = 2$, which seems to be an interesting problem.

We next consider the second case of our main result.

Theorem 4.4. *Let X be the topological sum of spaces X_n, each of them Lindelöf, weight $\leqslant \mathfrak{c}$ and of cardinality greater than $\mathfrak{c}$. Then each X_n satisfies $(*)_n$ and X has an ω-filter.*

Since a pairwise disjoint family is certainly point countable, this theorem follows easily from Lemma 4.7 below and an application of Theorem 2.1.

Lemma 4.5. *Let Y be a space with $|Y| > \mathfrak{c}$ and weight $\leqslant \mathfrak{c}$. In addition, let C be the set of those $y \in Y$ having a neighborhood of size at most $\mathfrak{c}$. Then $B = Y \setminus C$ has size greater than $\mathfrak{c}$.*

Proof. Since C is open it can be covered by $\leqslant \mathfrak{c}$ open sets each of cardinality $\leqslant \mathfrak{c}$. (Here we use that Y has weight $\leqslant \mathfrak{c}$.) As a consequence, $|C| \leqslant \mathfrak{c}$, hence $|B| > \mathfrak{c}$. □

Lemma 4.6. *Let Y be a normal space, and let $\mathcal{D}$ be a countable collection closed G_δ-subsets of Y. Then $Y \setminus \bigcup \mathcal{D}$ can be covered by $\leqslant \mathfrak{c}$ closed G_δ-subsets of Y.*

Proof. This is obvious since for every $D \in \mathcal{D}$, $Y \setminus D$ is an F_σ-subset of Y, hence by normality can be covered by countably many closed G_δ-subsets of Y. □

Lemma 4.7. *Let Y be a normal space with $|Y| > \mathfrak{c}$ and weight $\leqslant \mathfrak{c}$. Then Y contains a family $\mathcal{D}$ consisting of ω_1 pairwise disjoint closed G_δ-subsets, each of cardinality greater than $\mathfrak{c}$.*

Proof. Let $\mathcal{E}_0 = \{Y\}$ and let B be as in Lemma 4.5. Pick an arbitrary $p \in B$, and let $\mathcal{E}_1$ be a family of closed G_δ-subsets of Y, maximal with respect to the properties of being pairwise disjoint, contained in $Y \setminus \{p\}$ and of cardinality greater than $\mathfrak{c}$. Since B is infinite, $|\mathcal{E}_1| \geqslant 1$. Assume for a moment that $\mathcal{E}_1$ is finite. Then $Y \setminus \bigcup \mathcal{E}_1$ is a neighborhood of p, and contains a closed neighborhood V of p. Then $|V| > \mathfrak{c}$ and by Lemma 4.5 we may pick $q \in V \setminus \{p\}$ and a closed neighborhood W of q with $W \subseteq V \setminus \{q\}$ such that $|W| > \mathfrak{c}$. Without loss of generality, W is a G_δ-subset of Y. But this contradicts the maximality of $\mathcal{E}_1$. We conclude that $\mathcal{E}_1$ is infinite. Now if $\mathcal{E}_1$ is uncountable, we are done and we may stop. Suppose therefore that $\mathcal{E}_1$ is countably infinite. Then $Y \setminus \bigcup \mathcal{E}_1$ is of cardinality $\leqslant \mathfrak{c}$ by Lemma 4.6.

Observe that every $E \in \mathcal{E}_1$ is a closed G_δ-subset of Y and is of cardinality greater than $\mathfrak{c}$. So we may repeat the same procedure in every $E \in \mathcal{E}_1$, thus obtaining the family $\mathcal{E}_2$. Observe that $\mathcal{E}_2$ consists of G_δ-subsets of Y since a G_δ-subset of a G_δ-subset is a G_δ-subset. Now if $\mathcal{E}_2$ is uncountable, we are done. So suppose that this is not true. Then $\mathcal{E}_2$ is countably infinite, and every $E \in \mathcal{E}_1$ is being "split" at least infinitely often. In addition, by the same reasoning as above, $|Y \setminus \mathcal{E}_2| \leqslant \mathfrak{c}$.

If this process continues to a limit α, we let $\mathcal{F}$ be the family consisting of all intersections of chains which have exactly one element from each of the $\mathcal{E}_\beta$, $\beta < \alpha$. There are at most $\omega^\omega = \mathfrak{c}$ such intersections that are potentially non-empty, and each of those is a G_δ-subset of Y. Observe that $|Y \setminus \bigcup \mathcal{F}| \leqslant \mathfrak{c}$, and hence there has to be at least one element in $\mathcal{F}$ which has cardinality greater than $\mathfrak{c}$. Let $\mathcal{E}_\alpha$ consist of all those elements of $\mathcal{F}$ which have cardinality $> \mathfrak{c}$.

So the process can be continued, for $\alpha < \omega_1$, for of course we can stop if any of the $\mathcal{E}_\alpha$ are uncountable. It then follows that the uncountable family of closed sets, $\bigcup\{\mathcal{E}_\alpha\colon \alpha < \omega_1\}$ forms a tree when ordered by $\supset$. For each $\alpha < \omega_1$, $Y \setminus \bigcup \mathcal{E}_\alpha$ has cardinality at most $\mathfrak{c}$, hence there is some $y \in Y$ such that the chain consisting of those $E_\alpha \in \mathcal{E}_\alpha$ which contain y is uncountable.

For every ordinal $\gamma < \omega_1$, recall that $E_{\gamma+1}$ is a member of an infinite pairwise disjoint family of subsets of E_γ hence we can pick a "successor" F_γ of E_γ that is disjoint from $E_{\gamma+1}$. Then $\mathcal{D} = \{F_\gamma\colon \gamma < \omega_1\}$ is as required. □

Theorem 4.8 (CH). *If X is Lindelöf and nowhere of weight at most $\mathfrak{c}$, then X has property $(*)_n$ for all n.*

Proof. We may assume that X is embedded into $[0,1]^\kappa$ for some κ. We may also assume that X is ccc. Fix an elementary submodel M closed under ω-sequences, with cardinality $\mathfrak{c}$ and which includes X and its topology. We define π_M to be the projection mapping from $[0,1]^\kappa$ onto $[0,1]^{M\cap\kappa}$. Two sets of interest are $\pi_M(X)$ and $\pi_M(X\cap M)$. If it happens that $\pi_M(X)$ has cardinality greater than $\mathfrak{c}$, then we are done by applying Theorem 4.4 to $\pi_M(X)$. On the other hand, if $\pi_M(X)$ has cardinality $\mathfrak{c}=\omega_1$, and if $\pi_M(X)$ is not separable, then similar to Corollary 3.2, X has $(*)_n$ for all n (use only nowhere separable ccc regular closed sets as in the proof of Corollary 3.2). Of course X itself is nowhere separable because it is nowhere of weight at most $\mathfrak{c}$.

We next observe that in a Lindelöf space, the closure of the union of each countable family of closed sets of weight at most $\mathfrak{c}$ will also have weight at most $\mathfrak{c}$. Indeed, a Lindelöf space of weight at most $\mathfrak{c}$ has at most $\mathfrak{c}$ continuous real-valued functions. A countable union of spaces, each of which has at most $\mathfrak{c}$ continuous real-valued functions will also have at most $\mathfrak{c}$ continuous real-valued functions. Finally, a regular space with a dense set which has at most $\mathfrak{c}$ continuous real-valued functions will again have at most $\mathfrak{c}$ continuous real-valued functions.

Now, we are assuming that $\pi_M(X)$ has cardinality $\mathfrak{c}$ and that $\pi_M(X)$ is separable. Fix a countable subset $\{x_n\colon n\in\omega\}\subseteq X$ so that $\pi_M(\{x_n\colon n\in\omega\})$ is dense in $\pi_M(X)$. By elementarity and the fact that $M^\omega\subset M$ we know that $\pi_M(X\cap M)$ is nowhere separable. Therefore we may assume that for each n, $\pi_M(x_n)\notin\pi_M(X\cap M)$. We can choose a sufficiently large countable set $J\subseteq M\cap\kappa$ such that for each n, if there is a G_δ in M containing x_n such that the set has weight at most $\mathfrak{c}$, then $[x_n\restriction J]=\{x\in X\colon x\restriction J=x_n\restriction J\}$ will have weight at most $\mathfrak{c}$. Let A denote those n such that $[x_n\restriction J]$ has weight at most $\mathfrak{c}$. As noted above, the closure of $\bigcup\{[x_n\restriction J]\colon n\in A\}$ has weight at most $\mathfrak{c}$, hence has nowhere dense union in X. This family is in M, hence a witness to its non-denseness is in M.

Therefore, it follows that we have an $x\in X$, namely one of the x_n's with $n\notin A$, such that that $\pi_M(x)\notin\pi_M(X\cap M)$, and for each G_δ in M which contains x, the weight of that G_δ is greater than $\mathfrak{c}$.

Let k be minimal such that there is a closed G_δ, K, in M which contains x but which does not have property $(*)_{k+1}$. Set $\mathcal{G}$ to be those G_δ's which have property $(*)_k$.

Claim. There is a G_δ $K'\subseteq K$ with $x\in K'$ and $K'\in M$ such that $\mathcal{G}\cap\mathcal{P}(K')$ has the finite intersection property.

Otherwise inductively choose $\{K_\alpha\colon\alpha\in\omega_1\}$, working in M, all of which are in $\mathcal{G}$ and such that for each α there is a $J_\alpha\subseteq K_\alpha$ which is in $\mathcal{G}$ and which is disjoint from $K_{\alpha+1}$. At limit stages $K_\alpha=\bigcap_{\beta<\alpha}K_\beta$; since $x\in K_\alpha$, it follows that $K_\alpha\in\mathcal{G}$. Given K_α and $\mathcal{G}\cap\mathcal{P}(K_\alpha)$ not having the finite intersection property (in M), there is a $J_\alpha\in\mathcal{G}\cap\mathcal{P}(K_\alpha)\cap M$ such that $x\notin J_\alpha$. Since J_α is a G_δ which is in M, there is a $K_{\alpha+1}\in M$ so that $x\in K_{\alpha+1}$ and $K_{\alpha+1}$ is disjoint from J_α. The definition of k gives a contradiction, because the family $\{J_\alpha\colon\alpha\in\omega_1\}$ witness that K has property $(*)_{k+1}$.

By the Claim we have that $\mathcal{G}\cap\mathcal{P}(K)$ generates a filter. Also, by the minimality of k, $x\in K'$ for each $K'\in M\cap\mathcal{G}\cap\mathcal{P}(K)$. It follows similarly that $\mathcal{G}\cap M$ is countably complete, hence, by elementarity, that $\mathcal{G}$ is countably complete. Since X is Lindelöf and $\mathcal{G}$ is countably complete, there will be a point y which is a member of G for all $G\in\mathcal{G}$.

Since $\mathcal{G}$ is in M, there will be such a point y in M. Clearly though, we will then have that $\pi_M(y) = \pi_M(x)$—which contradicts that $\pi_M(x) \notin \pi_M(X \cap M)$. $\square$

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