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Note on Weakly n-Dimensional Spaces

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Abstract. Weakly n-dimensional spaces were first distinguished by Karl Menger. In this note we shall discuss three topics concerning this class of spaces: universal spaces, products, and the sum theorem. We prove that there is a universal space for the class of all weakly n-dimensional spaces, present a simpler proof of Tomaszewski's result about the dimension of a product of weakly n-dimensional spaces, and show that there is an n-dimensional space which admits a pairwise disjoint countable closed cover by weakly n-dimensional subspaces but is not weakly n-dimensional itself.

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1. Introduction

We shall consider *separable metrizable spaces* only. Our terminology follows Kuratowski [3]. We assume that the reader is familiar with the basic notions and results in dimension theory. For details, see [3] and [1].

Given a space X, we shall denote by $X_{(n)}$ the set of all points in X that have arbitrarily small neighborhoods with at most $(n-1)$-dimensional boundaries, [3, §25, III]. If $1 \leqslant \dim X < \infty$, then the set

$$\Lambda(X) = X \backslash X_{(n-1)}, \quad n = \dim X, \tag{1}$$

is called the *dimensional kernel* of X. Menger's classical theorem asserts that

$$\dim \Lambda(X) \in \{\dim X, \dim X - 1\}, \tag{2}$$

cf. [9].

A space X is *weakly n-dimensional*, $n \geqslant 1$, if $\dim X = n$ and $\dim \Lambda(X) = n-1$, cf. [3, §27, VI], [1, p. 39]. The first example of a weakly one-dimensional space was constructed by Sierpiński [12], before the notion of dimension was defined. For $n \geqslant 2$, the existence of weakly n-dimensional spaces was demonstrated by Mazurkiewicz [6], thereby solving a problem of Menger. Simpler constructions are due to Tomaszewski [13] and van Mill and Pol [10].

In this note we shall discuss three topics concerning weakly n-dimensional spaces: universal spaces, products, and the sum theorem.

This note was written during the second author's visit to the Vrije Universiteit (Amsterdam). He would like to express his thanks for the hospitality.

In Section 2 we shall prove the following:

Theorem 1.1. *For each $n \geqslant 1$ there exists a weakly n-dimensional space E such that any weakly n-dimensional space embeds in E.*

Menger [8] asked whether there exists a weakly one-dimensional space X such that $\dim X^n = n$ for every n. This problem was solved by Tomaszewski [13], who proved the following interesting result:

Theorem 1.2. *(Tomaszewski). If X is weakly n-dimensional and Y is weakly m-dimensional then* $\dim(X \times Y) \leqslant n + m - 1$.

Section 3 is devoted to a simpler proof of this result. The basic idea of the proof follows [13], but our approach is more direct and elementary.

We do not know if for any weakly one-dimensional space X its cube X^3 must be one-dimensional.

In the last section we prove, refining a reasoning from [10], the following:

Theorem 1.3. *Let K be an $(n+1)$-dimensional compact space, $n \geqslant 1$. Then K contains an n-dimensional subspace X such that $X \backslash X_{(0)}$ is covered by the union of a countable family pairwise disjoint (relatively) closed weakly $(n-1)$-dimensional subsets of X.*

Let us comment on the assertion of Theorem 1.3, assuming $n \geqslant 2$. Let $\mathscr{E}$ be a countable pairwise disjoint collection of closed weakly $(n-1)$-dimensional subspaces of X covering $X \backslash X_{(0)}$, and let E denote its union. Since $X_{(0)}$ is zero-dimensional, $E_{(n-2)} \subseteq X_{(n-1)}$, and so

$$\Lambda(X) = X \backslash X_{(n-1)} \subseteq E \backslash E_{(n-2)} = \Lambda(E).$$

Since $\dim E = n-1, E \backslash E_{(n-2)}$ is the dimensional kernel of E, and so we get

$$\dim \Lambda(E) \geqslant \dim \Lambda(X) \geqslant n - 1.$$

So this yields:

Corollary 1.4. *For each $m \geqslant 1$ there is a space E which is not weakly m-dimensional, whereas it admits a countable pairwise disjoint closed cover consisting of weakly m-dimensional subspaces.*

For dimension one, such an example was constructed by Mazurkiewicz [4] (in response to a question of Sierpiński concerning "quasi-connectivity"). We did not find analogous examples for higher dimensions in the literature.

Mazurkiewicz [5] also constructed a space which is the union of two weakly one-dimensional closed subspaces but is not weakly one-dimensional itself. It is an obvious question whether our construction also yields such examples. This is not true unfortunately, since one can check that for any n-dimensional space Y with $n \geqslant 2$ the set $Y \backslash Y_{(0)}$ cannot be covered by finitely many weakly $(n-1)$-dimensional closed subspaces (the proof of this assertion is implicit in our proof of Theorem 1.3).

Problems concerning sum theorems for weakly n-dimensional spaces were raised by Menger, cf. [7], [9].

2. Proof of Theorem 1.1

We shall denote by $\mathbb{I}^\infty$ and 2^∞ the Hilbert cube and the Cantor set, respectively; let $p : 2^\infty \times \mathbb{I}^\infty \to 2^\infty$ be the projection. As usual, $\mathbb{N}$ denotes the set of natural numbers. We shall let $\mathscr{K}(\mathbb{I}^\infty)$ denote the space of nonempty compact sets in $\mathbb{I}^\infty$, endowed with the Vietoris topology [3, §42].

For a fixed natural number n, let $\mathscr{M}$ be the subspace of the product

$$\mathscr{K}(\mathbb{I}^\infty) \times \mathscr{K}(\mathbb{I}^\infty)^{\mathbb{N}} \times (\mathscr{K}(\mathbb{I}^\infty) \times \mathscr{K}(\mathbb{I}^\infty))^{\mathbb{N}},$$

consisting of all elements of the form

$$(K, \langle L_i : i \in \mathbb{N}\rangle, \langle (C_i, D_i) : i \in \mathbb{N}\rangle) \tag{1}$$

satisfying the following conditions:

$$L_1 \subseteq L_2 \subseteq \cdots \subseteq K, \quad \dim L_i \leqslant n-1 \quad (i \in \mathbb{N}), \tag{2}$$

$$K = C_i \cup D_i, \quad \dim(C_i \cap D_i) \leqslant n-2 \quad (i \in \mathbb{N}). \tag{3}$$

There is a subspace T in 2^∞ and a continuous mapping φ from T onto the set $\mathscr{M}$. It will be convenient to denote φ as follows:

$$t \mapsto (K(t), \langle L_i(t) : i \in \mathbb{N}\rangle, \langle (C_i(t), D_i(t)) : i \in \mathbb{N}\rangle). \tag{4}$$

For any $t \in T$, let

$$\begin{aligned} H(t) = \{z \in K(t) \setminus \textstyle\bigcup_{i=1}^\infty L_i(t) : \ & \text{for any neighborhood } U \text{ of } z \text{ in } \mathbb{I}^\infty \\ & \text{there is an } i \in \mathbb{N} \text{ with } z \in C_i(t) \subseteq U \\ & \text{and } z \notin D_i(t)\}. \end{aligned} \tag{5}$$

For $t \in T$, put

$$E(t) = H(t) \cup \bigcup_{i=1}^\infty L_i(t). \tag{6}$$

Finally, let $E \subseteq 2^\infty \times \mathbb{I}^\infty$ be the subspace of all points of the form (t, z), where $t \in T$ and $z \in E(t)$.

We shall show that

$$E \text{ is weakly } n\text{-dimensional}, \tag{7}$$

and

$$\text{any weakly } n\text{-dimensional space embeds in } E. \tag{8}$$

To this end let, us consider the sets

$$K^* = \{(t,z) : t \in T, z \in K(t)\}, \qquad L_i^* = \{(t,z) : t \in T, z \in L_i(t)\},$$
$$C_i^* = \{(t,z) : t \in T, z \in C_i(t)\}, \qquad D_i^* = \{(t,z) : t \in T, z \in D_i(t)\}.$$

Then for every $i \in \mathbb{N}$ we have

$$\dim L_i^* \leqslant n-1, E \subseteq C_i^* \cup D_i^*, \quad \dim(C_i^* \cap D_i^*) \leqslant n-2. \tag{9}$$

Indeed, observe that the projection p restricted to L_i^* or $C_i^* \cap D_i^*$ is perfect since φ is continuous. So (9) is a consequence of (2) and (3).

By (9), $\dim(\bigcup_{i=1}^{\infty} L_i^*) \leqslant n-1$. We claim that

$$E \setminus \bigcup_{i=1}^{\infty} L_i^* \subseteq E_{(n-1)}. \tag{10}$$

This implies among other things that the set $E \setminus E_{(n-1)}$ is at most $(n-1)$-dimensional. So if $\dim E > n$, then we contradict Menger's theorem (see (2) in §1). This implies that $\dim E \leqslant n$ and also by the same argument that E is weakly n-dimensional provided that E is n-dimensional. The latter fact will follow once we established (8). Simply observe that there are weakly n-dimensional spaces (cf. the introduction) which embed into E by (8). Hence E is at least n-dimensional since it contains n-dimensional subsets.

Let us consider any point $a = (t, z)$ from the set on the left hand side of the inclusion (10), i.e., $z \in H(t)$ by (6). Let W be a neighborhood of a in $2^\infty \times \mathbb{I}^\infty$, and let

$$U = \{x \in \mathbb{I}^\infty : (t, x) \in W\}.$$

Then U is a neighborhood of z; let $i \in \mathbb{N}$ be an index given by (5). The projection $p[C_i^* \setminus W]$ is closed in T and misses t since $C_i(t) \subseteq U$. Let V be an open-and-closed neighborhood of t in T, disjoint from $p[C_i^* \setminus W]$. Then $C_i^* \cap p^{-1}[V] \subseteq W$ and therefore $C_i^* \cap D_i^* \cap p^{-1}[V]$ is an $(n-2)$-dimensional partition between the point a and the set $E \setminus W$, cf. (9).

Having justified (10), let us consider now an arbitrary weakly n-dimensional space X. We shall find an element $t \in T$ such that X embeds in $E(t)$. The dimensional kernel $\Lambda(X)$ is an F_σ-set in X ([3, §27, VI]). Hence there exist closed sets $Z_1 \subseteq Z_2 \subseteq \cdots$ in X such that $\dim Z_i \leqslant n-1$ for every i and $X \setminus \bigcup_{i=1}^{\infty} Z_i \subseteq X_{(n-1)}$.

Kuratowski's theorem [2, Sect. 3], [3, §45, VII, Remark (ii)], provides an embedding $h : X \to \mathbb{I}^\infty$ such that $\dim \overline{h[Z_i]} \leqslant n-1$, for $i = 1, 2, \ldots,$ and $h(x) \in (\overline{h[X]})_{(n-1)}$ for any $x \in X \setminus \bigcup_{i=1}^{\infty} Z_i$. Let

$$K = \overline{h[X]}, \quad L_i = \overline{h[Z_i]}. \tag{11}$$

Then

$$h[X] \setminus \bigcup_{i=1}^{\infty} L_i \subseteq K_{(n-1)}. \tag{12}$$

From (12), there exist pairs (C_i, D_i), $i \in \mathbb{N}$, of compact sets in K, with $K = C_i \cup D_i$ and $\dim(C_i \cap D_i) \leqslant n-2$, such that for any $z \in h[X] \setminus \bigcup_{i=1}^{\infty} L_i$ and any neighborhood U of z in $\mathbb{I}^\infty$, there is an $i \in \mathbb{N}$ with $z \in C_i \subseteq U$ and $z \notin D_i$. The sets K and L_i in (11), together with the pairs (C_i, D_i) determine a point in the set $\mathscr{M}$, cf. (1), (2) and (3). So there exists $t \in T$ such that

$$K = K(t), \quad L_i = L_i(t), \quad C_i = C_i(t), \quad D_i = D_i(t). \tag{13}$$

Let us consider the section $E(t)$ of the set E, cf. (6). According to formula (5),

$$h[X] \setminus \bigcup_{i=1}^{\infty} L_i \subseteq H(t),$$

and hence by (13), $h[X] \subseteq E(t)$.

This completes the proof of (8) and ends the proof of the theorem.

3. Proof of Tomaszewski's Theorem 1.2

To begin with, we shall first prove that the product of two weakly-one dimensional spaces X and Y is one-dimensional.

Pick an arbitrary point $(x, y) \in (X \times Y)\backslash(\Lambda(X) \times \Lambda(Y))$. We claim that $X \times Y$ is at most one-dimensional at (x, y). We first show that this suffices. Striving for a contradiction, assume that $X \times Y$ is two-dimensional. Then by our claim,

$$\Lambda(X \times Y) \subseteq \Lambda(X) \times \Lambda(Y)$$

and since $\dim\,(\Lambda(X) \times \Lambda(Y)) \leqslant 0$ this contradicts Menger's result (see (2) in §1) that $\Lambda(X \times Y)$ has to be at least one-dimensional.

We assume without loss of generality that $y \notin \Lambda(Y)$. If in addition $x \notin \Lambda(X)$ then $X \times Y$ is zero-dimensional at (x, y). So we assume further that $x \in \Lambda(X)$. Let U and V be arbitrary open subsets of X and Y, respectively, such that $x \in U$ and $y \in V$. We will construct an open subset $E \subseteq X \times Y$ such that $(x, y) \in E \subseteq U \times V$ while moreover $\dim \operatorname{Fr} E \leqslant 0$. Indeed, since $y \notin \Lambda(Y)$, we may assume without loss of generality that V is open-and-closed. Let U' be an open neighborhood of x such that $\overline{U'} \subseteq U$ and $\operatorname{Fr} U' \subseteq X\backslash\Lambda(X)$. It is possible to pick U' since $\Lambda(X)$ is zero-dimensional. If $\dim V \leqslant 0$ then $\operatorname{Fr}(U' \times V) = \operatorname{Fr} U' \times V$ is at most zero-dimensional and so we are done. Assume therefore that V is one-dimensional. Then V is weakly one-dimensional, and so we may assume without loss of generality that $V = Y$. Put $A = \operatorname{Fr} U'$ and let U'' be an open subset of X such that $A \subseteq U'' \subseteq \overline{U''} \subseteq U$.

We claim that for every $n \in \mathbb{N}$ there exist pairwise disjoint open-and-closed subsets $U_{1n}, U_{2n}, \ldots$ of X such that

(1) $U_{in} \cap A \neq \emptyset$ for every i,
(2) $\operatorname{diam} U_{in} < 1/n$ for every i,
(3) $A \subseteq \bigcup_{i=1}^{\infty} U_{in} \subseteq \overline{\bigcup_{i=1}^{\infty} U_{in}} \subseteq U''$,
(4) $\operatorname{Fr}(\bigcup_{i=1}^{\infty} U_{in}) \subseteq \Lambda(X)$.

Indeed, for every $x \in X\backslash\Lambda(X)$ pick an open-and-closed neighborhood C_x of diameter at most $1/n$ such that either $C_x \cap A = \emptyset$ or $C_x \cap (X\backslash U'') = \emptyset$. A countable subcollection of the C_x cover $X\backslash\Lambda(X)$, say $\mathscr{C}$. Since the C_x are open-and-closed we may assume without loss of generality that $\mathscr{C}$ is pairwise disjoint. At most countably many elements of $\mathscr{C}$ intersect A, say $\{C_{x_i} : i \in \mathbb{N}\}$. An easy check shows that the sets $U_{in} = C_{x_i}, i \in \mathbb{N}$, are as required.

Now for every $n \in \mathbb{N}$ let $\mathscr{E}_n$ denote the collection of all open-and-closed subsets of Y of diameter at most $1/n$ and put

$$B_n = Y\backslash \bigcup \mathscr{E}_n. \tag{5}$$

Observe that if $b \in Y$ is such that for every n there exists an open-and-closed subset C_n of Y containing b and of diameter at most $1/n$ then Y is zero-dimensional at b. So if $b \in \Lambda(Y)$ then there exists $n \in \mathbb{N}$ such that $b \in B_n$.

We next claim that for every n there is a decreasing sequence $V_{in}, i \in \mathbb{N}$, of open-and-closed subsets of Y such that $\bigcap_{i=1}^{\infty} V_{in} = B_n$. Indeed, pick a countable subcollection $\mathscr{F}_n \subseteq \mathscr{E}_n$ with $\bigcup \mathscr{F}_n = \bigcup \mathscr{E}_n$. We may assume without loss of generality that the collection $\mathscr{F}_n$ is pairwise disjoint. Enumerate it as $\{F_{in} : i \in \mathbb{N}\}$ and put $V_{in} = Y \setminus \bigcup_{j=1}^{i} F_{in}$. It is clear that the V_{in} are as required.

Now put

$$W = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} U_{in} \times V_{in}. \tag{6}$$

We will show that

$$A \times \Lambda(Y) \subseteq W. \tag{7}$$

To this end, pick an arbitrary point $(a,b) \in A \times \Lambda(Y)$. There exists $n \in \mathbb{N}$ such that $b \in B_n$. Since $a \in A \subseteq \bigcup_{i=1}^{\infty} U_{in}$ there also exists $i \in \mathbb{N}$ such that $a \in U_{in}$. We conclude that $(a,b) \in U_{in} \times B_n \subseteq U_{in} \times V_{in} \subseteq W$.

We will next show that

$$\operatorname{Fr} W \subseteq (\Lambda(X) \times \Lambda(Y)) \cup (A \times Y). \tag{8}$$

To this end, let $(a,b) \in \operatorname{Fr} W$ and let $(a_k, b_k) \in W, k \in \mathbb{N}$, be a sequence converging to (a,b). For every $k \in \mathbb{N}$ pick $i_k, n_k \in \mathbb{N}$ such that

$$(a_k, b_k) \in U_{i_k n_k} \times V_{i_k n_k}.$$

Let us first assume that the set $\{n_k : k \in \mathbb{N}\}$ is infinite. Then by (1) and (2) it follows that for infinitely many $k \in \mathbb{N}$ we have $\varrho(a_k, A) < 1/k$, whence $(a,b) \in A \times Y$.

Assume next that the set $\{n_k : k \in \mathbb{N}\}$ is finite. By passing to a subsequence if necessary, we may even assume that it consists of a single element, say n. The set $K = \{i_k : k \in \mathbb{N}\}$ is infinite since each of the open-and-closed sets $U_{in} \times V_{in}$ contains finitely many (a_k, b_k) only.

We claim that $b \in B_n \subseteq \Lambda(Y)$. Striving for a contradiction, assume that $b \notin B_n$, and pick $i \in \mathbb{N}$ such that $b \notin V_{in}$. Since V_{in} is open-and-closed and $b_k \to b, k \to \infty$, all but finitely many b_k do not belong to V_{in}. But this contradicts K being infinite.

Since the collection $\{U_{in} : i \in \mathbb{N}\}$ consists of pairwise disjoint open-and-closed subsets of X each of which contains finitely many a_k only, it follows by (4) that $a \in \operatorname{Fr}(\bigcup_{i=1}^{\infty} U_{in}) \subseteq \Lambda(X)$. So we conclude that $(a,b) \in \Lambda(X) \times \Lambda(Y)$, as required.

Now put

$$E = W \cup (U' \times Y). \tag{9}$$

Then E is an open neighborhood of (x,y) and by (3) we find that $\overline{E} \subseteq U \times Y$. We claim that $\operatorname{Fr} E$ is zero-dimensional. First observe that

$$\operatorname{Fr} E \subseteq \operatorname{Fr} W \cup \operatorname{Fr}(U' \times Y) = \operatorname{Fr} W \cup (A \times Y). \tag{10}$$

Put $B_0 = \operatorname{Fr} E \cap (A \times Y)$ and $B_1 = \operatorname{Fr} E \setminus B_0$, respectively. Since by (7), $A \times \Lambda(Y) \subseteq W$ and $\operatorname{Fr} E \cap W = \emptyset$, it follows that

$$B_0 \subseteq A \times (Y \setminus \Lambda(Y)),$$

which is zero-dimensional. We conclude that B_0 is a zero-dimensional closed subspace of $\operatorname{Fr} E$. In addition, (10) and (4) imply that

$$B_1 \subseteq \operatorname{Fr} W \setminus (A \times Y) \subseteq \Lambda(X) \times \Lambda(Y),$$

which is also zero-dimensional. We conclude that B_1 is zero-dimensional as well. Since B_0 is closed, the Countable Sum Theorem now easily gives us that $\dim E \leqslant 0$, as desired.

To prove the general theorem from the just derived special case, one can use the straight-forward inductive argument in Tomaszewski [13] verbatim.

4. Proof of Theorem 1.3

We shall follow the main idea of the proof of Theorem 3.1 in [10]. The starting point is a continuous map $f : K \to \mathbb{I}$, a Cantor set $\triangle \subseteq (0, 1)$, and a compact set $Z \subseteq K$ such that

$$\dim f^{-1}(t) = n \quad \text{for} \quad t \in \triangle, \tag{1}$$

and

$$f[Z] = \triangle, \quad \text{and } \dim X = n \quad \text{for every} \quad X \subseteq Z \quad \text{with} \quad f[X] = \triangle. \tag{2}$$

As in [10], one can get such a map f using a construction in Rubin, Schori and Walsh [11] (property (2) follows from Theorem 4.2 in [11]). Let

$$T = \{t \in \triangle : \dim\,(Z \cap f^{-1}(t)) \geqslant 1\}. \tag{3}$$

Then, cf. [3, §45, IV], [10, Lemma 2.1],

$$T \quad \text{is} \quad \sigma\text{-compact and} \quad f^{-1}[\triangle \backslash T] \cap Z \subseteq Z_{(0)}. \tag{4}$$

We shall find compact sets $Z_i \subseteq Z$ such that

$$Z_i \cap Z_{(0)} = \emptyset, \quad \dim Z_i \leqslant n - 1, \tag{5}$$

while moreover,

$$\begin{aligned} &\text{(a)} \quad \text{the sets} \quad T_i = f[Z_i] \quad \text{are pairwise disjoint,} \\ &\text{(b)} \quad \triangle = \bigcup_{i=1}^{\infty} T_i \cup f[Z_{(0)}]. \end{aligned} \tag{6}$$

Let G be a zero-dimensional G_δ-set in $Z\backslash Z_{(0)}$ such that $Z\backslash(Z_{(0)} \cup G)$ is at most $(n-1)$-dimensional, cf. [3, §27, II, Cor. 2d]. Since G is zero-dimensional, (3) guarantees that for every $t \in T$ we have $\big(Z\backslash(Z_{(0)} \cup G)\big) \cap f^{-1}(t) \neq \emptyset$. We conclude that $T = f[Z\backslash(Z_{(0)} \cup G)]$.

Since $Z_{(0)}$ is a G_δ-set, cf. [3, §26, II], $Z\backslash(Z_{(0)} \cup G)$ can be written as $\bigcup_{i=1}^{\infty} L_i$, where $L_1 \subseteq L_2 \subseteq \cdots$ are compact.

So by (4) we get,

$$\triangle = \bigcup_{i=1}^{\infty} f[L_i] \cup f[Z_{(0)}]. \tag{7}$$

Since $L = \bigcup_{i=1}^{\infty} f[L_i]$ is zero-dimensional and the collection $\{f[L_i] : i \in \mathbb{N}\}$ is increasing, we can split L into pairwise disjoint compact sets $\{T_i : i \in \mathbb{N}\}$ such that for every i there is an index $n(i)$ with $T_i \subseteq f[L_{n(i)}]$.

For $i \in \mathbb{N}$, put $Z_i = f^{-1}[T_i] \cap L_{n(i)}$. Then by (7), conditions (5) and (6) are met.

We shall now choose for every i a subset S_i of Z such that,

$$S_i \subseteq Z_i, \quad f[S_i] = T_i, \quad \dim(S_i \backslash (S_i)_{(0)}) \leqslant n-2. \tag{8}$$

Assume first that $\dim Z_i = n-1$ and write $Z_i \backslash (Z_i)_{(0)}$ as $C_i \cup D_i$, where $\dim C_i \leqslant n-2$ and $\dim D_i \leqslant 0$. We let $S_i = C_i \cup (Z_i)_{(0)}$. The last part of (8) is clear, and so it remains to show that for any $t \in T_i, S_i \cap f^{-1}(t) \neq \emptyset$. Let $N_i = \{t \in T_i : \dim(f^{-1}(t) \cap Z_i) \geqslant 1\}$, cf. (3). If $t \notin N_i$ then $f^{-1}(t) \cap Z_i \subseteq (Z_i)_{(0)}$ by the same reasoning as the one above. In addition, if $t \in T_i \backslash N_i$, then $f^{-1}(t) \cap Z_i$ is not contained in the zero-dimensional set D_i, hence $f^{-1}(t) \cap S_i \neq \emptyset$.

If $\dim Z_i < n-1$ then we put $S_i = Z_i$.

Having defined the S_i, let us consider $S = \bigcup_{i=1}^{\infty} S_i$ and put

$$X = S \cup Z_{(0)}. \tag{9}$$

The S_i are pairwise disjoint, relatively closed sets in X by (5), (6), and (8).

Observe that by (2), (7), and (8),

$$\dim X = n \quad \text{and} \quad X \backslash X_{(0)} \subseteq S. \tag{10}$$

We claim that

$$J = \{i \in \mathbb{N} : \dim S_i = n-1\} \quad \text{is infinite.} \tag{11}$$

Striving for a contradiction, assume that this is not true, and let us consider the set $A = \bigcup_{i \in J} S_i$. Then A is closed in S and $\dim(S \backslash A) \leqslant n-2$. Observe that $Z_{(0)} \subseteq X_{(0)}$. This implies by (9) that

$$\dim(X \backslash A) \leqslant \dim(S \backslash A) + \dim X_{(0)} + 1 \leqslant n-1. \tag{12}$$

This shows among other things that the dimensional kernel $\Lambda(X)$ is contained in A since X is n-dimensional (see (10)) and A is closed. By (12) it also follows that $A_{(0)} \subseteq X_{(n-1)}$ (cf. [3, §27, II, Cor. 1.d.]). So $\Lambda(X)$ is a subset of $A \backslash A_{(0)}$ which is at most $(n-2)$-dimensional by (8). But this again contradicts Menger's theorem, cf. (2) in Section 1.

To complete the proof, let us list the elements of J as $k(1) < k(2) < \cdots$, and let $E_i = \cup\{S_j : k(i-1) < j \leqslant k(i)\}$, where $k(0) = 0$. Then the E_i are pairwise disjoint closed sets in X. Observe that they are all $(n-1)$-dimensional by (11), and weakly $(n-1)$-dimensional by (8). Since X is n-dimensional, we are done.

Remark. The sets E_i we have constructed have the additional property $\dim(E_i \backslash (E_i)_{(0)}) \leqslant n-2$.

Note added in Proof

In section 3 of the present paper we presented a proof of the fact that the product of two weakly 1-dimensional spaces is 1-dimensional. At the end of the section we state that for proving Tomaszewski's Theorem from the just derived special case, it suffices to follow the inductive argument in the paper [13] verbatim. However, we recently discovered a flaw in that inductive argument in [13]. It is unclear to us whether Tomaszewski's Theorem that the product of a weakly n-dimensional space and a weakly m-dimensional space is at most $(n+m-1)$-dimensional, is true. However, as was proved in [13] as well as in the present paper, it is certainly correct in the special case $n = m = 1$. In section 1 of the present paper we state that we do not know whether X^3 is 1-dimensional if X is weakly 1-dimensional. We recently proved that the product of an arbitrary family of weakly 1-dimensional spaces is 1-dimensional, which solves this problem.

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