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# Three-point sets

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**Abstract**

We prove that a planar set which meets each line in exactly three points cannot contain a continuum and cannot be an $F_\sigma$-set. We also present some results on extending and splitting $n$-point sets. Our results imply that there is a four-point set which contains an arc. 

*Keywords:* Planar set; Two-point set; Three-point set; Continuum; Arc; $F_\sigma$-set

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## 1. Introduction

Let $\kappa \geqslant 2$ be a cardinal number less than or equal to the continuum $2^{\aleph_0}$. A planar set is called a *$\kappa$-point set* if every line intersects the set in exactly $\kappa$ points. Obviously, the plane is a continuum-point set. So this concept is of interest for cardinals smaller than the continuum only. Sierpiński [13, p. 447] gave the following explicit example of an $\aleph_0$-point set: the union of all circles $x^2 + y^2 = n^2$ where $n = 1, 2, \ldots$. For finite $\kappa$ it is not this easy to construct a $\kappa$-point set. In fact, no explicit constructions are known. Using a well-ordering argument, Mazurkiewicz [10] proved that two-point sets exist. The same argument shows that $n$-point sets exist for every $n$ (see, e.g., Bagemihl [1] and Sierpiński [12]). It is unknown whether there is an $n$-point set which is a Borel subset of the plane. This has proved to be quite a difficult and interesting problem (see, e.g., Mauldin [9] for more details).

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Larman [7] presented the claim that no two-point set can be an $F_\sigma$-subset of the plane. His proof is in two steps. He argues that any two-point $F_\sigma$-subset of the plane contains an arc and that no two-point set contains an arc. Unfortunately, the proof of his first statement is incorrect, as was pointed out and corrected by Baston and Bostock [2]. It was shown by Miller [11] that in the constructible universe there is a two-point set which is coanalytic. So far these seem to be the only results on the descriptive complexity of $n$-point sets. It is even unknown whether a two-point set can be a $G_\delta$-subset of the plane. Observe that Sierpiński's $\aleph_0$-point set is closed and hence is $G_\delta$ as well as $F_\sigma$.

Kulesza [6] proved that a two-point set is zero-dimensional. The Sierpiński example from above shows that for $\aleph_0$-point sets this need not hold.

The aim of this paper is to try to generalize the above results for two-point sets to $n$-point sets for $n$ larger than 2. We show that no three-point set is an $F_\sigma$-subset of the plane. This is done by following Larman's program, although the technicalities are much more complicated. Much to our surprise, the situation for $n > 3$ is totally different. We do not know whether a four-point set can be an $F_\sigma$-subset of the plane, but we construct an example of a four-point set which contains an arc. So, Larman's program cannot be followed for $n > 3$.

Three-point sets turn out to be much more complicated to deal with than two-point sets. We do not know whether every three-point set is zero-dimensional but we will show that so-called strong three-point sets are zero-dimensional.

## 2. Notation

An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$ and a space $C$ is called *arcwise connected* if for every pair $x_1, x_2$ of distinct points of $C$ there exists a homeomorphic embedding $h : [0, 1] \to C$ satisfying $h(0) = x_1$ and $h(1) = x_2$. A *continuum* is a nonempty, compact, connected metric space. A topological space is *rim-finite* if there exists a base for the topology of the space which consists of sets having finite boundaries. It is easily seen that an $n$-point set, and obviously, each subspace of an $n$-point set is rim-finite.

The line through two distinct points $x \neq y$ in the plane shall be denoted by $L(x, y)$. If $\ell$ is an arbitrary line in the plane then a side of $\ell$ is a component of the complement of $\ell$, e.g., $x$ and $y$ are on the same side of $\ell$ means $x$ and $y$ are elements of the same component of $\mathbb{R}^2 \setminus \ell$.

Let $\varepsilon$ be a fixed positive number, $A$ an arbitrary set in $\mathbb{R}^2$, and $n \in \mathbb{N}$. Let

$$\begin{aligned} P_n^\varepsilon(A) = \bigl\{x \in \mathbb{R}\colon \bigl|\bigl(\{x\} \times \mathbb{R}\bigr) \cap A\bigr| = n \text{ and} \\ \text{if } (x, a), (x, b) \in A \text{ and } a \neq b \text{ then } |a - b| \geqslant \varepsilon\bigr\}. \end{aligned}$$

We let the functions

$$y_i : P_n^\varepsilon(A) \to \mathbb{R}, \quad 1 \leqslant i \leqslant n,$$

be defined by the properties: $(P_n^\varepsilon(A) \times \mathbb{R}) \cap A$ is the union of the graphs of the $y_i$'s and for each $x \in P_n^\varepsilon(A)$, $y_1(x) < y_2(x) < \cdots < y_n(x)$. Observe that in this case by definition of $P_n^\varepsilon(A)$, we have $y_i(x) \geqslant y_{i-1}(x) + \varepsilon$ for $i = 2, \ldots, n$.

A planar set is called a *partial n-point set* if every line intersects the set in *at most n* points. It is an easy exercise to show that the circle $x^2 + y^2 = 1$ is an example of a partial two-point set that is not a subset of any two-point set. We say that it cannot be *extended* to a two-point set. As will be shown in Section 4, it also cannot be extended to a three-point set. Interestingly, it *can* be extended to a four-point set, see Section 5 for details.

For more information on sets that can or cannot be extended to two-point sets, see Dijkstra, Kunen and van Mill [4].

The cardinality of a set $X$ is denoted by $|X|$ and, as usual, we let $\mathfrak{c}$ abbreviate $2^{\aleph_0}$.

## 3. Arcs in $n$-point sets

As we stated in the introduction, the flaw in Larman's proof was first pointed out and corrected by Baston and Bostock. However, it is also possible to do this by an unpublished method communicated to us by Mauldin [8]. Since it will be used by us later, we will present it in detail here. None of the results in this section is due to the authors of the present paper.

**Lemma 3.1.** *Let $\varepsilon > 0$, $n \in \mathbb{N}$, and let $F$ be a compact subset of $\mathbb{R}^2$ such that every vertical line intersects $F$ in at most $n$ points. Then $P_n^\varepsilon(F)$ is compact and for each $i \in \{1, \dots, n\}$, $y_i$ is continuous on $P_n^\varepsilon(F)$.*

**Proof.** We shall use induction with respect to $n$. For $n = 1$, let $(x_m)_m$ be a sequence in $P_1^\varepsilon(F)$ converging to $x \in \mathbb{R}$. By compactness of $F$, $\limsup_{m\to\infty} y_1(x_m) = a$ and $\liminf_{m\to\infty} y_1(x_m) = b$ exist. And so, $(x, a)$ and $(x, b)$ are points of $F$ which gives $a = b$. Hence, $P_1^\varepsilon(F)$ is compact and $y_1$ is continuous on $P_1^\varepsilon(F)$.

Suppose now that the statement of the lemma is true for some $n \geqslant 1$. Let $F$ be a compact subset in the plane that intersects every vertical line in at most $n + 1$ points. Let $(x_m)_m$ be a sequence in $G = P_{n+1}^\varepsilon(F)$ converging to $x \in \mathbb{R}$. Let $\liminf_{m\to\infty} y_{n+1}(x_m) = b$ and note that $(x, b) \in F$. There exists a subsequence $(x_{m_j})_j$ of $(x_m)_m$ such that for each $j \in \mathbb{N}$,

$$\left|y_{n+1}(x_{m_j}) - b\right| < \varepsilon/2$$

and

$$\lim_{j\to\infty} y_{n+1}(x_{m_j}) = b.$$

Define the compact set

$$F' = F \cap \Big(\big(\{x\} \cup \{x_{m_j}\colon\ j \in \mathbb{N}\}\big) \times (-\infty, b - \varepsilon/2]\Big).$$

Since

$$b - \varepsilon/2 < y_{n+1}(x_{m_j}) < b + \varepsilon/2,$$

we have, by definition of $F'$, that $(x_{m_j}, y_{n+1}(x_{m_j})) \notin F'$ and that $(x_{m_j}, y_n(x_{m_j})) \in F'$. It follows that for each $x_{m_j}$, $|(\{x_{m_j}\} \times \mathbb{R}) \cap F'| = n$ which implies that $x_{m_j} \in P_n^\varepsilon(F')$.

Observe now that $(x, b) \in F \setminus F'$ and that $|(\{x\} \times \mathbb{R}) \cap F'| \leqslant n$, and so for each $z \in \mathbb{R}$, $|(\{z\} \times \mathbb{R}) \cap F'| \leqslant n$. So, it follows by our inductive hypothesis that $P_n^\varepsilon(F')$ is compact and hence $x \in P_n^\varepsilon(F')$. We have now

$$\lim_{j\to\infty} y_{n+1}(x_{m_j}) = b$$

and

$$y_{n+1}(x_{m_j}) - y_n(x_{m_j}) \geqslant \varepsilon.$$

If we apply the induction hypothesis to $F'$ and $n$ we obtain

$$\lim_{j\to\infty} y_n(x_{m_j}) = y_n(x)$$

and hence

$$b - y_n(x) \geqslant \varepsilon.$$

This fact and the fact that $x \in P_n^\varepsilon(F')$ give $x \in G$. We may conclude that $G$ is compact. Let now $\limsup_{m\to\infty} y_{n+1}(x_m) = a$ and suppose that $a > b$. Then $(x, a) \in F$, $(x, b) \in F$, and $a > b > y_n(x) > \cdots > y_1(x)$ which contradicts the fact that there are only $n + 1$ points in $(\{x\} \times \mathbb{R}) \cap F$. Hence, $a = b$ and $y_{n+1}$ is continuous on $G$ follow immediately.

Define the compact set

$$F'' = \{(x, y) \in (G \times \mathbb{R}) \cap F \colon\ y \leqslant y_{n+1}(x) - \varepsilon\},$$

and observe that the set $F''$ satisfies the conditions of the lemma for $n$. So, by induction, $y_1, \ldots, y_n$ are continuous functions on $P_n^\varepsilon(F'') = G$. $\square$

**Proposition 3.2.** *Let $X$ be an $F_\sigma$-set in $\mathbb{R}^2$ and let $n \in \mathbb{N}$ such that for each $x \in \mathbb{R}$, $|(\{x\} \times \mathbb{R}) \cap X| = n$. Then there exist a nondegenerate interval $[a, b]$ and continuous functions $f_1 < f_2 < \cdots < f_n$ from $[a, b]$ into $\mathbb{R}$ such that $X$ contains the graph of every $f_i$. In particular, $X$ contains arcs.*

**Proof.** Let $X = \bigcup_{i=1}^\infty F_i$ with $F_1 \subset F_2 \subset \cdots$ and for each $i \in \mathbb{N}$, $F_i$ is compact. Let for each $i \in \mathbb{N}$,

$$\begin{aligned} H_i = \{x \in \mathbb{R}\colon\ &\big|\big(\{x\} \times \mathbb{R}\big) \cap F_i\big| = n \text{ and} \\ &\text{if } (x, a), (x, b) \in F_i \text{ and } a \neq b \text{ then } |a - b| \geqslant 1/i\}. \end{aligned}$$

Then, by Lemma 3.1, we have for each $i \in \mathbb{N}$, that $H_i$ is compact. Observe that $\bigcup_{i=1}^\infty H_i = \mathbb{R}$ so that by the Baire Category Theorem, there exists an $i \in \mathbb{N}$ such that $H_i$ contains an interval $[a, b]$ with $a < b$ in $\mathbb{R}$. By Lemma 3.1 again, for each $k \leqslant n$, $y_k{\restriction}_{[a,b]}$ is a continuous function, and so the graph of $y_k{\restriction}_{[a,b]}$ is an arc. We conclude that $X$ is as required. $\square$

## 4. A three-point set cannot be $F_\sigma$

In this section we will prove the main result in this paper that a three-point set is not an $F_\sigma$-subset of the plane.

**Lemma 4.1.** *Let $A$ be an arc in the plane with end points $p$ and $q$ and let $m$ be a line parallel to $L(p,q)$ that intersects $A$ and with maximum distance towards $L(p,q)$. If $|A \cap m| \geqslant 2$, then for some line $\ell$, $|A \cap \ell| \geqslant 4$.*

**Proof.** If $L(p,q) = m$ then $A \cap m$ is infinite and we are finished. So, we may assume that $L(p,q)$ and $m$ are distinct.

Let $\alpha : [0,1] \to A$ be a homeomorphism with $\alpha(0) = p$ and $\alpha(1) = q$. Since $|A \cap m| \geqslant 2$ there are points $a$ and $c$ with $0 < a < c < 1$ and $\alpha(a)$ and $\alpha(c)$ in $m$. If $\alpha((a,c)) \subset m$ then $A \cap m$ is infinite so we may assume that there is a $b \in (a,c)$ with $\alpha(b) \notin m$. Since $m$ has maximal distance towards $L(p,q)$ the point $\alpha(b)$ lies in the half plane of $m$ that contains $L(p,q)$ and since $\alpha(b)$ can be chosen arbitrarily close to $\alpha(a)$ we may assume that $\alpha(b)$ lies between $m$ and $L(p,q)$. Select a line $\ell$ that is parallel to $m$ and lies between $m$ and $\alpha(b)$. Then $p$, $\alpha(b)$, and $q$ lie on one side of $\ell$ and $\alpha(a)$ and $\alpha(c)$ on the other side. So, $\ell$ cuts the arcs $\alpha([0,a])$, $\alpha([a,b])$, $\alpha([b,c])$, and $\alpha([c,1])$ and hence $\ell$ intersects $A$ in at least four points.

**Lemma 4.2.** *Let $A$ be an arc in the plane with end points $p$ and $q$. If some line $\ell$ intersects $A$ in three points such that $p$ and $q$ are on the same side of $\ell$ then there is a line $\ell'$ that intersects $A$ in at least four points.*

**Proof.** Let $\alpha : [0,1] \to A$ be a homeomorphism with $\alpha(0) = p$ and $\alpha(1) = q$. By assumption we have $|\ell \cap A| = 3$, which means that there are points $a$, $b$, and $c$ with $0 < a < b < c < 1$ and such that $\alpha(a)$, $\alpha(b)$, and $\alpha(c)$ are all on the line $\ell$. Select $e \in (a,b)$ and $f \in (b,c)$ and note that $\alpha(e)$ and $\alpha(f)$ are in $\mathbb{R}^2 \setminus \ell$.

To prove that there is a line $\ell'$ that intersects $A$ in at least four points, we distinguish three cases:

(1) $\alpha(e)$, $p$, and $q$ are on the same side of $\ell$. Let $\ell'$ be a line that is parallel to $\ell$ and that separates $\ell$ from $p$, $q$, and $\alpha(e)$. Then $p$, $\alpha(e)$, and $q$ are on one side of $\ell'$ and $\alpha(a)$ and $\alpha(b)$ are on the other side. So $\ell'$ cuts the arcs $\alpha([0,a])$, $\alpha([a,e])$, $\alpha([e,b])$, and $\alpha([c,1])$ and hence $\ell'$ intersects $A$ in at least four points.

(2) If $\alpha(f)$, $p$, and $q$ are on the same side of $\ell$, we argue as in (1).

(3) $\ell$ separates $p$ and $q$ from $\alpha(e)$ and $\alpha(f)$. Select a line $\ell'$ that is parallel to $\ell$ and that separates $\alpha(e)$ and $\alpha(f)$ from $\ell$. Then, $\alpha(a)$, $\alpha(b)$, and $\alpha(c)$ are on one side of $\ell'$ and $\alpha(e)$ and $\alpha(f)$ on the other side. so $\ell'$ cuts the arcs $\alpha([a,e])$, $\alpha([e,b])$, $\alpha([b,f])$, and $\alpha([f,c])$ and hence $\ell'$ intersects $A$ in at least four points.

The proof is complete. □

**Lemma 4.3.** *No three-point set contains arcs.*

**Proof.** Let $X$ be a three-point set, let $A$ be an arc, and suppose $A \subset X$. We may assume without loss of generality that the end points of $A$ are represented by $p = (0,0)$ and $q = (1,0)$ and that $A \setminus \{p,q\} \subset \mathbb{R} \times (0,\infty)$. Let $m$ be a line parallel to $L(p,q)$ that intersects $A$ and with maximum distance towards the line $L(p,q)$. Then either $|A \cap m| = 1$

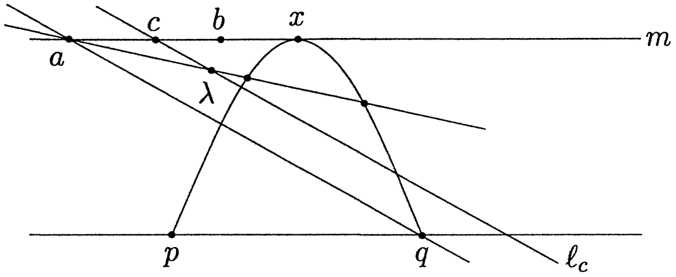


Fig. 1.

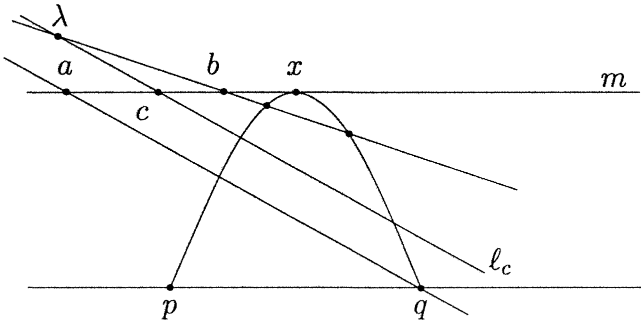


Fig. 2.

or $|A \cap m| = 2$ or $|A \cap m| = 3$. If $2 \leqslant |A \cap m| \leqslant 3$, then by Lemma 4.1, for some line $\ell$ there will be at least four points of intersection with $A$ and hence with $X$, contradicting the three-point property of $X$. So we may assume that $|A \cap m| = 1$.

Let $A \cap m = \{x\}$ with $x = (x_1, x_2)$ and note that $m = \mathbb{R} \times \{x_2\}$. Let $a$ and $b$ be the other distinct points of $m \cap X$. We take that $a = (a_1, x_2)$, $b = (b_1, x_2)$ and $a_1 < b_1$. We distinguish two cases:

(1) $x$ is not between $a$ and $b$ on the line $m$. Without loss of generality, we may assume that $b_1 < x_1$. Let $c$ be a point in $(a_1, b_1) \times \{x_2\}$ and $\ell_c$ be a line through $c$ that is parallel to $L(a, q)$. Then, by Lemma 4.2, the line $\ell_c$ intersects $A$ in at most two points and so there is a point $\lambda = (\lambda_1, \lambda_2)$ in $(\ell_c \cap X) \setminus A$. We study three subcases:

  (a) $\lambda_2 < x_2$. Then $x$ lies to the right of $L(a, \lambda)$ and $p$ and $q$ lie to the left of $L(a, \lambda)$ so $L(a, \lambda)$ intersects $A$ in two points. Since $a$ and $\lambda$ are two points of $X \setminus A$ we have $L(a, \lambda) \cap X$ contains at least four points, a contradiction. See Fig. 1.

  (b) $\lambda_2 = x_2$. Then $\lambda = c$ and hence $m \cap X$ contains at least four points; $a$, $\lambda$, $b$, $x$.

  (c) $\lambda_2 > x_2$. Now $x$ lies to the right of $L(\lambda, b)$ and $p$ and $q$ lie to the left of $L(\lambda, b)$. So, we have that $L(\lambda, b)$ intersects $A$ in two points. Since $\lambda$ and $b$ are two points of $X \setminus A$ we have $L(\lambda, b) \cap X$ contains at least four points, a contradiction. See Fig. 2.

(2) $a_1 < x_1 < b_1$. The point of intersection of the lines $L(a, q)$ and $L(b, p)$, say $\alpha$, is between the $x$-axis $L(p, q)$ and $m$. Let $m'$ be a line strictly between $m$ and $\alpha$, and

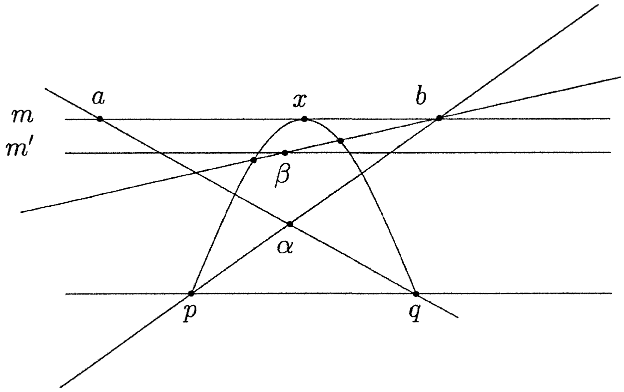


Fig. 3.

parallel to $m$. Then by Lemma 4.2 the line $m'$ intersects $A$ in at most two points and hence there is a point $\beta$ in $m' \cap (X \setminus A)$. We note that the sides of $L(p,b)$ and $L(q,a)$ that contain $x$ cover $m'$. See Fig. 3.

By symmetry we may assume that $\beta$ and $x$ are on the same side of, say, $L(p,b)$. Then $L(b,\beta)$ separates both $p$ and $q$ from $x$ so $|L(b,\beta) \cap X| \geqslant 4$, a contradiction.

By (1) and (2), the fact that $X$ contains an arc leads to a contradiction. We conclude that a three-point set contains no arcs. □

**Theorem 4.4.** *No three-point set contains nontrivial continua.*

**Proof.** Let $C$ be a nontrivial continuum that is contained in $X$, a three-point set. It is obvious that $C$ is a rim-finite continuum and so by [14, Lemma 1] it is arcwise connected. By Lemma 4.3, $C$ cannot be contained in $X$ since $X$ contains no arcs. Hence the set $X$ contains no continuum. □

We now come to the main result in this section.

**Theorem 4.5.** *A three-point set cannot be an $F_\sigma$-set in the plane.*

**Proof.** If we apply Proposition 3.2 for $n = 3$ and Lemma 4.3 we are done. □

It will be shown in the next section that there are $k$-point sets that contain arcs for every $k \geqslant 4$. So the method used here to show that three-point sets cannot be $F_\sigma$-sets does not work for $k > 3$. As was mentioned in the introduction, it is easy to give an example of a closed set $F$ in $\mathbb{R}^2$ that intersects every line in $\aleph_0$ points.

## 5. Extending and splitting $n$-point sets

We now present results on extending and splitting $n$-point sets. Our first result is very simple.

**Theorem 5.1.** *No $n$-point set is contained in an $(n+1)$-point set.*

**Proof.** Let $X$ be an $n$-point set, $Y$ an $(n+1)$-point set, and suppose $X \subset Y$. Then $Y \setminus X$ is a "one-point set" which clearly does not exist.

The following result is more interesting.

**Theorem 5.2.** *For each $n \geqslant 2$, for each $k \geqslant n+2$, and for each partial $n$-point set $X$ there exists a $k$-point set $Y$ such that $X \subset Y$.*

**Proof.** Let $X$ be a partial $n$-point set with $n \geqslant 2$ and let $k \geqslant n+2$. Let $\{\ell_\alpha\colon \alpha < \mathfrak{c}\}$ enumerate all lines in the plane. We shall construct by transfinite induction a nondecreasing sequence $\{E_\alpha\colon \alpha < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2 \setminus X$ with induction hypotheses:

(1) $|E_\alpha| < |\alpha| + \aleph_0$,

(2) $X \cup E_\alpha$ is a partial $k$-point set,

(3) $|(X \cup E_{\alpha+1}) \cap \ell_\alpha| = k$.

Put $E_0 = \emptyset$ and if $\lambda \leqslant \mathfrak{c}$ is a limit ordinal then $E_\lambda = \bigcup_{\alpha<\lambda} E_\alpha$. Observe that for limits (2) is trivially satisfied because the sequence of $E_\alpha$'s is nondecreasing. Assume that $E_\alpha$ has been constructed and consider the line $\ell_\alpha$.

Define $i = |(X \cup E_\alpha) \cap \ell_\alpha|$ and note that $0 \leqslant i \leqslant k$. Put

$$\mathcal{L} = \bigl\{L(a,b)\colon a, b \in E_\alpha, a \neq b\bigr\} \setminus \{\ell_\alpha\}.$$

Note that

$$\Bigl|\Bigl(X \cup E_\alpha \cup \bigcup \mathcal{L}\Bigr) \cap \ell_\alpha\Bigr| \leqslant k + |E_\alpha|^2 \leqslant |\alpha| + \aleph_0 < \mathfrak{c},$$

and hence we may select $k$ points $x_1, \ldots, x_k$ from

$$\ell_\alpha \setminus \Bigl(X \cup E_\alpha \cup \bigcup \mathcal{L}\Bigr).$$

We define $E_{\alpha+1} = E_\alpha \cup \{x_1, \ldots, x_{k-i}\}$.

It is obvious that $E_{\alpha+1}$ satisfies hypotheses (1) and (3). We verify (2): Assume that some line $\ell$ intersects $X \cup E_{\alpha+1}$ in at least $k+1$ points. Then $\ell_\alpha \neq \ell$ so $|\ell_\alpha \cap \ell| \leqslant 1$ and hence $\ell$ contains at most one of the $x_j$'s. So, $|(X \cup E_\alpha) \cap \ell| \geqslant k$. Since $k \geqslant n+2$ and $|\ell \cap X| \leqslant n$ we have $|\ell \cap E_\alpha| \geqslant 2$. So $\ell \in \mathcal{L}$ and hence $\ell$ contains no $x_i$. So

$$\bigl|(X \cup E_{\alpha+1}) \cap \ell\bigr| = \bigl|(X \cup E_\alpha) \cap \ell\bigr| \leqslant k,$$

which contradicts our assumption.

Now if we let $Y = X \cup E_{\mathfrak{c}}$ we are done. □

Observe that a two-dimensional subset of the plane has nonempty interior by [5, Theorem 1.8.11] and so it cannot be an $n$-point set. An $n$-point set is therefore either zero- or one-dimensional. (This also follows from the trivial observation that an $n$-point set is rim-finite.) As we stated in the introduction, two-point sets are zero-dimensional and we

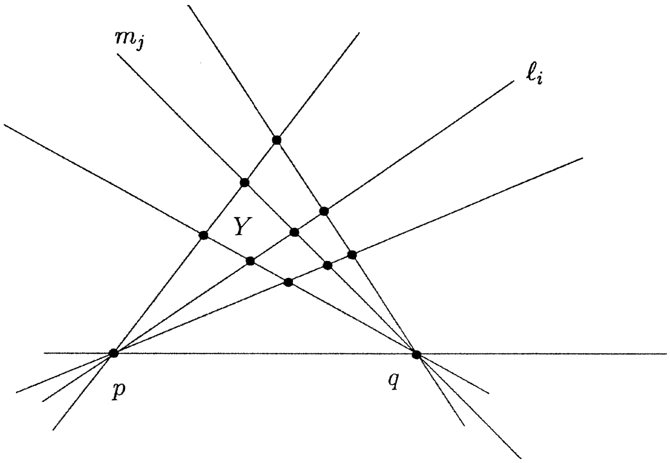


Fig. 4.

do not know whether three-point sets share this property. To our surprise, four-point sets can be one-dimensional, as the following result shows.

**Corollary 5.3.** *There exists for every $k \geqslant 4$ a $k$-point set that contains a circle* (*and hence it is one-dimensional*).

**Proof.** A circle is a partial two-point set. Now apply Theorem 5.2. □

Theorem 5.2 gives us many examples of $k$-point sets that are unions of $n$-point sets for $n < k$. This observation leads immediately to the question whether every $k$-point set ($k \geqslant 4$) can be "split" in this way. This question is answered in the negative in Dijkstra [3] where a four-point set is constructed that does not contain any two-point sets. Here we present a more general counterexample. We first need a technical lemma.

**Lemma 5.4.** *Given an $n \geqslant 4$, distinct points $p$ and $q$ in the plane, and a partial $n$-point set $X$ with $p, q \in X$ and $|X| < \mathfrak{c}$, there exists a finite planar set $Y$ such that $X \cup Y$ is a partial $n$-point set and for every partition $(A, B)$ of $X \cup Y$ such that for some $k$ the set $A$ is a partial $k$-point set and $B$ is a partial $(n-k)$-point set, we have that both $p$ and $q$ are in the same partition element.*

**Proof.** Since $|X| < \mathfrak{c}$ we can find distinct lines $\ell_1, \ldots, \ell_{n-1}$ that all contain $p$ and intersect no other point of $X$. Consider the set

$$Z = \bigcup \bigl\{\ell_i \cap L(a, b)\colon\ i \leqslant n-1,\ a, b \in X,\ \text{and } a \neq b\bigr\}.$$

Note that $|Z| \leqslant (n-1)|X|^2 < \mathfrak{c}$ so we can find distinct lines $m_1, \ldots, m_{n-1}$ that all contain $q$ and contain no point of $Z$. In addition, we may assume that none of the $m_i$'s is parallel to any of the $\ell_j$'s. Let $Y$ consist of the $(n-1)^2$ points of intersection of the $\ell_j$'s with the $m_i$'s. Note that $Y$ and $L(p, q)$ are disjoint and see Fig. 4.

In order to show that $X \cup Y$ is a partial $n$-point set let $\xi$ be a line that intersects $X \cup Y$ in at least $n+1$ points. If $|\xi \cap X| \geqslant 2$ then $\ell_i \cap \xi \subset Z$ for each $i$ and hence $\xi \cap Y = \emptyset$. So then we have $|\xi \cap X| \geqslant n+1$, a contradiction. We conclude that $|\xi \cap X| \leqslant 1$ and hence $|\xi \cap Y| \geqslant n$. Since $Y$ is contained in the set $\bigcup_{i=1}^{n-1} \ell_i$ we have with the pigeonhole principle that $\xi$ has two points in common with some $\ell_i$ and hence $\xi = \ell_i$. But $\ell_i$ contains precisely $n$ points of $X \cup Y$: $p \in X$ and the intersections with the $n-1$ lines $m_j$.

Now let $(A, B)$ be a partition of $X \cup Y$ such that $A$ is a partial $k$-point set and $B$ is a partial $(n-k)$-point set. Assume that this partition separates $p$ from $q$. By symmetry we may assume that $p \in A$ and $q \in B$. Note that every $m_j$ intersects $X \cup Y$ in precisely $n$ points so precisely $k$ of these points must be in $A$. Since $q \in B$ we have

$$m_j \cap A \subset Y \quad \text{and} \quad |A \cap Y| = \sum_{j=1}^{n-1} |m_j \cap A| = (n-1)k.$$

These $(n-1)k$ points are distributed over $n-1$ $\ell_i$'s so by the pigeonhole principle some $\ell_i$ contains at least $k$ points of $A \cap Y$. Since $p \in A$ we have that $\ell_i$ contains at least $k+1$ points of $A$, a contradiction. □

This lemma allows us to construct examples of 'peculiar' $n$-point sets.

**Theorem 5.5.** *For each $n \geqslant 4$ there exists an $n$-point set that fails to contain a $k$-point set for any $k < n$.*

**Proof.** Let $X_1$ consist of the points $1, 2, \ldots, n$ on the $x$-axis. Use Lemma 5.4 to find a $Y_1$ for the points $(1, 0)$ and $(2, 0)$ in $X_1$ and put $X_2 = X_1 \cup Y_1$. We proceed inductively to find a $Y_{i-1}$ for $(i-1, 0)$ and $(i, 0)$ in $X_{i-1}$. Put $X_i = X_{i-1} \cup Y_{i-1}$. We consider the finite partial $n$-point set $X_n$. It is implicit in Mazurkiewicz' proof of the existence of two-point sets that every partial $n$-point set with cardinality less than $\mathfrak{c}$ is extendable to an $n$-point set. Let $X$ be such an extension of $X_n$. Let $A$ be a subset of $X$ such that $A$ is a $k$-point set with $k < n$. It is obvious that $B = X \setminus A$ is then an $(n-k)$-point set and hence $2 \leqslant k \leqslant n-2$. Let $1 \leqslant i \leqslant n-1$ and note that $A' = A \cap X_{i+1}$ and $B' = B \cap X_{i+1}$ form a partition of $X_{i+1} = X_i \cup Y_i$ into a partial $k$-point set respectively a partial $(n-k)$-point set. Consequently, both $(i, 0)$ and $(i+1, 0)$ belong to the same partition element. Since this is true for every $i$ we have $X_1 \subset A$ or $X_1 \subset B$, a contradiction. □

## 6. Strong three-point sets

As we said before, we do not know whether three-point sets are zero-dimensional. In this section we introduce the so-called *strong three-point sets* which turn out to be zero-dimensional.

If $a$, $b$, and $c$ are three distinct points in $\mathbb{R}^2$, then the uniquely determined circle or line that contains $\{a, b, c\}$ is denoted by $C(a, b, c)$. A set $X \subset \mathbb{R}^2$ is called a *strong three-point set* if it meets each line and each circle in exactly three points. Every strong three-point set is obviously a three-point set.

The one-point compactification of the plane is the two-sphere $\mathbb{S}^2$. Every line in the plane corresponds to a circle in $\mathbb{S}^2$. So a strong three-point set is a set which meets every circle in $\mathbb{S}^2$ in precisely three points. So strong three-point sets are a natural generalization of two-point sets, more so than ordinary three-point sets.

The following result follows from a theorem of Bagemihl [1]. For completeness sake we include a direct proof.

**Theorem 6.1.** *There exists a strong three-point set.*

**Proof.** Let the set of all circles and lines of $\mathbb{R}^2$ be enumerated by

$$\mathcal{C} = \{C_\alpha \colon \alpha < \mathfrak{c}\}.$$

We shall construct by transfinite induction a nondecreasing sequence $(E_\alpha)_{\alpha \leqslant \mathfrak{c}}$ of subsets of $\mathbb{R}^2$ with induction hypotheses:

(1) $|E_\alpha| \leqslant |\alpha| + \aleph_0$,
(2) $|E_\alpha \cap C| \leqslant 3$, for each $C \in \mathcal{C}$,
(3) $|C_\alpha \cap E_{\alpha+1}| = 3$.

Put $E_0 = \emptyset$ and if $\lambda \leqslant \mathfrak{c}$ is a limit ordinal then $E_\lambda = \bigcup_{\alpha<\lambda} E_\alpha$. Observe that for limits (2) is automatically satisfied since the sequence of $E_\alpha$'s is nondecreasing. Let $\alpha < \mathfrak{c}$ be a fixed ordinal and consider $E_\alpha$ and $C_\alpha$. To define $E_{\alpha+1}$, we will find three appropriate points $y_1$, $y_2$, and $y_3$ in $C_\alpha$ that can be added to $E_\alpha$ without violating the partial three-point property.

(a) Let $E_\alpha(0) = E_\alpha$, let

$$\mathcal{L}(E_\alpha) = \bigl\{C(a,b,c) \colon a, b, c, \in E_\alpha, a \neq b, a \neq c, b \neq c\bigr\} \setminus \{C_\alpha\},$$

and let

$$H_\alpha(0) = \Bigl(\bigcup \mathcal{L}(E_\alpha) \cap C_\alpha\Bigr) \cup E_\alpha(0).$$

Since for each $C \in \mathcal{L}(E_\alpha)$, $|C \cap C_\alpha| \leqslant 2$ we have

$$\bigl|H_\alpha(0)\bigr| \leqslant 2\bigl|\mathcal{L}(E_\alpha)\bigr| + |E_\alpha| \leqslant 2|E_\alpha|^3 + |E_\alpha| \leqslant |\alpha| + \aleph_0 < \mathfrak{c}$$

we can find a point $y_1$ in $C_\alpha \setminus H_\alpha(0)$.

(b) Let $E_\alpha(1) = E_\alpha \cup \{y_1\}$ and $H_\alpha(1) = (\bigcup \mathcal{L}(E_\alpha(1)) \cap C_\alpha) \cup E_\alpha(1)$. As in (a) we find that $|H_\alpha(1)| \leqslant |\alpha| + \aleph_0 < \mathfrak{c}$ and so there is a point $y_2$ in $C_\alpha \setminus H_\alpha(1)$.

(c) With the same procedure as under (b), if we define $E_\alpha(2) = E_\alpha(1) \cup \{y_2\}$ and $H_\alpha(2) = (\bigcup \mathcal{L}(E_\alpha(2)) \cap C_\alpha) \cup E_\alpha(2)$, we can find a point $y_3$ in $C_\alpha \setminus H_\alpha(2)$ so let $E_\alpha(3) = E_\alpha(2) \cup \{y_3\}$.

We now let

$$E_{\alpha+1} = E_\alpha\bigl(|E_\alpha \cap C_\alpha|\bigr).$$

It is obvious that $E_{\alpha+1}$ satisfies (1) and (3). To prove (2) assume that there is a $C \in \mathcal{C}$ with $|E_{\alpha+1} \cap C| \geqslant 4$. Let $k = \min\{i \colon |E_\alpha(i) \cap C| \geqslant 4\}$. Then since $E_\alpha(0) = E_\alpha$ we have $k \geqslant 1$. Since $E_\alpha(k) \setminus E_\alpha(k-1) = \{y_k\}$ we have $y_k \in C$ and $|E_\alpha(k-1) \cap C| \geqslant 3$. This fact implies that $C \in \mathcal{L}(E_\alpha(k-1))$ and contradicts the choice of $y_k$. The induction is complete.

We conclude that $E_\mathfrak{c}$ is a strong three-point set. □

We will now present some properties of strong three-point sets. Using the Mazurkiewicz technique, it is easy to construct $n$-point sets which miss a given bounded open ball in the plane. Hence $n$-point sets need not be dense. But strong three-point sets are clearly dense.

**Theorem 6.2.** *Any strong three-point set is dense in* $\mathbb{R}^2$.

Our next aim is to prove that strong three-point sets are zero-dimensional. We need a technical lemma first.

**Lemma 6.3.** *Let* $T = \{(a,b,c) \in (\mathbb{R}^2)^3\colon\ a, b,$ *and* $c$ *are not colinear*$\}$. *Let* $M, r : T \to \mathbb{R}^2$ *be the functions that assign to every* $(a,b,c) \in T$ *the centre respectively the radius of* $C(a,b,c)$. *Then* $M$ *and* $r$ *are continuous functions.*

**Proof.** Let $(a,b,c) \in T$ and let $M = M(a,b,c)$ be the center of $C(a,b,c)$. It is easily seen that $M$ is the unique solution of the linear equations

$$M \cdot (b-a) = \frac{\|b\|^2 - \|a\|^2}{2} \quad \text{and} \quad M \cdot (b-c) = \frac{\|b\|^2 - \|c\|^2}{2},$$

where we used the dot product and length for vectors in the plane. Note that the two components of $M$ are rational functions of the components of $a$, $b$, and $c$, and hence continuous. The function $r$ is the distance between $M$ and $a$ and so also continuous. □

**Theorem 6.4.** *Every strong three-point set is zero-dimensional.*

**Proof.** Let $X$ be a strong three point set. Let $z$ be an arbitrary point of $X$. Without loss of generality we may assume that $z$ is the origin. Let $\ell_z$ be the $x$-axis. Since $\ell_z \cap X$ contains only two points other than $z$ we may select an $\varepsilon > 0$, arbitrarily small, such that $([-2\varepsilon, 2\varepsilon] \times \{0\}) \cap X = \{z\}$. Consider the circle $x^2 + y^2 = \varepsilon^2$ which we may represent as $C((\varepsilon,0),(0,\varepsilon),(-\varepsilon,0)) = C((\varepsilon,0),(0,-\varepsilon),(-\varepsilon,0))$. By Theorem 6.2 and Lemma 6.3 there exist $a$, $b$, and $c$ in $\mathbb{R} \times (0,\infty)$ close enough to $(\varepsilon,0)$, $(0,\varepsilon)$, and $(-\varepsilon,0)$ such that the radius of $C(a,b,c)$ is between $3\varepsilon/4$ and $5\varepsilon/4$ and the center of $C(a,b,c)$ is within $\varepsilon/4$ of $z$.

In the same way we can find points $a'$, $b'$, and $c'$ in $\mathbb{R} \times (-\infty,0)$ close enough to $(\varepsilon,0)$, $(0,-\varepsilon)$, and $(-\varepsilon,0)$ such that the radius of $C(a',b',c')$ is between $3\varepsilon/4$ and $5\varepsilon/4$ and the center of $C(a',b',c')$ is within $\varepsilon/4$ of $z$. Observe that $C(a,b,c)$ and $C(a',b',c')$ are in $\{u\colon\ \|u\| < 2\varepsilon\}$ and that $z$ is on the inside of both $C(a,b,c)$ and $C(a',b',c')$. Let $C(a,b,c) \cap (\mathbb{R} \times \{0\}) = \{(p,0),(q,0)\}$ with $p < 0 < q$ and let $C(a',b',c') \cap (\mathbb{R} \times \{0\}) = \{(s,0),(t,0)\}$ with $s < 0 < t$. Let now $S$ be the union of the interval from $p$ to $s$, the interval from $q$ to $t$ (on the $x$-axis), $C(a,b,c) \cap (\mathbb{R} \times (-\infty,0))$ and $C(a',b',c') \cap (\mathbb{R} \times (0,\infty))$, see Fig. 5.

It is easily seen that $S$ separates $z$ from $\{u\colon\ \|u\| \geqslant 2\varepsilon\}$ in $\mathbb{R}^2$ and that $S$ and $X$ are disjoint. Since $\varepsilon$ was chosen arbitrarily small we conclude that for each open set $O$ that contains $x$ there exists an open-and-closed set in $X$ that is contained in $O$. We conclude that $X$ is zero-dimensional. □

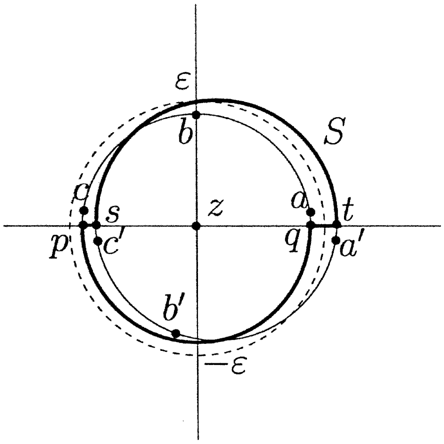


Fig. 5.

**Note Added in Proof.** Bouhjar, Dykstra and Mauldin recently showed that no $n$-point set ($n \geqslant 2$) in the plane is $F_\sigma$. This generalizes Theorem 4.5 in the present paper.

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