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Easier proofs of coloring theorems

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Abstract

We present a simpler proof of the known theorem that a fixed-point free homeomorphism on an n-dimensional paracompact space can be colored with $n+3$ colors.

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1. Introduction

Let X be a *normal* space and let $f : X \to X$ be a fixed-point free continuous function. A *coloring* of f is a finite closed cover $\mathcal{A}$ of X such that $A \cap f[A] = \emptyset$ for every $A \in \mathcal{A}$. Since finite open covers can be shrunk to closed covers, and finite closed covers can be swelled to open covers, the closedness of the coloring is irrelevant. Finite open covers do equally well.

It is a natural question to ask for the minimum number of colors needed to color a fixed-point free continuous function. For homeomorphisms, the following definitive answer is known:

Theorem 1.1. *Let X be a paracompact Hausdorff space with* $\dim X \leqslant n$. *Then any fixed-point free autohomeomorphism of X can be colored with $n+3$ colors.*

The number $n+3$ in this theorem is best possible. We will comment on this, and on related things, in Section 5.

The proof of Theorem 1.1 in the literature goes as follows. The first basic result is due to van Douwen [3]: he proved (among other things) that any fixed-point free

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autohomeomorphism on a finite-dimensional paracompact space can be colored with *finitely* many colors. We call such a function *finitely colorable*. Then Aarts et al. [1], using van Douwen's Theorem, proved that the number of colors on an n-dimensional *metrizable* space can be reduced to $n+3$. Then van Hartskamp and Vermeer [6], following a suggestion due to Hart and using again van Douwen's Theorem, proved that a fixed-point free autohomeomorphism on an n-dimensional paracompact Hausdorff space is semi-conjugated to a fixed-point free autohomeomorphism on some $\leqslant n$-dimensional metrizable space. So by applying the earlier result on metrizable spaces, and by pulling back the obtained coloring, one obtains a coloring of the original homeomorphism.

The aim of this note is to present a self-contained proof of Theorem 1.1, using standard facts from dimension theory only.

For all undefined notions, see Engelking [4,5].

2. From finite to $n+3$

Let $\mathcal{E}$ be a collection of sets. As usual, we say that the *order* of $\mathcal{E}$ is less than or equal to n, abbreviated $\operatorname{ord}(\mathcal{E}) \leqslant n$, if for every subfamily $\mathcal{F}$ of $\mathcal{E}$ of cardinality $n+2$ we have $\bigcap \mathcal{F} = \emptyset$.

Let X be a normal space, and let $f : X \to X$ be a fixed-point free homeomorphism. In this section we will prove that if f is finitely colorable and $\dim X \leqslant n$, then f can be colored with $n+3$ colors. The main technical tool is the following theorem, which is interesting in its own right.

Theorem 2.1. *Let X be a normal space with $\dim X \leqslant n$, $n < \omega$, and let $\mathcal{U}$ be a finite open cover of X with $|\mathcal{U}| \geqslant n+3$. In addition, let $f : X \to X$ be a homeomorphism. Then there is an open shrinking $\mathcal{V} = \{V_U\colon U \in \mathcal{U}\}$ of $\mathcal{U}$ such that for any $\mathcal{F} \subseteq \mathcal{U}$ with $|\mathcal{F}| = n+3$ we have*

$$\bigcap_{U \in \mathcal{F}} V_U \cup f^{-1}[V_U] = \emptyset.$$

It is tempting to think of the shrinking $\mathcal{V}$ in this result as a cover for which

$$\operatorname{ord}\bigl\{V_U \cup f^{-1}[V_U]\colon U \in \mathcal{U}\bigr\} \leqslant n+1. \tag{2.1}$$

But since simple examples show that for different U's the sets $V_U \cup f^{-1}[V_U]$ can be the same, the property of $\mathcal{V}$ stated in the theorem is stronger than the one in (2.1).

Before presenting the proof of Theorem 2.1, we will first show how we can use it to reduce the number of colors of a homeomorphism. We employ a technique due to Błaszczyk and Kim Dok Yong [2, Lemma 2], and Krawczyk and Steprāns [7, Lemma 2.1].

Theorem 2.2. *Let X be a normal space with $\dim X \leqslant n$, and let $f : X \to X$ be a fixed-point free homeomorphism. If f can be colored with finitely many colors, then it can be colored with $n+3$ colors.*

Proof. Let k be the minimum cardinality of a coloring of f, and let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a coloring. Assume that $k > n + 3$. We will derive a contradiction. Observe that $U_i \neq U_j$ if $i \neq j$. By Theorem 2.1 we may assume without loss of generality that for any $\mathcal{F} \subseteq \mathcal{U}$ of cardinality $n + 3$ we have

$$\bigcap_{U \in \mathcal{F}} U \cup f^{-2}[U] = \emptyset.$$

Since f is a homeomorphism, this means that for any $\mathcal{F} \subseteq \mathcal{U}$ of cardinality $n + 3$ we have

$$\bigcap_{U \in \mathcal{F}} f[U] \cup f^{-1}[U] = \emptyset. \tag{2.2}$$

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a closed shrinking of $\mathcal{U}$. For $i \leqslant k - 1$, define

$$B_i = F_i \cup \big(F_k \cap f[X \setminus U_i] \cap f^{-1}[X \setminus U_i]\big).$$

We claim that $\mathcal{B} = \{B_1, \ldots, B_{k-1}\}$ is a coloring of f. To see that $f[B_i] \cap B_i = \emptyset$ observe that

$$f[B_i] = f[F_i] \cup \big(f[F_k] \cap f^2[X \setminus U_i] \cap (X \setminus U_i)\big).$$

But this gives us what we want since $F_i \cap f[F_i] = \emptyset$, $F_i \cap (X \setminus U_i) = \emptyset$ and $F_k \cap f[F_k] = \emptyset$. To see that $\mathcal{B}$ covers, first note that

$$\bigcup_{i=1}^{k-1} F_i \subseteq \bigcup_{i=1}^{k-1} B_i.$$

Hence it suffices to show that F_k is covered. To this end, pick an arbitrary $x \in F_k$ and consider the collection

$$\big\{f[U_i] \cup f^{-1}[U_i]\colon\ i \leqslant k - 1\big\}.$$

Since $k - 1 > n + 2$, (2.2) implies that there exists $i \leqslant k - 1$ such that

$$x \notin f[U_i] \cup f^{-1}[U_i].$$

But then $x \in B_i$. This contradiction completes the proof of the theorem. □

Before turning to the proof of Theorem 2.1, we make the following remarks.

Recall that we wish to shrink an open cover $\mathcal{U}$ to an open cover $\mathcal{V} = \{V_U\colon\ U \in \mathcal{U}\}$ such that for any subfamily $\mathcal{W} \subseteq \mathcal{U}$ of cardinality $n + 3$ we have

$$\bigcap_{U \in \mathcal{W}} V_U \cup f^{-1}[V_U] = \emptyset.$$

A moments reflection shows that our task is to construct $\mathcal{V}$ in such a way that for any subfamily $\mathcal{W}$ of $\mathcal{V}$ of cardinality $n + 3$ and any partition $\mathcal{F} \cup \mathcal{G}$ of $\mathcal{W}$ we have

$$\bigcap \mathcal{F} \cap \bigcap_{G \in \mathcal{G}} f^{-1}[G] = \emptyset.$$

We first prove a simple lemma.

Lemma 2.3. *Let $A, A_1, \ldots, A_n$ be closed subsets of a space X. Moreover, let $V_1, \ldots, V_n$ be open subsets of X with $A \cap A_i \subseteq V_i$ for every $i \leqslant n$. If $\widehat{A} = A \setminus \bigcup_{i=1}^n V_i$ and $\widehat{A}_i = A_i \cup \overline{V}_i$ for $i \leqslant n$ then*

(a) $A \cup \bigcup_{i=1}^n A_i \subseteq \widehat{A} \cup \bigcup_{i=1}^n \widehat{A}_i$,

(b) $\widehat{A} \cap \bigcap_{i=1}^n \widehat{A}_i \subseteq \bigcap_{i=1}^n \operatorname{Fr} V_i$.

Proof. Observe that (a) is trivial. We will prove (b) by induction on n. If $n = 1$ then

$$\widehat{A} \cap \widehat{A}_1 = (A \setminus V_1) \cap (A_1 \cup \overline{V}_1) = \big[(A \cap A_1) \setminus V_1\big] \cup \big[A \cap (\overline{V}_1 \setminus V_1)\big] \subseteq \operatorname{Fr} V_1,$$

since $A \cap A_1 \subseteq V_1$. Now assume the lemma to be true for $m \leqslant n-1$, $n \geqslant 2$. Observe that by our inductive assumptions,

$$\begin{aligned}\widehat{A} \cap \bigcap_{i=1}^{n} \widehat{A}_i &= \left(A \setminus \bigcup_{i=1}^{n-1} V_i\right) \cap \bigcap_{i=1}^{n-1} (A_i \cup \overline{V}_i) \cap \big[(A \setminus V_n) \cap (A_n \cup \overline{V}_n)\big] \\ &\subseteq \bigcap_{i=1}^{n-1} \operatorname{Fr} V_i \cap \operatorname{Fr} V_n,\end{aligned}$$

which is as required. □

Corollary 2.4. *Let X be a normal space with $\dim X \leqslant n$. Let $F_i \subseteq U_i$ for $i \leqslant m$, with F_i closed and U_i open. In addition, Let $G_j \subseteq U_j'$ for $j \leqslant k$, with G_j closed and U_j' open. If $m + k \geqslant n + 3$ then there exist closed subsets $\widehat{F}_1, \ldots, \widehat{F}_m, \widehat{G}_1, \ldots, \widehat{G}_k$ such that*

(a) $\widehat{F}_i \subseteq U_i$ *for* $i \leqslant m$, *and* $\widehat{G}_j \subseteq U_j'$ *for* $j \leqslant k$,

(b) $\bigcup_{i=1}^m F_i \subseteq \bigcup_{i=1}^m \widehat{F}_i$, *and* $\bigcup_{j=1}^k G_j \subseteq \bigcup_{j=1}^k \widehat{G}_j$,

(c) $\bigcap_{i=1}^m \widehat{F}_i \cap \bigcap_{j=1}^k \widehat{G}_j = \emptyset$.

Proof. We assume without loss of generality that $m + k = n + 3$. Suppose first that $m \geqslant n + 2$. Since $\dim X \leqslant n$, the Theorem on Partitions [4, Theorem 7.2.15], implies that there exist open sets $V_1, \ldots, V_{n+1}$ such that

(1) $F_i \subseteq V_i \subseteq \overline{V}_i \subseteq U_i$ for $i \leqslant n+1$,

(2) $\bigcap_{i=1}^{n+1} \operatorname{Fr} V_i = \emptyset$.

So we get what we want by a direct application of Lemma 2.3 (let $A_i = F_i$ for $i \leqslant n+1$ and $A = F_{n+2}$). We may therefore assume without loss of generality that $m \leqslant n+1$ and, similarly, that $k \leqslant n+1$. Since $m + k = n + 3$, this implies that $k, m \geqslant 2$.

Observe that for $i \leqslant m-1$ and $j \leqslant k-1$ we have

$$F_i \cap F_m \subseteq U_i \quad \text{and} \quad G_j \cap G_k \subseteq U_j'.$$

Since $\dim X \leqslant n$ and $(m-1) + (k-1) = n+1$, the Theorem on Partitions [4, 7.2.15], implies that there exist open sets $V_1, \ldots, V_{m-1}, W_1, \ldots, W_{k-1}$ such that

(3) $F_i \cap F_m \subseteq V_i \subseteq \overline{V_i} \subseteq U_i$ for $i \leqslant m-1$,

(4) $G_j \cap G_k \subseteq W_j \subseteq \overline{W}_j \subseteq U_j'$ for $j \leqslant k-1$,

(5) $\bigcap_{i=1}^{m-1} \operatorname{Fr} V_i \cap \bigcap_{j=1}^{k-1} \operatorname{Fr} W_j = \emptyset$.

So we again get what we want by a direct application of Lemma 2.3. □

Proof of Theorem 2.1. Let $\mathcal{F}$ be a closed shrinking of $\mathcal{U}$. Fix $n+3$ different elements of $\mathcal{U}$, say,

$$\mathcal{G} = \{U_1, \ldots, U_m, U'_1, \ldots, U'_k\},$$

where $m + k = n + 3$. Let

$$\mathcal{H} = \{F_1, \ldots, F_m, G_1, \ldots, G_k\},$$

be the elements of $\mathcal{F}$ corresponding to the elements in $\mathcal{G}$. So $F_i \subseteq U_i$ for $i \leqslant m$ and $f^{-1}[G_j] \subseteq f^{-1}[U'_j]$ for $j \leqslant k$. Since $m + k = n + 3$, by Corollary 2.4, there exist closed sets $A_1, \ldots, A_m, B_1, \ldots, B_k$ such that

(1) $A_i \subseteq U_i$ for $i \leqslant m$, and $B_j \subseteq f^{-1}[U'_j]$ for $j \leqslant k$,

(2) $\bigcup_{i=1}^m F_i \subseteq \bigcup_{i=1}^m A_j$, and $\bigcup_{j=1}^k f^{-1}[G_j] \subseteq \bigcup_{j=1}^k B_j$,

(3) $\bigcap_{i=1}^m A_i \cap \bigcap_{j=1}^k B_j = \emptyset$.

Put $A'_j = f[B_j]$ for $j \leqslant k$. In $\mathcal{F}$ replace F_i by A_i for $i \leqslant m$ and G_j by A'_j for $j \leqslant k$. The other elements of $\mathcal{F}$ are not being replaced. We claim that the collection $\mathcal{F}'$ obtained in this way covers X, and hence is a closed shrinking of $\mathcal{U}$. To see that it covers, pick an arbitrary $x \in X$. If $x \in \bigcup_{i=1}^m F_i$ then we are done by the first part of (2). If $x \in \bigcup_{j=1}^k G_j$ then

$$f^{-1}(x) \in \bigcup_{j=1}^k f^{-1}[G_j] \subseteq \bigcup_{j=1}^k B_j$$

by the second part of (2). As a consequence, for some $j \leqslant k$, $x \in A'_j$. Finally, if x does not belong to any element of $\mathcal{H}$ then the element of $\mathcal{F}$ that contains x belongs to $\mathcal{F}'$. Observe next that

$$\bigcap_{i=1}^m A_i \cap \bigcap_{j=1}^k f^{-1}[A'_j] = \emptyset.$$

Let $\mathcal{V}$ be an appropriate open swelling of the $\mathcal{F}'$ which is simultaneously a shrinking of $\mathcal{U}$ [4, Theorem 7.1.4].

So we arrive at the conclusion that $\mathcal{U}$ has an open shrinking $\mathcal{V}$ such that the V's and $f^{-1}[V]$'s corresponding to $\mathcal{G}$ have empty intersection. The same procedure can be repeated with any subfamily of $\mathcal{V}$ of cardinality $n+3$. So after finitely many steps, we arrive at the required shrinking of $\mathcal{U}$. $\square$

3. Coloring continuous functions

If X is a space then βX denotes its Čech–Stone compactification. If U is an open subset of X then

$$\operatorname{Ex} U = \beta X \setminus \overline{(X \setminus U)}$$

is the largest open subset of βX that contains U. If $\mathcal{U} = \{U_1, \ldots, U_k\}$ is a finite open cover of X then $\{\operatorname{Ex} U_1, \ldots, \operatorname{Ex} U_k\}$ is an open cover of βX [4, Lemma 7.1.13]. In addition, $\dim X = \dim \beta X$ [4, Theorem 7.1.17].

Let X be a normal space with $\dim X \leqslant n$, and let $f : X \to X$ be a continuous function without fixed-points. Assume that f is finitely colorable. What is the least number of colors needed to color f? An inspection of the proof of Theorem 2.2 shows that the only place where we used that the map under consideration is a homeomorphism, is in the proof of Theorem 2.1. So the question naturally arises whether Theorem 2.1 is also true for continuous maps instead of homeomorphisms.

For $n = 0$ there are no problems. To see this, assume that X is zero-dimensional, $\mathcal{U}$ is a finite open cover of X, and $f : X \to X$ is continuous. Since $\dim X = 0$, there is an open shrinking $\mathcal{V} = \{V_U\colon U \in \mathcal{U}\}$ of $\mathcal{U}$ with $\mathrm{ord}(\mathcal{V}) \leqslant 0$, i.e., $\mathcal{V}$ is pairwise disjoint. But then $\mathcal{V}$ is as required. For let $x \in X$, and observe that there is precisely one element of $\mathcal{V}$ that contains x. Similarly, there is precisely one element of $f^{-1}(\mathcal{V})$ that contains x.

But for larger n, Theorem 2.1 does not hold for continuous functions, as the following simple example shows.

Example 3.1. Let X be the topological sum of the spaces $\mathbb{I}_n$, $n < \omega$, where each $\mathbb{I}_n$ is a copy of $[0, 1]$. In addition, let $\{d_i\colon i \geqslant 1\}$ be a countable dense subset of $\mathbb{I}_0$. Now define $f : X \to X$ as follows. $f \restriction \mathbb{I}_0$ is a homeomorphism from $\mathbb{I}_0$ onto $\mathbb{I}_1$. In addition, $f \restriction \mathbb{I}_i$ is the function with constant value d_i for $i \geqslant 1$. Then f is clearly continuous (and has no fixed-point). For every $i < \omega$ let E_i and F_i be open subsets of $\mathbb{I}_i$ such that $E_i \cup F_i = \mathbb{I}_i$ and $\mathbb{I}_i \setminus E_i \neq \emptyset \neq \mathbb{I}_i \setminus F_i$. Define

$$U_1 = E_0, \quad U_2 = F_0, \quad U_3 = \bigcup_{i \geqslant 1} E_i \quad \text{and} \quad U_4 = \bigcup_{i \geqslant 1} F_i.$$

Then $\mathcal{U} = \{U_1, \dots, U_4\}$ is an open cover of X. Let $\mathcal{V} = \{V_1, \dots, V_4\}$ be an arbitrary open shrinking of $\mathcal{U}$. Then V_1 and V_2 cover $\mathbb{I}_0$, and are clearly proper subsets of $\mathbb{I}_0$. So by connectivity of $\mathbb{I}_0$, $V_1 \cap V_2 \neq \emptyset$. Pick $i \geqslant 1$ with $d_i \in V_1 \cap V_2$. Since V_3 and V_4 cover $\bigcup_{i \geqslant 1} \mathbb{I}_i$, the same argument shows that $V_3 \cap V_4 \cap \mathbb{I}_i \neq \emptyset$. We conclude that

$$f^{-1}[V_1] \cap f^{-1}[V_2] \cap V_3 \cap V_4 \neq \emptyset,$$

as required.

This example shows that if one wishes to color continuous functions, the method used in the previous section does not work. Fortunately, a beautiful trick due to R. Pol using the coloring result for homeomorphisms to obtain a coloring result for continuous functions, does the job for us.

Theorem 3.2. *Let X be a normal space and let $f : X \to X$ be fixed-point free and continuous. If f is finitely colorable and* $\dim X \leqslant n$*, then f can be colored with $n + 3$ colors.*

Proof. Let $\mathcal{U}$ be a finite coloring of f. We may assume without loss of generality that X is compact. Simply observe that we can extend $f : X \to X$ to a continuous function $\beta f : \beta X \to \beta X$, $\mathcal{U}$ to an open cover $\mathrm{Ex}\,\mathcal{U} = \{\mathrm{Ex}\,U\colon U \in \mathcal{U}\}$ of βX, and that $\dim X =$

$\dim \beta X$. Also, βf has no fixed-point since $\mathcal{U}$ is finite. So a "good" shrinking of $\mathrm{Ex}\,\mathcal{U}$ corresponding to βf traces to a "good" shrinking of $\mathcal{U}$ corresponding to f.

We can now follow the proof due to Pol in [1, Section 2.3] verbatim to obtain the desired result. $\square$

4. From infinite to finite

The results in the previous section show that the "coloring of maps"—problem boils down to the problem of which maps are finitely colorable. The first result on this problem is due to van Douwen [3]. He proved that if X is finite-dimensional and paracompact, then any fixed-point free closed mapping $f : X \to X$ having the property that for some $n < \omega$, $|f^{-1}(x)| \leqslant n$ for every $x \in X$, is finitely colorable. For related results for compact spaces, see also Kim Dok Yong [10].

In this section we will present a simpler proof of van Douwen's Theorem, based on his ideas though. We will present our proof for homeomorphisms only. We leave it to the reader to check that the same proof also works for maps of finite bounded order.

Lemma 4.1. *Let X be a normal space with $\dim X \leqslant n$, let $f : X \to X$ be a homeomorphism, and let $\mathcal{F}$ be a discrete collection of closed subsets of X with $f[F] \cap F = \emptyset$ for every $F \in \mathcal{F}$. Then there exist closed subsets $A_1, \ldots, A_{2n+3}$ of X such that*
 (a) *$f[A_i] \cap A_i = \emptyset$ for $i \leqslant 2n+3$,*
 (b) *$\bigcup \mathcal{F} \subseteq \bigcup_{i=1}^{2n+3} A_i$.*

Proof. If $|\mathcal{F}| \leqslant 2n+3$ then there is nothing to prove. So assume otherwise, and let $\{G_1, \ldots, G_{2n+3}\}$ be any subcollection of $\mathcal{F}$ of cardinality $2n+3$. Put $F_0 = \bigcup_{i \leqslant 2n+3} G_i$, and $A_k^0 = G_k$ for $k \leqslant 2n+3$. Enumerate $\mathcal{F} \setminus \{G_1, \ldots, G_{2n+3}\}$ as $\{F_\alpha\colon 1 \leqslant \alpha < \kappa\}$, without repetitions. By transfinite induction on $\alpha < \kappa$, we will construct closed subsets A_k^α of F_α for $k \leqslant 2n+3$ such that
 (1) $\operatorname{ord}\{A_k^\alpha\colon k \leqslant 2n+3\} \leqslant n$,
 (2) for every $k \leqslant 2n+3$,

$$f\left[\bigcup_{\beta \leqslant \alpha} A_k^\beta\right] \cap \bigcup_{\beta \leqslant \alpha} A_k^\beta = \emptyset.$$

Assume that the A_k^β are defined for $\beta < \alpha < \kappa$ and $k \leqslant 2n+3$ and put $B_k = \bigcup_{\beta < \alpha} A_k^\beta$. Then each B_k is closed since $\mathcal{F}$ is discrete and by (2), $f[B_k] \cap B_k = \emptyset$. For $k \leqslant 2n+3$, put

$$D_k = f[B_k] \cup f^{-1}[B_k].$$

Since $k \leqslant 2n+3$ and f is one-to-one, it easily follows from (1) that

$$\bigcap_{k \leqslant 2n+3} D_k = \emptyset.$$

As a consequence,

$$\{F_\alpha \setminus D_k\colon k \leqslant 2n+3\}$$

covers F_α and consists of open subsets of F_α. (Here we use that f is a closed map.) Since $\dim F_\alpha \leqslant n$, being a closed subset of X [4, Theorem 7.1.8], there is a collection $\mathcal{G} = \{A_k^\alpha\colon k \leqslant 2n+3\}$ consisting of closed subsets of F_α such that $A_k^\alpha \cap D_k = \emptyset$ for $k \leqslant 2n+3$, $\operatorname{ord}(\{A_k^\alpha\colon k \leqslant 2n+3\}) \leqslant n$, and $\bigcup_{k\leqslant 2n+3} A_k^\alpha = F_\alpha$. We claim that these sets are as required. This is easy. Simply observe that for $k \leqslant 2n+3$,

$$\begin{aligned} f\Bigl[\bigcup_{\beta\leqslant\alpha} A_k^\alpha\Bigr] \cap \bigcup_{\beta\leqslant\alpha} A_k^\alpha &= f[B_k \cup A_k^\alpha] \cap (B_k \cup A_k^\alpha) \\ &= \bigl(f[B_k] \cap B_k\bigr) \cup \bigl(f[B_k] \cap A_k^\alpha\bigr) \cup \bigl(f[A_k^\alpha] \cap B_k\bigr) \cup \bigl(f[A_k^\alpha] \cap A_k^\alpha\bigr) = \emptyset \end{aligned}$$

since

$$f[F_\alpha] \cap F_\alpha = \emptyset, \quad A_k^\alpha \cap D_k = \emptyset \quad \text{and} \quad f[A_k^\alpha] \cap D_k = \emptyset.$$

Now for $k \leqslant 2n+3$, put $A_k = \bigcup_{\alpha<\kappa} A_k^\alpha$. □

We now come to the main result in this section.

Theorem 4.2. *Let X be finite-dimensional and paracompact. Then every fixed-point free autohomeomorphism of X is finitely colorable.*

Proof. Let $f : X \to X$ be a fixed-point free autohomeomorphism. We will show that f is $(n+1)\cdot(2n+3)$ colorable, where $n = \dim X$. Since f is fixed-point free, and X is paracompact and n-dimensional, there is a locally finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of X such that $\operatorname{ord}(\mathcal{U}) \leqslant n$ while moreover, $f[U_s] \cap U_s = \emptyset$ for every $s \in S$. There is an open cover $\mathcal{V}$ of X which can be represented as the union of $n+1$ families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n+1}$, where $\mathcal{V}_i = \{V_{i,s}\}_{s\in S}$ is pairwise disjoint and $V_{i,s} \subseteq U_s$ for $s \in S$ and $i \leqslant n+1$ [5, Theorem 3.2.4]. Let $\mathcal{F}$ be a closed shrinking of $\mathcal{V}$ [4, Theorem 1.5.18]. It is clear that we can represent $\mathcal{F}$ as the union of $n+1$ families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{n+1}$, where $\mathcal{F}_i = \{F_{i,s}\}_{s\in S}$ is such that $F_{i,s} \subseteq V_{i,s}$ for $s \in S$ and $i \leqslant n+1$. As a consequence, $\mathcal{F}_i$ is discrete for every $i \leqslant n+1$ since $\mathcal{U}$ is locally finite. Clearly, $f[F] \cap F = \emptyset$ for every $F \in \mathcal{F}_i$. From this it follows that for every i, "$f \restriction \bigcup \mathcal{F}_i$ can be colored with $2n+3$ colors" (Lemma 4.1). We conclude that f can be colored with $(n+1)\cdot(2n+3)$ colors. □

5. Remarks

As noted in [1,6], by results of Steinlein [9,8], Theorem 1.1 is sharp for all n.

There exists a fixed-point free continuous function on the space of irrational numbers $\mathbb{P}$ which is not finitely colorable by a result of Mazur, see Krawczyk and Steprāns [7, Theorem 3.4]. This shows that Theorem 4.2 is best possible.

Several results in this note can also be formulated and proved for Tychonoff spaces instead of normal spaces. This is left as an exercise to the reader.

In [1] it was shown that the number of colors needed to color a fixed-point free involution on an n-dimensional space is $n+2$. This can be shown by our methods as well.

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