ELSEVIER

Topology and its Applications 85 (1998) 281–285

TOPOLOGY AND ITS APPLICATIONS

An example of t_p^*-equivalent spaces which are not t_p-equivalent

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Received 15 October 1996

Abstract

We construct an example of two countable spaces X and Y such that the spaces $C_p^*(X)$ and $C_p^*(Y)$ are homeomorphic and the spaces $C_p(X)$ and $C_p(Y)$ are not homeomorphic.

Keywords: Function space; $C_p(X)$; Ultrafilter

AMS classification: 54C35

1. Introduction

All spaces are completely regular.

For a space X, $C_p(X)$ denotes the space of all continuous real valued functions on X with the pointwise convergence topology. $C_p^*(X)$ is the subspace of $C_p(X)$ consisting of bounded functions.

Recently, Banakh and Cauty [2] proved that if X is countable and nondiscrete then $C_p^*(X)$ is homeomorphic to $C_p(X) \times \sigma$, where σ denotes the linear span of the standard basis in ℓ^2. This interesting result has several nontrivial consequences, among them the statement that if $C_p(X)$ and $C_p(Y)$ are homeomorphic then so are $C_p^*(X)$ and $C_p^*(Y)$. This result suggests the natural question of whether the reverse implication holds. The aim of this note is to answer this question in the negative: there exist countable spaces X

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0166-8641/98/$19.00
PII S0166-8641(97)00154-5

and Y for which $C_p^*(X) \approx C_p^*(Y)$ (i.e., X and Y are t_p^*-equivalent) but $C_p(X) \not\approx C_p(Y)$ (i.e., X and Y are not t_p-equivalent). For a related result, see [1].

Given a filter F on an infinite countable set T, we denote by N_F the space $T \cup \{\infty\}$, where $\infty \notin T$, topologized by isolating the points of T and using the family $\{A \cup \{\infty\}\colon A \in F\}$ as a neighborhood base at ∞.

Recall that a filter F is a P-filter if for every sequence (U_n) of sets from F we can find an $A \in F$ which is almost contained in every U_n, i.e., $A \setminus U_n$ is finite. P-ultrafilters are also called P-points. For the notions from infinite-dimensional topology that we are using, we refer the reader to [9].

2. The example

Lemma 2.1. *Let F be a filter on ω which is not a P-filter. Then the intersection $C_p(N_F) \cap [0,1]^{N_F}$ contains a closed nonempty subset R which is an absolute retract and a Z_σ-space.*

Proof. Since F is not a P-filter we can find a partition of ω into disjoint infinite subsets A_k, $k \in \omega$, with the following properties:

(a) $(\forall i \in \omega)$ $[U_i = \bigcup_{k \geqslant i} A_k \in F]$,

(b) $(\forall A \in F)(\exists i \in \omega)$ $[A \setminus U_i$ is infinite$]$.

The condition (b) is obviously equivalent to

(b′) $(\forall A \in F)(\exists k \in \omega)$ $[A \cap A_k$ is infinite$]$.

To simplify the notation we may assume that F is a filter on $\omega \times \omega$ and $A_k = \{(k,n)\colon n \in \omega\}$.

Let P be the subset of $[0,1]^{N_F}$ consisting of all functions f with the following properties:

(1) $(\forall k,n,i \in \omega)$ $[(i \leqslant n) \Rightarrow (f(k,i) \leqslant f(k,n)]$,

(2) $(\forall k,n,i \in \omega)$ $[(k \leqslant i) \Rightarrow (f(k,n) \geqslant f(i,n)]$,

(3) $f(\infty) = 0$.

Put $R = P \cap C_p(N_F)$. Obviously, the set R is closed in $C_p(N_F) \cap [0,1]^{N_F}$. Since R is a convex subset of the product $\mathbb{R}^{N_F}$ it is an absolute retract.

Consider an arbitrary $f \in R$. From the continuity of f at ∞ it follows that, for every $\varepsilon > 0$, there is an $A \in F$ such that $f(k,n) \leqslant \varepsilon$ for all $(k,n) \in A$. By (b′) there is $k \in \omega$ such that $f(k,n) \leqslant \varepsilon$ for infinitely many $n \in \omega$. Then the condition (1) implies that $f(k,n) \leqslant \varepsilon$ for all $n \in \omega$. From (2) it follows that $f(i,n) \leqslant \varepsilon$ for all $n \in \omega$ and all $i \geqslant k$. Hence we have

$$R = \{f \in P\colon (\forall \varepsilon > 0)(\exists k \in \omega)(\forall i \geqslant k)(\forall n \in \omega)\ [f(i,n) \leqslant \varepsilon]\}.$$

Let

$$R_k = \{f \in R\colon (\forall i \geqslant k)(\forall n \in \omega)\ [f(i,n) \leqslant 1/2]\},$$

for $k \in \omega$. Then each R_k is a closed subset of R and $R = \bigcup\{R_k\colon k \in \omega\}$. One can easily verify that all R_k are Z-sets in R. Indeed, it is enough to observe that, for a fixed k, the sequence of maps $\varphi_j : R \to R \setminus R_k$, $j \in \omega$, defined by

$$\varphi_j(f)(i,n) = \begin{cases} 1 & \text{for } i \leqslant k \text{ and } n \geqslant j, \\ f(i,n) & \text{otherwise} \end{cases}$$

is uniformly convergent to the identity on R (uniformly with respect to any metric on the product $[0,1]^{N_F}$). Therefore R is a Z_σ-space. □

Example 2.2. There exist countable spaces X and Y such that the spaces $C_p^*(X)$ and $C_p^*(Y)$ are homeomorphic and the spaces $C_p(X)$ and $C_p(Y)$ are not homeomorphic.

Let F be an ultrafilter on ω which is not a P-point. We take $X = \omega \times N_F$. Let $S = \{0, 1, 1/2, 1/3, \ldots\}$ (a convergent sequence). The space Y is the topological sum of the spaces X and S. We have the following:

Lemma 2.3. *The spaces $C_p(X)$ and $C_p(Y)$ are not homeomorphic.*

Proof. By [6, Example 7.1], the space $C_p(N_F)$ is a Baire space. Since the space $C_p(X)$ is homeomorphic to $C_p(N_F)^\omega$ it is also a Baire space, see [10].

On the other hand, it is known that the space $C_p(S)$ is homeomorphic to σ^ω (see [4]) and therefore it is of the first category. Hence the space $C_p(Y)$ which is homeomorphic to $C_p(X) \times C_p(S)$ is also of the first category. □

Lemma 2.4. *The spaces $C_p^*(X)$ and $C_p^*(Y)$ are homeomorphic.*

Proof. Both spaces $C_p^*(X)$ and $C_p^*(Y)$ are σ-precompact (i.e., they lie in the σ-compact subsets of their completions). By [2, Corollary 2.7], it is enough to show that each of these spaces embeds as a closed set into the other. The space $C_p^*(Y)$ is homeomorphic to $C_p^*(X) \times C_p^*(S) = C_p^*(X) \times C_p(S) \approx C_p^*(X) \times \sigma^\omega$. Obviously $C_p^*(Y)$ contains a closed copy of $C_p^*(X)$. Since $C_p^*(X)$ is homeomorphic to $C_p^*(X) \times C_p^*(X)$ it remains to prove that $C_p^*(X)$ contains a closed copy of σ^ω. From Lemma 1 it follows that the space $T = C_p(N_F) \cap [0,1]^{N_F}$ contains a closed subset R which is an absolute retract and a Z_σ-space. By Lemma 5.3 from [5], the product R^ω contains a closed copy of σ^ω. Since it is obvious that the product T^ω can be embedded as a closed subset of $C_p^*(X)$, we are done. □

3. Remarks

Remark 3.1. Using the results from [3, Lemma 4.11] and [2, Lemma 3.1] it is possible to give slightly simpler examples of spaces X and Y as in Example 2.2. It is enough to consider $X = N_F$, where F is an ultrafilter on ω which is not a P-point, and again take $Y = X \oplus S$. But in this case the proof of the properties of X and Y is much more involved.

During the 8th Prague Topological Symposium, S.P. Gul'ko announced the following result:

Theorem 3.2 (Gul'ko, Sokolov). *Let F be an ultrafilter on ω. The following are equivalent:*

(i) *F is not a P-point,*

(ii) *$C_p(N_F)$ contains a closed copy of the rationals $\mathbb{Q}$ (equivalently $C_p(N_F)$ is not a hereditary Baire space),*

(iii) *$C_p(N_F)$ contains a closed copy of the space σ.*

Hence we may use this result (together with Lemma 3.1 from [2]) for the proof of the properties of our example, instead of Lemma 2.1. But we decided to include this lemma to make our paper more self-contained.

The result of Gul'ko and Sokolov shows that the existence of P-points in ω^* (which follows from the Continuum Hypothesis) implies the existence of hereditary Baire spaces $C_p(N_F)$. This fact has some other interesting consequences for the function spaces $C_p^*(X)$. We have the following simple observations:

Proposition 3.3. *Let X be a countable space such that $C_p(X)$ is a hereditary Baire space. Then the space $C_p^*(X)$ does not contain a closed copy of the space σ^ω. In particular $C_p^*(X)$ is not homeomorphic to $C_p^*(X)^\omega$.*

Proof. We have $C_p^*(X) = \bigcup_{n=1}^{\infty} C_p(X) \cap [-n,n]^X$. The Hurewicz theorem implies that, in our case, every closed absolute Borel subset of $C_p(X) \cap [-n,n]^X$ is an absolute G_δ, for every n. Therefore every closed absolute Borel subset of $C_p^*(X)$ is an absolute $G_{\delta\sigma}$. It is well known that the space σ^ω is not such a space. The last part of the proposition follows from the fact that $C_p^*(X)$ always contains a closed copy of σ. (This can be seen by a direct argument, but also follows from the result of Banakh and Cauty quoted in the introduction.) □

Let us point out that the space $C_p^*(N_F)$ is always homeomorphic to all its finite powers $(C_p^*(N_F))^n$. If X is nondiscrete and $C_p^*(X)$ is analytic then it is homeomorphic to $C_p(X)$ and contains a closed copy of σ^ω (see [2, Section 3]). Proposition 3.3 implies the following fact:

Proposition 3.4. *Let F be a filter such that $C_p(N_F)$ is a hereditary Baire space. Then the space $C_p^*(\omega \times N_F)$ is not homeomorphic to $(C_p^*(N_F))^\omega$.*

Proof. By Lemma 4.11 from [3] the space $C_p(\omega \times N_F) \approx (C_p(N_F))^\omega$ can be embedded as a closed subset of $C_p(N_F)$. Therefore $C_p(\omega \times N_F)$ is a hereditary Baire space and by Proposition 3.3 $C_p^*(\omega \times N_F)$ does not contain a closed copy of the space σ^ω. On the other hand $C_p^*(N_F)$ always contains a closed copy of σ, so $(C_p^*(N_F))^\omega$ contains a closed copy of σ^ω. □

Observe that if X is the topological sum of the spaces X_i, $i \in I$, then $C_p(X)$ is canonically (linearly) homeomorphic to $\prod_{i\in I} C_p(X_i)$. By [2, Theorem 3.2], if $I = \omega$ and, for all $i \in \omega$, X_i is nondiscrete and $C_p(X_i)$ is analytic, then also

$$C_p^*(X) \approx \prod_{i\in\omega} C_p^*(X_i).$$

Our result shows that this cannot be extended to the general case. □

Remark 3.5. In [8, Lemma 4.1] it has been proved that a closed zero-dimensional subset of the space $C_p(N_F)$ can be embedded in F^ω as a closed subset, for every filter F (here, we consider F as a subspace of the Cantor set $2^\omega \approx \mathcal{P}(\omega)$). The theorem of Gul'ko and Sokolov shows that this is not the case for $C_p^*(N_F)$. This space always contains a closed copy of the rationals $\mathbb{Q}$. But, if F is a P-point then the product F^ω is hereditary Baire. This follows from the fact that F^ω can be embedded as a closed subset in $C_p(N_F)$, see [3, Lemma 4.11] and [7, Theorem 2.1].

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