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Monotone normality, measures and hyperspaces

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Abstract

We show that a compact Hausdorff, hereditarily Lindelöf, monolithic, monotonically normal space has a monolithic hyperspace. This generalises a result of M. Bell for ordered spaces. A consistent example of a nonmonotonically normal space with a monolithic hyperspace is given. We also show that every monotonically normal compact space is measure separable in the sense of Kunen and Džamonja. © 1998 Elsevier Science B.V.

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1. Introduction and notation

A key role in this paper will be played by the notion of monotone normality. Monotonically normal spaces are a common generalisation of both metric and ordered spaces and have recently received quite a lot of attention in the literature. We will first recall the definition: A space X is called *monotonically normal* (see [15]) if X is T_1 and there exists for every pair x and U, where $x \in U$ and U is an open subset of X, an open set $\mu(x,U)$ such that $x \in \mu(x,U) \subset U$ and the following two properties hold:

$$\text{If } U \subset V \text{ then } \mu(x,U) \subset \mu(x,V), \tag{1.1}$$

$$\mu(x, X \setminus \{y\}) \cap \mu(y, X \setminus \{x\}) = \emptyset. \tag{1.2}$$

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Such a μ is called a monotone normality operator for X. In this paper we will only use the following important property of such a monotone operator, which easily follows from the two above:

$$\text{If } \mu(x,U)\cap\mu(y,V)\neq\emptyset \quad \text{then } x\in V \text{ or } y\in U. \tag{1.3}$$

In fact, this property alone would suffice to define a monotone operator, as (1.2) follows at once from (1.3) and we can always assume that μ fulfills (1.1) by defining a new operator, using unions (letting $\widehat{\mu}(x,U)$ be the union of all $\mu(x,V)$, where $x\in V\subset U$ and V open in X). It is well known that monotonically normal spaces are hereditarily collectionwise normal and that every stratifiable space and every generalized ordered space is monotonically normal. See [7] for details. Let X be a topological space and let κ be an infinite cardinal. X is called *κ-monolithic* [1] if for every subset A of X such that $|A|\leqslant\kappa$ we have that $\mathrm{nw}(\overline{A})\leqslant\kappa$. Here nw denotes the net weight of a space. If X is compact Hausdorff, we can use weight instead of net weight in the above definition, as $\mathrm{w}(\overline{A})=\mathrm{nw}(\overline{A})$ in this case. X is called *monolithic* if it is κ-monolithic for all cardinals κ. Monolithicity is a hereditary and $\aleph_0$-productive property. Examples of monolithic spaces include: all metric spaces and all spaces of countable net weight.

By $H(X)$ we will denote the hyperspace of closed nonempty subsets of X, endowed with the Vietoris topology. We will use the following notation for the standard subbasis elements of $H(X)$:

$$\langle U\rangle=\{F\in H(X)\colon\ F\subset U\}\quad\text{and}\quad [U]=\{F\in H(X)\colon\ F\cap U\neq\emptyset\}.$$

Here U is an arbitrary nonempty open subset of X. We will also use the notation $\langle U_1,\dots,U_n\rangle$ for

$$\bigcap_{i=1}^{n}[U_i]\cap\left\langle\bigcup_{i=1}^{n}U_i\right\rangle,$$

where the U_i's are nonempty open subsets of X. These sets form a base for the topology of $H(X)$.

Arhangel'skiĭ asked in [2] when $H(X)$ is monolithic. Murray Bell, in [3], obtained the two following results concerning this question:

Theorem 1. *Let X be a T_1-space. If $H(X)$ is monolithic then X is monolithic, hereditarily Lindelöf and compact.*

Theorem 2. *Let X be a compact orderable space. Then $H(X)$ is monolithic if and only if X is monolithic and hereditarily Lindelöf.*

In fact, Bell proved a somewhat stronger result. Looking at his proof of Theorem 1 we see that he in fact proved the following: If $H(X)$ is $\aleph_0$-monolithic and X is T_1, then X is $\aleph_0$-monolithic, compact and hereditarily Lindelöf. We will use this later on. Also, we will be using the following simple fact from [3]:

Fact 1. *Let F be a closed subset of a compact Hausdorff space X. If there exists a collection $\mathcal{U}$ of open subsets of X that T_1-separates the points of F, then* $\mathrm{w}(F)\leqslant|\mathcal{U}|$.

(Recall that a family is called (strongly) T_1-separating for the points of F if for every two distinct points x, y of F there is a member U of the family such that $x \in U$ and $y \notin U$ ($y \notin \overline{U}$).)

In the first section of our paper we will extend Bell's Theorem 2 to the class of monotonically normal spaces. We will also show that the $\aleph_0$-monolithicity of $H(X)$ need not imply that X is monotonically normal. Our results show that, in general, the characterization of spaces with a ($\aleph_0$-)monolithic hyperspace is quite a difficult problem: this class contains the ($\aleph_0$-)monolithic, ccc, monotonically normal compacta, and is closed under closed continuous images and multiplication with a compact metric space. Whether it is closed under finite products (whenever these are ccc) is still open.

We will also need some definitions from measure theory. All the measures we consider will be finite Borel measures. We will call a (finite Borel) measure a *Radon measure* if it is inner regular for the compact sets, i.e., the measure of each measurable subset is the supremum of the measures of its compact subsets. The measure algebra of a Borel measure space (X, μ) is the Boolean algebra of the Borel sets modulo the μ-negligible sets. This can be made into a metric space in the case that μ is finite: let $d([A], [B]) = \mu(A \triangle B)$, where $[A]$ denotes the equivalence class of a Borel set A and $A \triangle B$ is the symmetric difference of A and B (it is easily checked that this definition does not depend on the representatives chosen, and that this indeed defines a metric). A measure is called *separable* if this metric space is separable.

In [9] Kunen and Džamonja introduced the class of *measure separable* spaces: A space is called measure separable if it is compact Hausdorff and every Radon measure on X is separable. They proved the following facts about this class of spaces: It is closed under countable products and continuous images onto Hausdorff spaces. Every compact metric and compact orderable space is measure separable. We will prove in Section 3 that all compact monotonically normal spaces are measure separable, generalizing their result for compact orderable spaces (of course, this fact is also a generalisation of the well-known fact that every compact metric space is measure separable, but this can be proven more directly by using the countable base of such a space).

Finally, for more information on cardinal functions and hyperspaces we refer the reader to [6]. We will use the notation $\mathrm{hl}(X)$ for the hereditary Lindelöf number of X, as defined there.

2. Monolithicity of $H(X)$

In this section we will generalize Murray Bell's Theorem 2. We will prove the same result for compact monotonically normal spaces. For this we will first prove the following theorem:

Theorem 3. *Let X be a compact, monotonically normal space. Put* $\mathrm{hl}(X) = \lambda$ *and let $\mathcal{F}$ be a family in $H(X)$ of (infinite) cardinality κ. Suppose that X is $\kappa \cdot \lambda$-monolithic. Then the closure of $\mathcal{F}$ in $H(X)$ has weight less than or equal to $\kappa \cdot \lambda$.*

Proof. We will first fix some notation: Using the fact that $\mathrm{hl}(X) = \lambda$ and the compactness of X, we choose for every $F \in \mathcal{F}$ a local base of open neighborhoods $(U_\alpha(F))_{\alpha<\lambda}$. In particular we will have that

$$\bigcap_{\alpha<\lambda} U_\alpha(F) = F.$$

Now fix an F and a $U_\alpha(F)$ for the time being, and consider the cover $\{\mu(x, U_\alpha(F)) \colon x \in F\}$ of F. By compactness there exists a finite $F_\alpha \subset F$ such that

$$F \subset \bigcup_{x \in F_\alpha} \mu(x, U_\alpha(F)).$$

Now let

$$A(F) = \bigcup_{\alpha<\lambda} F_\alpha \quad \text{and} \quad A = \bigcup_{F \in \mathcal{F}} A(F).$$

It is obvious that $|A(F)| \leqslant \lambda$ and $|A| \leqslant \kappa \cdot \lambda$. So, putting $B = \overline{A}$, it is clear that $\mathrm{w}(B) \leqslant \kappa \cdot \lambda$, using the $\kappa \cdot \lambda$-monolithicity of X.

We will need the following two lemmas:

Lemma 1. *Let $F \in \mathcal{F}$ and $p \notin F \cup B$. Then:*

$$\mu(p, X \setminus B) \cap F = \emptyset.$$

Proof. Suppose not. Then for every $\alpha < \lambda$ there exists an $a_\alpha \in F_\alpha$ such that $\mu(p, X \setminus B) \cap \mu(a_\alpha, U_\alpha(F)) \neq \emptyset$. Now, because $a_\alpha \in A \subset B$, and using the second property of μ we may conclude that for all $\alpha < \lambda$ we have that $p \in U_\alpha(F)$. This yields that $p \in F$, contradicting our assumptions. □

Lemma 2. *Let $G \in \overline{\mathcal{F}}^{H(X)}$ and let O be an open neighbourhood of G and $p \in G \setminus B$. Then there exist $\alpha < \lambda$, $F \in \mathcal{F}$ and $x \in F_\alpha$ such that $p \in \mu(x, U_\alpha(F))$ and $\mu(x, U_\alpha(F)) \subset O$.*

Proof. We have that $G \in \langle O \rangle \cap [\mu(p, X \setminus B)]$, so there exists an $F \in \mathcal{F}$ such that $F \subset O$ and $F \cap \mu(p, X \setminus B) \neq \emptyset$. From Lemma 1 it follows that $p \in F$. Now we can choose an $\alpha < \lambda$ such that $p \in F \subset U_\alpha(F) \subset O$, and for this α we can find an $x \in F_\alpha$ such that $p \in \mu(x, U_\alpha(F))$. Obviously we now also have $\mu(x, U_\alpha(F)) \subset O$. □

We shall now construct a family of open subsets of $H(X)$ which will be T_1-separating for $\overline{\mathcal{F}}^{H(X)}$. This family will have cardinality not exceeding $\kappa \cdot \lambda$ and by Fact 1 in the introduction, this will show that the weight of $\overline{\mathcal{F}}^{H(X)}$ does not exceed $\kappa \cdot \lambda$. Let $\mathcal{V}$ be a collection of open subsets of X which is strongly T_1-separating for B, such that $|\mathcal{V}| \leqslant \kappa \cdot \lambda$. Also choose a family $(V_\alpha)_{\alpha<\lambda}$ of open neighborhoods of B such that $\bigcap_{\alpha<\lambda} \overline{V}_\alpha = B$. Put:

$$\mathcal{U}' = \mathcal{V} \cup \{V_\alpha\}_{\alpha<\lambda} \cup \{\mu(x, U_\alpha(F)) \colon F \in \mathcal{F},\ \alpha < \lambda,\ x \in F_\alpha\}. \tag{2.1}$$

Now we make this collection closed under finite unions and intersections, call the resulting collection $\mathcal{U}$. It is clear that both $\mathcal{U}'$ and $\mathcal{U}$ have cardinality not exceeding $\kappa \cdot \lambda$. Finally we put:

$$\mathcal{W} = \{\langle U\rangle, \langle X \setminus \overline{U}\rangle, [U], [X \setminus \overline{U}]:\ U \in \mathcal{U}\} \tag{2.2}$$

and we will show that this family is the required T_1-separating family for $\overline{\mathcal{F}}^{H(X)}$ (in fact it is even T_2-separating).

Let G and H be two distinct elements of $\overline{\mathcal{F}}^{H(X)}$. Without loss of generality we may assume that there is an $x \in G \setminus H$. We will consider three cases.

Case 1(a). Suppose $x \in B$ and $H \cap B = \emptyset$. Using the compactness of H and the fact that $\mathcal{U}$ is closed under finite intersections, we can choose a $V \in \mathcal{U}$ such that $x \in B \subset V$ and $\overline{V} \cap H = \emptyset$. Then $[V]$ and $\langle X \setminus \overline{V}\rangle$ are disjoint hyperspace neighborhoods of G and H, respectively, and obviously these are members of $\mathcal{W}$.

Case 1(b). Suppose $x \in B$ and $H \cap B \neq \emptyset$. Now, for every $y \in H \cap B$ we can find a $U_y \in \mathcal{V}$ such that $y \in U_y$ and $x \notin \overline{U_y}$. We can find finitely many of these U_y's that cover $H \cap B$, call the union of these U_y's U. We then have that $U \in \mathcal{U}$ and $x \notin \overline{U}$. If we now have that $H \subset U$ then we can separate G and H by $[X \setminus \overline{U}]$ and $\langle U\rangle$, respectively. So assume this is not the case. Now choose an open neighbourhood O of H such that $x \notin \overline{O}$. Because $H \setminus U \subset X \setminus B$ and $H \setminus U$ is compact, we can find, using Lemma 2, finitely many $\alpha < \lambda$, $F \in \mathcal{F}$ and a_α such that the $\mu(a_\alpha, U_\alpha(F))$'s cover $H \setminus U$, and are all contained in O. Let V be the union of the previously found U and these finitely many $\mu(a_\alpha, U_\alpha(F))$'s. Then $V \in \mathcal{U}$ and $x \notin \overline{V}$. Now $[X \setminus \overline{V}]$ and $\langle V\rangle$ are disjoint neighbourhhoods of G and H, respectively.

Case 2. Suppose that $x \notin B$. First choose a V_α such that $x \notin \overline{V_\alpha}$. If we now have that $H \subset V_\alpha$, we can separate G and H by $\langle V_\alpha\rangle$ and $[X \setminus \overline{V_\alpha}]$. If not, we can find, as in the previous case, a finite number of $\mu(a_\alpha, U_\alpha(F))$'s that cover $H \setminus V_\alpha$, that stay inside an open neighbourhood O of H whose closure does not contain x. Let V be the union of V_α and the $\mu(a_\alpha, U_\alpha(F))$'s, and this is a member of $\mathcal{U}$. Now again $\langle V\rangle$ and $[X \setminus \overline{V}]$ will be the required separating neighborhoods of G and H. □

This theorem has some corollaries:

Corollary 1. *Let X be a compact, monotonically normal space. If $\kappa \geqslant \mathrm{hl}(X)$ and X is κ-monolithic, then $H(X)$ is κ-monolithic as well.*

Corollary 2. *Let X be a hereditarily Lindelöf, monotonically normal, compact space. If X is (κ-)monolithic then $H(X)$ will be (κ-)monolithic as well.*

Now we can easily prove the announced generalisation of Bell's Theorem 2:

Theorem 4. *Let X be a monotonically normal space. Then $H(X)$ is monolithic if and only if X is monolithic, compact and hereditarily Lindelöf.*

Proof. Necessity follows from Theorem 1. Sufficiency follows from Corollary 2. □

Using the remark following Theorem 2 we have the following:

Theorem 5. *Let X be a monotonically normal space. Then $H(X)$ is $\aleph_0$-monolithic if and only if X is $\aleph_0$-monolithic, compact and hereditarily Lindelöf.*

In [2] it was asked whether X must be metrizable when its hyperspace is monolithic. As Bell showed, this question is undecidable in ZFC. For the class of monotonically normal spaces we have:

Theorem 6. *The following are equivalent*:

(1) *There exists a nonmetric, monotonically normal space with an $\aleph_0$-monolithic hyperspace.*

(2) *There is a Souslin line* ($\neg$SH).

Proof. If (1) holds, then we know that X is hereditarily Lindelöf, compact and $\aleph_0$-monolithic. So X is not separable. This implies (2) by a result of Williams and Zhou [16].

For the other direction, if there is a Souslin line, by standard techniques we can make it compact, and such that every separable subspace is second-countable. Then Bell's Theorem 2 (or our Corollary 2) imply that its hyperspace is also $\aleph_0$-monolithic. $\square$

We will now present a nice preservation result for spaces with monolithic hyperspaces. This will have as a corollary the fact that there are (consistent) nonmonotonically normal spaces with a monolithic hyperspace. This is of some relevance, as up to now all spaces with monolithic hyperspaces were monotonically normal, and we have shown [4,5] that Kunen's compact L-space from CH has a nonmonolithic hyperspace. This latter space cannot be monotonically normal, as it carries a nonseparable Radon measure (see the next section).

Theorem 7. *Let X be a compact space such that $H(X)$ is κ-monolithic and let Y be a compact space of weight $\leqslant \kappa$. Then $H(X \times Y)$ is κ-monolithic.*

Proof. We will first prove the theorem for the case that Y is a zero-dimensional space. So let $(U_\alpha)_{\alpha\in\kappa}$ be a clopen base for Y. For every closed set F in $X \times Y$ we define F_α to be $F \cap (X \times U_\alpha)$. Let $\mathcal{F}$ be an arbitrary family of closed sets of $X \times Y$ of cardinality less than or equal to κ and denote by $\mathcal{F}_\alpha$ the family $\{F_\alpha\colon\ F \in \mathcal{F},\ F_\alpha \neq \emptyset\} \subset H(X \times U_\alpha)$.

Claim. *If $F \in \overline{\mathcal{F}}^{H(X\times Y)}$ and $F_\alpha \neq \emptyset$ then $F_\alpha \in \overline{\mathcal{F}_\alpha}$ (in $H(X \times U_\alpha)$).*

For a proof of this, let $U = \langle V_1, \ldots, V_k\rangle$ be an arbitrary neighbourhood of F in $H(X \times U_\alpha)$, where $V_1, \ldots, V_k$ are open subsets of $X \times U_\alpha$ (so also open in $X \times Y$). If $F_\alpha = F$ then $F \in U$ so there exists a $G \in \mathcal{F}$ such that $F' \in U$. But then $G \subset X \times U_\alpha$, so $G = G_\alpha \in \mathcal{F}_\alpha$. If $F_\alpha \neq F$ then $F \in \langle V_1, \ldots, V_k, X \times U_\alpha^c\rangle$, which is open in $X \times Y$, so there is an $F' \in \langle V_1, \ldots, V_k, X \times U_\alpha^c\rangle \cap \mathcal{F}$. But then $F'_\alpha \in U \cap \mathcal{F}_\alpha$, as required.

Denote, for all $\alpha \in \kappa$, the projection mapping from $H(X \times U_\alpha)$ to $H(X)$ by π_α. By continuity of π_α we have that $\pi_\alpha[\overline{\mathcal{F}_\alpha}] \subset \overline{\pi_\alpha[\mathcal{F}_\alpha]}$, so $\pi_\alpha[\overline{\mathcal{F}_\alpha}]$ has weight $\leqslant \kappa$, by κ-monolithicity of $H(X)$. So we can choose a family $\mathcal{V}_\alpha$ of open subsets of $H(X)$ of cardinality $\leqslant \kappa$ which is T_1-separating for the points of $\pi_\alpha[\overline{\mathcal{F}_\alpha}]$. Now we define the following family of open subsets of $H(X \times Y)$:

$$\begin{aligned}\mathcal{W} = &\big\{\langle V_1 \times U_\alpha, \ldots, V_k \times U_\alpha\rangle\colon\ \alpha \in \kappa,\ \langle V_1, \ldots, V_k\rangle \in \mathcal{V}_\alpha\big\}\\ &\cup \big\{\langle V_1 \times U_\alpha, \ldots, V_k \times U_\alpha, X \times U_\alpha^c\rangle\colon\ \alpha \in \kappa,\ \langle V_1, \ldots, V_k\rangle \in \mathcal{V}_\alpha\big\}\\ &\cup \big\{\langle X \times U_\alpha^c\rangle, [X \times U_\alpha]\colon\ \alpha \in \kappa\big\}.\end{aligned}$$

Obviously, $|\mathcal{W}| \leqslant \kappa$. We will now show that $\mathcal{W}$ is T_1-separating for the points of $\overline{\mathcal{F}}$, thereby showing that $\mathrm{w}(\overline{\mathcal{F}}) \leqslant \kappa$. So let F and G be two distinct elements of $\overline{\mathcal{F}}$. Without loss of generality there is a point $x \in F \setminus G$. We can find a U_α and an open subset V of X such that $x \in V \times U_\alpha$ and $(V \times U_\alpha) \cap G = \emptyset$. If $G_\alpha = \emptyset$, then $F \in [X \times U_\alpha]$ and $G \in \langle X \times U_\alpha^c\rangle$ and these sets are disjoint. So in that case we are done. If $G_\alpha \neq \emptyset$, then by the claim we have that $F_\alpha, G_\alpha \in \overline{\mathcal{F}_\alpha}$ and $\pi_\alpha(F_\alpha) \neq \pi_\alpha(G_\alpha)$, as $\pi_\alpha(x) \in \pi_\alpha(F_\alpha) \setminus \pi_\alpha(G_\alpha)$. So we can find a set $\langle V_1, \ldots, V_k\rangle \in \mathcal{V}_\alpha$ such that $\pi_\alpha(F_\alpha) \in \langle V_1, \ldots, V_k\rangle$ and $\pi_\alpha(G_\alpha) \notin \langle V_1, \ldots, V_k\rangle$. It follows that F is either in $\langle V_1 \times U_\alpha, \ldots, V_k \times U_\alpha\rangle$ or in $\langle V_1 \times U_\alpha, \ldots, V_k \times U_\alpha, X \times U_\alpha^c\rangle$, depending on whether $F_\alpha = F$ or not. Any of these sets is in $\mathcal{W}$, and G is not an element of it, for if it were, $\pi_\alpha(G_\alpha)$ would be an element of $\langle V_1, \ldots, V_k\rangle$, as is easy to see. So $\mathcal{W}$ is T_1-separating for $\overline{\mathcal{F}}$, as required. This concludes the case that Y is zero-dimensional.

To finish the proof, let Y be an arbitrary compact space of weight $\leqslant \kappa$. Because $\{0,1\}^\kappa$ maps continuously onto the the Tychonoff cube of weight κ, it is easy to see that Y is the continuous image of some compact zero-dimensional space Z. Then we also have a continuous surjection from $X \times Z$ onto $X \times Y$, which, by compactness, induces a surjection between their hyperspaces. It is easily seen that κ-monolithicity is preserved by closed continuous maps, so the first part of the proof now yields the desired result. $\square$

Corollary 3. *Assuming* $\neg$SH, *there exists a nonmonotonically normal space with a* $\aleph_0$-*monolithic hyperspace.*

Proof. Let X be a compact, $\aleph_0$-monolithic Souslin line, as in Theorem 6. Then $X \times (\omega + 1)$ (or $X \times 2^\omega$) has an $\aleph_0$-monolithic hyperspace, by the preceding theorem. If it were monotonically normal, by a theorem from [15], we would have that X is (compact and) stratifiable, and hence metrizable (see [7]). $\square$

Note that in this corollary some additional axiom is required: as Bell noted, MA(ω_1) implies that every X with $H(X)$ $\aleph_0$-monolithic is metrizable (and hence monotonically normal).

We will end this section with two questions that we were unable to solve. The first is related to the above theorem. It is obvious that one cannot prove that the product of two monotonically normal spaces with a monolithic hyperspace has a monolithic hyperspace: look at the square of a (compact monolithic) Souslin line, which is not even ccc, let alone hereditarily Lindelöf. But maybe not being ccc is the only obstacle to this. We can at least prove the following:

Theorem 8. *Let X and Y be monotonically normal, $\aleph_0$-monolithic spaces, and suppose that $X \times Y$ is ccc. Then $X \times Y$ is hereditarily Lindelöf.*

Proof. First note that since $X \times Y$ is ccc, the same holds for X and Y. It then follows from a result of Ostaszewski [11] that X and Y are hereditary Lindelöf. Now, let Z be an arbitrary subset of $X \times Y$, and let $\mathcal{U}$ be an open cover of Z. Let $\mathcal{V}$ be a maximal disjoint family of (nonempty) sets of the form $Z \cap (U_1 \times U_2)$, where U_1 and U_2 are open sets in X and Y, respectively, and such that each $V \in \mathcal{V}$ is contained in some $U_V \in \mathcal{U}$. By ccc-ness of the product we have that $\mathcal{V}$ is countable, and by maximality it is easy to see that $\bigcup \mathcal{V}$ is dense in Z. Denote by $\mathcal{U}_1$ the collection $\{U_1\colon \exists U_2 \subset Y\colon Z \cap (U_1 \times U_2) \in \mathcal{V}\}$. Likewise we define the collection $\mathcal{U}_2$. For every $U \in \mathcal{U}_1$ we choose a maximal disjoint family $O_1(U)$ of elements of the form $\mu(x_i, U_i)$, where $x_i \in U_i$ and every U_i is open in U. This defines a set of x_i's of which the corresponding $\mu(x_i, U_i)$ form $O_1(U)$. We call this set X_U. Similarly we define sets $Y_U \subset Y$ for every U in $\mathcal{U}_2$. By maximality, we have that if an open subset of X meets $U \in \mathcal{U}_1$ in a nonempty set, it must meet a member of $O_1(U)$ in a nonempty set, and likewise for Y and $O_2(U)$. Now we define

$$S = \overline{\bigcup_{U \in \mathcal{U}_1} X_U} \quad \text{and} \quad T = \overline{\bigcup_{U \in \mathcal{U}_2} Y_U}.$$

Because all sets involved are countable and because X and Y are $\aleph_0$-monolithic, we have that S and T have countable net weight. We now claim that:

$$Z \setminus \bigcup \mathcal{V} \subset (S \times Y) \cup (X \times T).$$

To see this, suppose this is not the case. We then have $(x, y) \in Z \setminus \bigcup \mathcal{V}$ and $x \notin S$ and $y \notin T$. So we can pick open neighborhoods U_x and U_y of x and y such that $U_x \cap S = \emptyset = U_y \cap T$. The open neighborhood $\mu(x, U_x) \times \mu(y, U_y)$ of (x, y) now meets some $(U_1 \times U_2) \cap Z \in \mathcal{V}$ in a nonempty set. Moreover, we have that either $x \notin U_1$ or $y \notin U_2$ (or both), as $(x, y) \notin \bigcup \mathcal{V}$. So assume the former, as the other case is similar. We have that $\mu(x, U_x) \cap U_1 \neq \emptyset$, so for some $\mu(x_i, U_i) \in O(U_1)$ we have that $\mu(x, U_x) \cap \mu(x_i, U_i) \neq \emptyset$. But then we have that either $x \in U_i$ or $x_i \in U_x$ (or both). In the former case we would have that $x \in U_i \subset U$, in the latter case we would have that $U_x \cap S$ contains x_i. In both cases we have a contradiction. This proves the claim.

So now we can find a countable subcollection of $\mathcal{U}$: $\{U_V\colon V \in \mathcal{V}\}$ which covers $\bigcup \mathcal{V}$, while the other points of Z are contained in a union of two hereditarily Lindelöf spaces, as the product of a hereditarily Lindelöf space and a space with a countable net weight is again hereditarily Lindelöf. So $X \times Y$ is hereditarily Lindelöf. □

(For the case that X and Y are ordered spaces, the above theorem can be obtained from an easy adaptation of the proof of [12, Theorem 1]. Our result, and its proof, were inspired by this theorem as well.) So now there is a natural question, trying to generalize the above preservation result from the case of compact metric Y to arbitrary monotonically normal spaces:

Question 1. Let X and Y be compact, ($\aleph_0$-)monolithic, monotonically normal, hereditarily Lindelöf spaces, and let $X \times Y$ be ccc. Is it true that $H(X \times Y)$ has a ($\aleph_0$-)monolithic hyperspace?

Of course, the above is trivially true if there are no compact L-spaces. Nontrivial examples of the conditions in the question (using two Souslin lines) can, e.g., be constructed under $\diamond$.

The second question is related to the problem of whether every compact monotonically normal space is the continuous image of a compact linearly ordered space (see [10]). It is easy to see that the closed continuous image of a (κ-)monolithic space is (κ-)monolithic. So if the following question would be answered affirmatively, we would have a different proof of our Theorems 4 and 5, by using Bell's theorem for the ordered case and observing that a map from X onto Y induces a map from $H(X)$ onto $H(Y)$.

Question 2. Let X be a compact, ccc, monotonically normal and ($\aleph_0$-)monolithic space. Is it true that X is the continuous image of a ($\aleph_0$-)monolithic, compact ccc linearly ordered space?

3. Measure separability of X

In this section we prove a generalisation of Kunen and Džamonja's result that compact orderable spaces are measure separable. This will show in particular that every "Kunen space" (i.e., a space that is constructed like the compact L-space in [8], using inverse limits) is never monotonically normal. The authors showed in [4,5] that all Kunen spaces (which are compact hereditarily Lindelöf and monolithic spaces with a nonseparable measure) have a nonmonolithic hyperspace, thereby showing that these conditions are in general not sufficient for the hyperspace to be monolithic. In fact, we obtained in [4] that a certain modification of Kunen's space had a nonmonolithic hyperspace, and hence we could only conclude from our Theorem 4 that this particular space was not monotonically normal. The results in [5] now give another proof of the fact that no Kunen space is monotonically normal.

We now first state a lemma:

Lemma 3. *Let X be a monotonically normal compact space with a finite Radon measure λ. Suppose that all points have measure* 0 *and that all nonempty open sets have strictly positive measure. Then X is separable (as a topological space), and its measure algebra is separable.*

Proof. We will build a tree of open subsets of X like in the paper [16] of Williams and Zhou to show the separability of X. For the reader's convenience we will give a complete proof here. The following fact from [16] will be used below:

Fact 2. *Let X be monotonically normal, and let $Y \subset X$. Suppose that U is an open cover of Y. Then:*

$$\overline{\{\mu(y,U)\colon\ U \in \mathcal{U},\ y \in Y \cap U\}} \subset \overline{Y} \cup \bigcup \mathcal{U}. \tag{3.1}$$

To prove this, let x be an element of the left hand side, and suppose that x is not a member of any $U \in \mathcal{U}$. Let V be an arbitrary open neighbourhood of x. There exists an open $U \in \mathcal{U}$ and a point $y \in Y$, such that $\mu(x,V) \cap \mu(y,U) \neq \emptyset$. Because $x \notin U$, we have that $y \in V$. As V is arbitrary, this proves that $x \in \overline{Y}$.

We will now construct a tree $\mathcal{T}$ of countable height by induction. The order in the tree will be reverse inclusion. The first level of the tree, $\mathcal{T}_0$, will be just $\{X\}$. We first pick for every point $x \in X$ and every natural number $n \geqslant 1$ a neighbourhood $U_{x,n}$ such that $\lambda(U_{x,n}) < 1/n$. It is easy to see that such a neighbourhood exists for all x and n, because the measure of a point is 0 and the measure is outer regular for open sets. Suppose now that we have constructed $\mathcal{T}_n$, the nth level of our tree $\mathcal{T}$. Let O be an arbitrary member of $\mathcal{T}_n$. Now consider the following family $\mathcal{U}$:

$$\mathcal{U} = \{\mu(x, O_x \cap U_{x,n})\colon\ x \in O_x,\ \overline{O_x} \subset O,\ O_x \text{ is open in } X\}.$$

Let $\mathcal{U}(O)$ be a maximal pairwise disjoint subfamily of $\mathcal{U}$ with at least two members (this can be done, because the conditions on λ imply that X has no isolated points). This determines a set of x's of which the corresponding $\mu(x, O_x \cap U_{x,n})$'s form $\mathcal{U}(O)$. This set will be denoted by $M(O,n)$. Now level $n+1$ of $\mathcal{T}$ will be given by the union of all $\mathcal{U}(O)$'s for every $O \in \mathcal{T}_n$. This completes the construction of $\mathcal{T}_n$ for all $n \in \omega$. It is easy to see, by induction, that the union of every level of the tree is dense in X. Also, denote by M_n the union of all $M(O,n)$ for $O \in \mathcal{T}_n$. We will now prove that $D = \bigcup_{n\in\omega} M_n$ is dense in X. Obviously M is countable, because every level of $\mathcal{T}$ is countable (since X has ccc) and the height of the tree is countable. So suppose there exists a point $x \in X \setminus \overline{D}$. Then, using the density of the union of $\mathcal{T}_2$ and the fact above, we have that

$$x \in \overline{\bigcup\{O\colon\ O \in \mathcal{T}_2\}} \subset \overline{M_2} \cup \bigcup_{y\in M_2} (O_y \cap U_{y,2}) \subset \overline{D} \cup \bigcup_{y\in M_2} O_y.$$

Because we assumed that $x \notin \overline{D}$, we know that $x \in \bigcup_{y\in M_1} O_y$, so there is a unique $O_1 \in \mathcal{T}_1$ that contains x. Continuing the above argument inductively, we find a decreasing sequence of O_n's, all containing x and such that each O_n comes from the nth level of $\mathcal{T}$. So $x \in \bigcap_{n\in\omega} O_n$. The interior of this intersection is empty because the its measure is 0 (all sets on level n have measure smaller than $1/n$) and we assumed that nonempty open sets have positive measure. So for some n there will be $y \in M_n$ such that

$$\mu(x, X \setminus \overline{D}) \cap \mu(y, O_y \cap U_{y,n}) \neq \emptyset,$$

where this $\mu(y, O_y \cap U_{y,n})$ is a member of $\mathcal{T}_n$ different from O_n. But we then have that $y \in D$ and $x \notin O_y$, so this cannot happen. This contradiction proves that X is separable. But now a theorem of Mary Ellen Rudin (see [13,14]) shows that X is the continuous image of a compact ordered space, which is measure separable. So X is measure separable as well, by the results mentioned in the introduction. $\square$

Note that this lemma already implies that no "Kunen" space is monotonically normal, as it is not separable and its measure always is of the above type.

An inspection of the proof of Kunen and Džamonja shows that for ordered spaces having a measure of the above type there is a nice description of the measure algebra, resulting in a more or less explicitly given countable dense subset of the measure algebra. In the above proof we made an essential use of the highly nontrivial results from [13,14]. As a consequence, we did not find a nice description of the measure algebra as in the ordered case. This leaves open the natural question of whether such a description is possible, i.e., whether a more direct proof of the above lemma avoiding Rudin's results can be found.

Now we can show the main theorem in this section:

Theorem 9. *Let X be a monotonically normal space with a finite Radon measure λ. Then the measure algebra of λ is separable.*

Proof. We will first reduce the proof to the case that X is compact, as follows: Let $(C_n)_{n\in\omega}$ be a maximal family of disjoint compact subsets of X having strictly positive measure. Such a family exists because λ is a Radon measure and the family is countable, because the measure is finite. (Note that this also holds if λ were a sigma-finite measure.) Obviously, by maximality of $(C_n)_{n\in\omega}$ we have that

$$\lambda\Bigl(X \setminus \bigcup_{n\in\omega} C_n\Bigr) = 0.$$

We now consider every C_n with the restriction of λ to this subset. These are all compact monotonically normal spaces with a finite Radon measure. Let $\mathcal{M}_n$ be the measure algebra of C_n and let $\mathcal{M}$ be the measure algebra of X. One can show $\mathcal{M}$ to be homeomorphic to the topological product of the $\mathcal{M}_n$ (considered as metric spaces) by the map Φ which sends the equivalence class of a Borel subset A to the point $([A \cap C_n])_{n\in\omega}$ in $\prod_{n\in\omega} \mathcal{M}_n$. This map is obviously well-defined, continuous 1–1 and onto. Also the map that sends a point in the product to the class of the union of its components is a continuous inverse for this map. This shows that we only have to prove that a compact monotonically normal space is measure separable. So suppose this is not the case. We now use the following simple fact from [9]:

Fact 3. *If X is compact and λ is a nonseparable Radon measure on X, then there is a closed $K \subset X$ such that $\lambda(K) > 0$, for all subsets B of K having positive measure the restriction of λ to B is not separable, and every nonempty relatively open subset of K has strictly positive measure.*

So if a compact space X is not measure separable, and λ would be a nonseparable Radon measure, we would find K as above. It is easy to see that such a K satisfies the conditions of the preceding lemma: it is monotonically normal, and all points have measure 0 because the measure algebra of a one point set is separable. So we conclude that λ restricted to this K is separable, which is a contradiction. $\square$

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