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ON COLORINGS OF TOPOLOGICAL GROUPS

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Abstract

We study the question whether there exists an example of a strongly zero-dimensional topological group G and a fixed-point free homeomorphism $f\colon G \to G$ such that βf has a fixed point. It is known that such a group cannot be paracompact. We use the construction of the free group over a space to get such an example, but it is not normal. So we specialize the question by asking for a normal such group and consider two classical examples: sigma products and the groups constructed by Hajnal and Juhász. We show that those examples do not solve our problem. In the process, we obtain a new coloring theorem and a new non-group example.

1. Introduction

Let X be a space and let $f\colon X \to X$ be continuous and fixed-point free. We say that f can be *colored* provided that its Čech-Stone extension $\beta f\colon \beta X \to \beta X$ is fixed-point free as well. We will first make a few observations that explain this terminology. Since βf has no fixed point and βX is compact, there clearly exists a finite functionally open cover $\mathcal{A}$ of βX such that for

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every $A \in \mathcal{A}$ we have $\beta f[A] \cap A = \emptyset$. Since $\beta f \restriction X = f$, the collection $\mathcal{B} = \mathcal{A} \restriction X$ is a finite functionally open cover of X such that for every $B \in \mathcal{B}$ we have $f[B] \cap B = \emptyset$. Such a collection $\mathcal{B}$ is called a *coloring* of f. It is clear, by using the standard properties of the Čech-Stone compactification, that a homeomorphism f has a coloring if and only if its Čech-Stone extension βf is fixed-point free (cf. [4]).

Our first result will be to show that every fixed point free continuous map on a space X, for which X^2 is normal and pseudocompact, can be colored. Much earlier, in one of his posthumously published papers [4], van Douwen showed that if X is a finite-dimensional paracompact space then any fixed-point free autohomeomorphism $f\colon X \to X$ can be colored. He also showed that the assumption of finite-dimensionality is essential: if X is the topological sum of the n-spheres, $n < \omega$, then the involution $x \mapsto -x$ cannot be colored. The theorem is not true for arbitrary finite dimensional Tychonov spaces, as was shown by Błaszczyk and Kim (see [2]). Their example is the following one. Let $X = \{-1, 0, 1\}^{\omega_1} \setminus \{\underline{0}\}$ and let $f\colon X \to X$ be defined by $f(x) = -x$. Then X is strongly zero-dimensional, f has no fixed point but βf has one. Simply observe that $\beta X = \{-1, 0, 1\}^{\omega_1}$ and so βf sends $\underline{0}$ onto itself. But X is unfortunately not normal. Let us also remark that Mazur [13] showed that if $\mathbb{P}$ denotes the space of irrational numbers, then for some fixed-point free map $f\colon \mathbb{P} \to \mathbb{P}$ its Čech-Stone extension $\beta f\colon \beta\mathbb{P} \to \beta\mathbb{P}$ has a fixed point. This explains why van Douwen's result is formulated for homeomorphisms, and also why we will concentrate on homeomorphisms in the sequel.

One naturally wonders whether the assumption on paracompactness in van Douwen's result can be weakened, say e.g. to normality in some special classes of spaces. In the next section we present a partial answer to this question. We also present an example, based on Ostaszewskis construction, to illustrate the sharpness of both theorems. Several authors also looked at this problem, and for various "nice" classes of spaces counterexamples were constructed, some normal and some not. See for example Błaszczyk and Kim [2], Watson [19], Good [6] and van Hartskamp and van Mill [9].

One of the classes of spaces not considered so far is the class of topological groups. The question naturally arises whether fixed-point free homeomorphisms on finite dimensional, or even strongly zero-dimensional, topological groups can be colored. We prove rather easily using the free group functor that this is not the case but our example is not normal. Trying to find a *normal* strongly zero-dimensional group with a non-colorable homeomorphism, we consider three natural possibilities:

1. The free group over an 'ordinary' non-colorable space;
2. A sigma product of compact groups with countable tightness;
3. The hereditarily normal Hajnal-Juhász group from [7].

In all three cases we show that this leads nowhere. So we did not solve our problem, but at least we were able to show that certain natural candidates do not work.

Question 1.1. *Is there a normal strongly zero-dimensional topological group with a non-colorable fixed-point free homeomorphism?*

As usual, we will say that X is *zero-dimensional* if $\operatorname{ind} X = 0$, i.e. X has a clopen base, and X is called *strongly zero-dimensional* if $\dim X = 0$. By a space we mean a Tychonov space and by a group a topological group. For all undefined notions we refer to [5].

2. Coloring Normal Spaces

The following theorem generalizes and is motivated by the obvious fact that every compact space can be colored. It is not a corollary of earlier theorems (cf. [4, 1, 10, 15]).

Theorem 2.1. *Let X be a space such that X^2 is normal and pseudocompact. Then every continuous fixed-point free map $f\colon X \to X$ can be colored.*

In particular, by the Comfort and Ross theorem (the product of pseudocompact groups is pseudocompact [3]), it follows that every map on a pseudocompact group whose square is normal can be colored. Before we give the proof we illustrate the sharpness with some examples and give a corollary.

Corollary 2.2. *Let X be such that $|\beta X \setminus X| = 1$. If X^2 is normal, then every homeomorphism $f\colon X \to X$ has a fixed-point.*

Proof. Let X be such that $\beta X \setminus X = \{p\}$. Clearly X is locally compact. Also X is pseudocompact (see [5, 3.12.16(b)]) and it follows by local compactness that X^2 is pseudocompact (cf. [5, 3.10.13]). Now assume by contradiction, that $f\colon X \to X$ is fixed-point free, then by Theorem 2.1 βf is fixed-point free as well. However $\beta f(p) = p$. Contradiction. □

The Błaszczyk-Kim example mentioned in the introduction is a non-normal space with a pseudocompact square, but still a homeomorphism exists which cannot be colored.

Van Douwen's example, from the introduction, is a metrizable space, with a non-pseudocompact square for which a homeomorphism exists which cannot be colored.

The next example, under $\diamondsuit$, is even nicer.

Example 2.3. Recall that a topological space X is called an *Ostaszewski space* provided that X has cardinality ω_1 and moreover is first countable, locally countable, countably compact, hereditarily separable, perfectly normal, strongly zero-dimensional, locally compact and not Lindelöf. In [16], Ostaszewski showed that such spaces exist under $\diamondsuit$.

In [14, Page 991, "$X \times X$ nonnormal"] Laberge constructs a modification of the original Ostaszewski space and builds in an involution $\iota\colon X \to X$ without fixed points. Under CH (a consequence of $\diamondsuit$) it is easy to see that X has only one compactification, namely, its one-point compactification. This is due to Ostaszewski. Hence by Corollary 2.2 it follows that X^2 is not normal.

Let us now present some notation that we will use in the proof.

Notation. *If $h : Y \to Y$ is a map, then the graph:* $\operatorname{graph}(h) \subset y \times y$ *of h is defined by* $\operatorname{graph}(h) = \{\, (x, h(x)) : x \in y \,\}$.

Recall that the graph is closed if Y is a Hausdorff space. Moreover we need the following lemma, which is undoubtedly known.

Lemma 2.4. *Let $h : X \to X$ be a continuous map. Then*

$$\operatorname{cl}_{\beta X \times \beta X} \operatorname{graph}(h) = \operatorname{graph}(\beta h).$$

Proof. We prove $\subseteq$: The graph of βh is closed and clearly contains $\operatorname{graph}(h)$, hence its closure.

We prove $\supseteq$: Let $(x, \beta h(x)) \in \operatorname{graph}(\beta h)$ and let $W \times V$ be a basic open neighborhood of it in $\beta X \times \beta X$. By continuity there exists an open set $U \subseteq \beta X$ such that $x \in U \subseteq \beta h^{-1}[V] \cap W$. As X is dense in βX, there exists an $x' \in U \cap X$. Hence $V \ni \beta h(x') = h(x')$ and so $(x', h(x')) \in U \times V \subseteq W \times V$. So $(x, \beta h(x)) \in \operatorname{cl}_{\beta X \times \beta X} \operatorname{graph}(h)$. □

Now we can prove the theorem.

Proof of Theorem 2.1. As f is fixed-point free and X is Hausdorff, $\operatorname{graph}(\mathrm{Id})$ and $\operatorname{graph}(f)$ are disjoint closed subsets of X^2. By normality of X^2 the sets are completely separated and this implies that their closures in the Čech-Stone compactification $\beta(X^2)$ are also disjoint.

By Glicksbergs Theorem [5, 3.12.21(c)], which states that the pseudocompactness of X^2 implies $\beta(X^2) = (\beta X)^2$ and by the previous lemma it follows that

$$\begin{aligned}
\emptyset &= \mathrm{cl}_{\beta(X^2)}\,\mathrm{graph}(\mathrm{Id}) \;\cap\; \mathrm{cl}_{\beta(X^2)}\,\mathrm{graph}(f) \\
&= \mathrm{cl}_{(\beta X)^2}\,\mathrm{graph}(\mathrm{Id}) \;\cap\; \mathrm{cl}_{(\beta X)^2}\,\mathrm{graph}(f) \\
&= \mathrm{graph}(\beta \mathrm{Id}) \;\cap\; \mathrm{graph}(\beta f) \\
&= \mathrm{graph}(\mathrm{Id}_{\beta \mathrm{X}}) \;\cap\; \mathrm{graph}(\beta f).
\end{aligned}$$

Hence βf is fixed-point free. □

3. Free Groups

We will now consider the class of topological groups. Our aim is to construct a strongly zero-dimensional topological group with a non-colorable fixed-point free homeomorphism. It is quite natural to consider the free group $F(X)$ over one of the counterexamples X mentioned in the introduction, and to extend the "bad" homeomorphism $f\colon X \to X$ to a "bad" homeomorphism of $F(X)$. (The standard way of doing that does not work however.)

For the construction of the free group $F(X)$ in the next theorem we refer the reader to [11, Theorem 8.8]).

Theorem 3.1. *For every space X, there exists a topological group $F(X)$ with the following properties:*

- *X is a closed subspace of $F(X)$;*
- *algebraically, $F(X)$ is the free group generated by X;*
- *for every continuous mapping f of X into any topological group G there exists a unique continuous homomorphism $Ff\colon F(X) \to G$ such that $Ff \restriction X = f$.*

In particular if $f\colon X \to X$ is an involution, then so is Ff.

For us there is a problem with the function Ff since it is a homomorphism and consequently sends the neutral element of $F(X)$ onto itself. So the functions Ff are not fixed-point free.

Since $F(X)$ is algebraically the free group over X, to every $y \in F(X) \setminus \{e\}$, where e denotes the neutral element of $F(X)$, we can associate a so-called reduced word. This means that y can be uniquely written in the form $\prod_{i=1}^{k} y_i^{n_i}$, where the n_i are integers unequal to zero, and consecutive y_i are distinct elements of X. Using this notation, the homomorphisms Ff are easy to describe. If $f\colon X \to G$ is continuous and G is a group then, since $Ff \restriction X = f$, for every $y = \prod_{i=1}^{k} y_i^{n_i} \in F(X) \setminus \{e\}$ we have

$$\begin{aligned} Ff(y) &= f(\textstyle\prod_{i=1}^{k} y_i^{n_i}) \\ &= \textstyle\prod_{i=1}^{k} f(y_i)^{n_i}. \end{aligned}$$

The functions Ff can be used to prove quite easily that X is C-embedded in $F(X)$. Indeed, let $f\colon X \to \mathbb{R}$ be continuous. Since $(\mathbb{R}, +)$ is a topological group, we can extend f to the function $Ff\colon F(X) \to \mathbb{R}$. Since $Ff \restriction X = f$, we are done. This (well-known) observation will be used later.

Let X be a space. Take $f\colon X \to (\mathbb{Z}_2, +)$ to be the constant function 1, and let $Ff\colon F(X) \to \mathbb{Z}_2$ be its canonical extension. It is left as an easy exercise for the reader to show that

$$(Ff)^{-1}(\{0\}) = \{\, y = \textstyle\prod_{i=1}^{k} y_i^{n_i} \in F(X) : \sum_i n_i \text{ is even} \,\},$$

and, similarly, that

$$(Ff)^{-1}(\{1\}) = \{\, y = \textstyle\prod_{i=1}^{k} y_i^{n_i} \in F(X) : \sum_i n_i \text{ is odd} \,\}.$$

Let $E(X) = (Ff)^{-1}(\{0\})$ and $O(X) = (Ff)^{-1}(\{1\})$, respectively. From the continuity of Ff it follows that both $E(X)$ and $O(X)$ are clopen subsets of $F(X)$. Clearly, $E(X)$ and $O(X)$ are homeomorphic: if x is any point of X then the translation $y \mapsto x \cdot y$ maps $E(X)$ onto $O(X)$, and vice versa.

We now use the clopenness of $E(X)$ and $O(X)$ to get fixed-point free extensions of fixed-point free functions.

Lemma 3.2. *Let $F(X)$ be the free topological group over X. If $f\colon X \to X$ is a fixed-point free involution on X then there exists a fixed-point free involution $g\colon F(X) \to F(X)$ such that $g \restriction X = f$.*

Proof. By Theorem 3.1 the function $f\colon X \to X \subseteq F(X)$ can be extended to a continuous homomorphism $Ff\colon F(X) \to F(X)$.

We claim that the neutral element e is the unique fixed point of Ff. So assume that for some $y \in F(X) \setminus \{e\}$ we have $Ff(y) = y$. First write y in its reduced word form $y = \prod_{i=1}^{k} y_i^{n_i}$. Then

$$y = \prod_{i=1}^{k} y_i^{n_i} = \prod_{i=1}^{k} f(y_i)^{n_i}. \qquad (*)$$

Since for all $1 \leq i < k$, $y_i \neq y_{i+1}$, and f is injective, we get $f(y_i) \neq f(y_{i+1})$ and so the right-hand side of $(*)$ is also written in its reduced word form.

But as $F(X)$ is freely generated by X and $y \neq e$, it follows that for all $i \leq k$ we have $f(y_i) = y_i$. Since f is fixed-point free, we arrived at the desired contradiction.

In particular we obtain that $Ff \restriction O(X)$ is fixed-point free. Now fix $p \in X$ and define the map $g\colon F(X) \to F(X)$ by

$$g(x) = \begin{cases} Ff(x) & x \in O(X), \\ Ff(x \cdot p) \cdot p^{-1} & x \in E(X). \end{cases}$$

Then $g \restriction O(X)$ is obviously fixed-point free. Since the translation $x \to x \cdot p$ is a homeomorphism from $O(X)$ onto $E(X)$, the map $g \restriction E(X)$ is fixed-point free as well. Since $E(X)$ and $O(X)$ are clopen subsets of $F(X)$ it is clear that g is continuous and hence an involution and as $X \subset O(X)$ it obviously extends f. $\square$

Example 3.3. There exists a strongly zero-dimensional topological group with a non-colorable fixed-point free involution.

Proof. As in the introduction, let $X = \{-1,0,1\}^{\omega_1}$ and let $f\colon X \to X$ be defined by $f(x) = -x$. In addition, let $Y = \{-1,0,1\}^{\omega_1} \setminus \{\underline{0}\}$. Then $g = f \restriction Y$ is a fixed-point free involution. It is clear that Y is pseudocompact and strongly zero-dimensional. This implies that $F(Y)$ is strongly zero-dimensional by Tkachenko [18]. It has, by Lemma 3.2, a fixed-point free involution h extending g. Now since Y is C-embedded in $F(Y)$, the closure of Y in $\beta F(Y)$ is $\beta Y = X$. Since $\beta h \restriction Y = g$ it follows that $\beta h \restriction \beta Y = \beta g$, i.e. βh has a fixed point since $\beta g = f$. $\square$

Notice that $F(Y)$ is not normal since it contains a closed copy of the non-normal space Y. One could try to start with a different Y which is normal and try to prove that $F(Y)$ is normal. But by this approach we run into serious problems. With the help of the inductive topology Tkachenko [17] proved that $F(X)$ is normal provided every finite product of X is normal and pseudocompact. The same condition appeared in Hart and van Mill [8] for the construction of a normal separable group which is not Lindelöf.

So a strongly zero-dimensional 'non-colorable' space with normal and pseudocompact finite powers would result in the desired group. However Theorem 2.1 implies that such a space does not exist.

4. Sigma Products

Another natural class of examples to consider are the so-called sigma products. Such a space is a topological group if all of its factors are topological groups, and it is normal under mild conditions. We will prove that if Σ is a sigma product of uncountably many compact spaces with countable tightness, then every fixed-point free continuous map $f\colon \Sigma \to \Sigma$ can be colored. This is achieved by an application of Theorem 2.1.

Theorem 4.1. *Every fixed-point free continuous map*

$f\colon \Sigma \to \Sigma$ on a Σ-product of compact spaces with countable tightness can be colored.

Proof. The Σ-product Σ is normal ([12]), and pseudocompactness follows from ω-boundedness (cf. [5, 3.10.17]). Since Σ^2 is homeomorphic to Σ we are done by Theorem 2.1. $\square$

5. The Hajnal-Juhász Group

In the previous section we showed that sigma products do not settle our question. They are in some vague sense 'extremal': any two elements of a sigma product eventually agree in high coordinate directions. Another construction of a topological group with in some sense the opposite 'extremal' behavior is due to Hajnal and Juhász [7]. They constructed under CH a $G \subseteq 2^{\omega_1}$ which is simultaneously a countably compact dense subgroup and an ω-HFD. We say that $X \subseteq \prod_{\alpha<\omega_1} X_\alpha$ is an ω-HFD if for every countably infinite subset $A \subseteq X$ there exists an index $\beta < \omega_1$ such that $\pi_\beta[A]$ is dense in $\prod_{\alpha\in\omega_1\setminus\beta} X_\alpha$, where

$$\pi_\beta\colon \prod_{\alpha<\omega_1} X_\alpha \to \prod_{\alpha\in\omega_1\setminus\beta} X_\alpha$$

is the standard projection. So the coordinates of points in an ω-HFD must differ very strongly in high coordinate directions for otherwise their 'tails' will not be dense.

As we observed, sigma products of suitable spaces are normal, but they are usually not separable. An ω-HFD subset of compact metrizable spaces is hereditarily separable and hereditarily normal. So the Hajnal-Juhász group G that we discussed above is not Lindelöf, being countably compact.

It was shown in [9] that if $G \subseteq {\mathbb{Z}_3}^{\omega_1}$ is a countably compact and dense ω-HFD subgroup then $G \setminus \{\underline{0}\}$ is a strongly zero-dimensional normal space of which the involution $x \mapsto -x$ cannot be colored. So this example is very close to a topological group, it just lacks one point. In this section we consider products of the form $\Pi = \prod_{\alpha\in\omega_1} G_\alpha$, where every G_α is a group of the form $\mathbb{Z}_n$, a countably compact and dense ω-HFD subgroup $G \subseteq \Pi$ and a product map $f = \prod_{\alpha\in\omega_1} f_\alpha$ on Π. We prove that if f has a fixed point then so does $f \restriction G$. As a consequence, if $f \restriction G$ is fixed-point free then it can be colored since $f = \beta(f \restriction G)$ (see below).

This result is not as satisfying as our results for sigma products. For we can 'only' prove that fixed points exist in products of $\mathbb{Z}_n$'s and for a very special class of functions, namely product maps. We do not know whether our results hold for arbitrary (continuous) functions, or for products of finite Abelian groups. So it is still possible that a version of the Hajnal-Juhász group will solve our problem, but our results are not very promising.

Let $X = \prod_{\alpha<\omega_1} X_\alpha$, where X_α is a compact metrizable topological group for every $\alpha < \omega_1$. In addition, let G be a countably compact and dense ω-HFD in X. We wish to present a few obvious properties of G. First observe that for every every $\alpha < \omega_1$, the projection of G into $\prod_{\beta<\alpha} X_\beta$ is surjective. This can be seen as follows. The image of G under the projection is dense since G is. It is also pseudocompact since G is countably compact. So the conclusion is that it is a dense pseudocompact subspace of the compact metrizable space $\prod_{\beta<\alpha} X_\alpha$, hence it has to be equal to the full product. Second, since G is countably compact and dense it obviously intersects every non-empty G_δ-subset of X. This implies by a result of [3] that $\beta G = X$.

Theorem 5.1. *For every $\alpha < \omega_1$ let G_α be $\mathbb{Z}_{n(\alpha)}$ for a certain integer $n(\alpha)$, and let $f_\alpha \colon G_\alpha \to G_\alpha$. In addition, let $G \subseteq Z = \prod_{\alpha<\omega_1} G_\alpha$ be a countably compact and dense ω-HFD subgroup. If $f = \prod_{\alpha<\omega_1} f_\alpha$ and if $f \restriction G$ is fixed-point free and $f[G] \subseteq G$ then f is fixed-point free. As a consequence, $f \restriction G$ can be colored.*

In the remainder of this section, we shall adopt the notation introduced in this theorem.

We introduce some additional notation. If $f\colon X \to X$ we let $\mathrm{Fix}(f)$ denote the fixed point set of f, i.e. the set $\{\, x \in X : f(x) = x \,\}$. For $x \in G$ put

$$\Lambda_x = \{\, \alpha < \omega_1 : x(\alpha) \in \mathrm{Fix}(f_\alpha) \,\}.$$

The larger Λ_x, the better x approximates a fixed point of f. So the coordinate directions in Λ_x correspond to the 'good' coordinates of x.

The proof of Theorem 5.1 is in two parts, one of them being trivial. If for some $\alpha < \omega_1$ the map f_α is fixed-point free, then f is fixed-point free as well, and so there is nothing to prove. So we assume without loss of generality that $\mathrm{Fix}(f_\alpha) \neq \emptyset$ for every α. The remaining part of the proof shows that $f \restriction G$ has a fixed point.

The following lemma is the essential step in our argumentation. It shows that for a given point in G we can sometimes improve its 'good' coordinates.

Lemma 5.2. *Fix $x \in G$. Let p be a prime number and $h\colon \mathbb{Z}_p \to \mathbb{Z}_p$ a function with at least one fixed point. Put*

$$B = \{\, \alpha < \omega_1 : n(\alpha) = p \text{ and } f_\alpha = h \,\}.$$

Then there exists $z \in G$ such that

$$\begin{cases} z(\alpha) \in \mathrm{Fix}(f_\alpha) & (\alpha \in B), \\ z(\alpha) = x(\alpha) & (\alpha \in \Lambda_x). \end{cases}$$

As a consequence, $\Lambda_x \cup B \subseteq \Lambda_z$.

Proof. For $d \in \mathbb{Z}_p$ and $z \in G$ put

$$B_z^d = \{\, \alpha \in B : z(\alpha) = d \,\}.$$

Now fix an arbitrary $\hat{d} \in \mathbb{Z}_p$. We claim that there exists $y \in G$ such that

$$y \restriction B_x^d$$

is constant for every $d \in \mathbb{Z}_p$ while moreover

$$\Lambda_x \cup B_x^{\hat{d}} \subseteq \Lambda_y.$$

If $\hat{d} \in \mathrm{Fix}(h)$ then $B_x^{\hat{d}} \subseteq \Lambda_x$ and so $y = x$ will do. So assume that $\hat{d} \notin \mathrm{Fix}(h)$.

We let $y_0 = x - f(x) \in G$. Then $y_0 \restriction B_x^d$ is constant for every $d \in \mathbb{Z}_p$ since f is a product map. Also, $y_0(\alpha) = 0$ for all $\alpha \in \Lambda_x$ and $y_0(\alpha) = \hat{d} - h(\hat{d}) \neq 0$ for all $\alpha \in B_x^{\hat{d}}$. As $\hat{d} - h(\hat{d}) \in \mathbb{Z}_p \setminus \{0\}$ and p is prime, there exists $r \in \mathbb{Z}_p$ for which $r(\hat{d} - h(\hat{d})) = 1 \pmod{p}$. Now fix $q \in \mathrm{Fix}(h)$. Write $y_1 = (q - \hat{d}) \cdot r \cdot y_0$ (the multiplication is of course pointwise). As G is a group, $y_1 \in G$ and it is clear that $y_1 \restriction B_x^d$ is constant for every d. Define $y = y_1 + x \in G$. We claim that y is as required. For if $\alpha \in \Lambda_x$ then $y(\alpha) = x(\alpha) \in \mathrm{Fix}(f_\alpha)$, and if $\alpha \in B_x^{\hat{d}}$ then $y(\alpha) = (q - \hat{d}) \cdot 1 + \hat{d} = q \in \mathrm{Fix}(h) = \mathrm{Fix}(f_\alpha)$. That $y \restriction B_x^d$ is constant for every d is trivial, since both y_1 and x are.

Put $z_{-1} = x$. By induction on $d \in \mathbb{Z}_p$ we will construct $z_d \in G$. Assume that z_{d-1} has been constructed such that its restriction to $B_x^{d'}$ is constant for every $d' \in \mathbb{Z}_p$ while moreover

$$\Lambda_x \cup \bigcup_{e \leq d-1} B_x^e \subseteq \Lambda_{z_{d-1}}.$$

Now we construct z_d. Put $\hat{d} = z_{d-1}(\alpha)$, where α is an arbitrary element of B_x^d. If $\hat{d} \in \mathrm{Fix(h)}$, put $z_d = z_{d-1}$, otherwise apply the above with $x = z_{d-1}$ and $\hat{d}$, and let z_d be the y that we get out of our process. Observe that if $\alpha \in B_x^d$ then $z_{d-1}(\alpha) = \hat{d}$ and so $B_x^d \subseteq B_{z_{d-1}}^{\hat{d}}$. From the construction it therefore follows that $z_d(\alpha) \in \mathrm{Fix}(h) = \mathrm{Fix}(f_\alpha)$, as required.

Now $z = z_{p-1}$ clearly satisfies the requirements. $\square$

Proof of Theorem 5.1. Let $C = \{\alpha < \omega_1 : n(\alpha)$ is a composite number$\}$. We distinguish two cases.

Suppose that C is unbounded. Then there exists a composite number $n = r \cdot s$ (with $1 < r, s < n$) such that $n(\alpha) = n$ for cofinally many α. In $\mathbb{Z}_n$ consider the set

$$S = \{ i \cdot s : i \in \mathbb{N} \}.$$

Then $|S| = r < n$ so there exists $t \in \mathbb{Z}_n \setminus S$.

Our aim is to contradict ω-HFDness. Pick an arbitrary strictly increasing sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ in C and put $C_\omega = \{ \lambda_i : i \in \mathbb{N} \}$. Fix $y_n \in \prod_{\alpha \leq \lambda_n} \mathbb{Z}_{n(\alpha)}$ such that $y_n(\alpha) = 0$ for all $\alpha < \lambda_n$ and $y_n(\lambda_n) = 1$. Since G projects onto each countable subproduct of X, there exists $z_n \in G$ such that $z_n(\alpha) = y_n(\alpha)$ for every $\alpha \leq \lambda_n$. Now define $A = \{ s \cdot z_n : n \in \omega \}$. Observe that if $n < m$ then

$$s \cdot z_n(\lambda_n) = s \text{ and } s \cdot z_m(\lambda_n) = 0,$$

so that A is a countably infinite subset of G. So there exists $\nu < \omega_1$ such that beyond ν the set A is dense. Let $\mu \geq \nu$ be such that $n(\mu) = n$. Then

$$\{t\} \times \prod_{\alpha > \mu} \mathbb{Z}_{n(\alpha)},$$

is a nonempty open set in $\prod_{\alpha \geq \mu} Z_{n(\alpha)}$ which misses the image of A under the projection. So we arrived at the desired contradiction.

So C is bounded and let $\gamma < \omega_1$ be an upperbound. Since all projections from G onto initial products are surjective, there exists a $g \in G$ such that $g(\lambda) \in \mathrm{Fix}(f_\lambda)$ for all $\lambda \in C$. If $C = \emptyset$ let $g = \underline{0}$.

Since $|\mathbb{Z}_n| = n < \omega$ there are only finitely many maps $f \colon \mathbb{Z}_n \to \mathbb{Z}_n$ and the total number of such maps for all natural n is countable. Enumerate these maps in a 1-1 fashion as $\langle f_i \rangle_{i \geq 1}$.

For every i let

$$B_i = \{ \alpha < \omega_1 : f_\alpha = f_i \}.$$

Put $r_0 = g$. By Lemma 5.2 there exists $r_1 \in G$ such that $r_1(\alpha) \in \operatorname{Fix}(f_\alpha)$ for all $\alpha \in B_1$. By induction on i we obtain a sequence $\langle r_i \rangle_i$ in G such that r_i and r_{i-1} agree on $B_1 \cup \cdots \cup B_{i-1}$, while moreover $r_i(\alpha) \in \operatorname{Fix}(f_\alpha)$ for every $\alpha \in B_i$.

It is clear that the sequence $\langle r_i \rangle_i$ is convergent to, say, $r \in Z$. Since G is countably compact, $r \in G$. But since G is an ω-HFD, it cannot contain non-trivial convergent sequences. So the sequence $\langle r_i \rangle_i$ is eventually constant. At some finite stage of our induction there is nothing left to be done, and we obtained the desired fixed point. □

Question 5.3. *Does Theorem 5.1 hold for arbitrary products of finite Abelian groups?*

Recall that the interesting cases are for Hajnal-Juhász groups whose squares are not normal, if any, since then the answer does not follow from Theorem 2.1.

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