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Topology and its Applications 76 (1997) 1–7

TOPOLOGY AND ITS APPLICATIONS

# Tight points and countable fan-tightness

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Received 14 September 1995

**Abstract**

The countable spaces whose product with the sequential fan $S_\mathfrak{c}$ have countable tightness are characterized. As a consequence, it is shown that if $X \times S_\mathfrak{c}$ has countable tightness then $X$ has countable fan-tightness.

*Keywords:* Countable fan-tightness; Tight point; Sequential fan

*AMS classification:* 54A25

## 1. Introduction

The aim of this paper is to characterize the countable spaces whose product with the sequential fan $S_\mathfrak{c}$ have countable tightness. Our results answer questions posed by Arhangel'skiĭ and Bella [2].

Henceforth all spaces are assumed at least $T_1$. $\omega$ denotes the set (or the discrete space) of all the integers as well as the first infinite ordinal. $\mathfrak{c}$ denotes $2^\omega$.

$S_\mathfrak{c} = \bigcup_{\alpha<\mathfrak{c}}\{z_n^\alpha\colon n < \omega\}$. All the $z_n^\alpha$'s are distinct and isolated in $S_\mathfrak{c}$. If $f \in \omega^\mathfrak{c}$ then

$$V(f) = \{0\} \cup \bigcup_{\alpha<\mathfrak{c}}\{z_n^\alpha\colon n \geqslant f(\alpha)\}$$

is a basic open neighborhood of 0 in $S_\mathfrak{c}$. Observe that the sequential fan $S_\mathfrak{c}$ is nothing but the space obtained by identifying the limit points of the topological sum of continuum many convergent sequences.

If $F \subseteq \mathfrak{c}$ then we put $S_F = \{0\} \cup \bigcup_{\alpha\in F}\{z_n^\alpha\colon n < \omega\}$. This notation will be used without explicit reference later.

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[1] The author is indebted to the University of Catania for generous hospitality and support.


PII S0166-8641(96)00168-X

A space $X$ has *countable tightness* if whenever $A \subseteq X$ and $x \in \overline{A}$, there exists a countable set $B \subseteq A$ such that $x \in \overline{B}$. If $X$ and $Y$ are spaces with $X \subseteq Y$ then $X$ has countable tightness in $Y$ if whenever $A \subseteq Y$ and $x \in \overline{A} \cap X$, there exists a countable set $B \subseteq A$ such that $x \in \overline{B}$.

A space $X$ has *countable fan-tightness* if for any countable family $\{A_n\colon n \in \omega\}$ of subsets of $X$ satisfying $x \in \bigcap_{n\in\omega} \overline{A_n}$ it is possible to select finite sets $K_n \subseteq A_n$ in such a way that $x \in \overline{\bigcup_{n\in\omega} K_n}$.

Let $X$ be a space and let $p \in X$. We say that a family of subsets $\mathcal{E}$ of $X$ *clusters* at $p$ if for every neighborhood $U$ of $p$ there exists $E \in \mathcal{E}$ such that $|E \cap U| \geqslant \omega$. We say that $p$ is a *tight* point of $X$ if for every family $\mathcal{E}$ of subsets of $X$ that clusters at $p$ there exists a *countable* subfamily $\mathcal{F}$ of $\mathcal{E}$ that clusters at $p$. Observe that if $p$ is an isolated point of $X$ then $p$ is tight for trivial reasons.

## 2. Tight points

Let $X$ be a space. If $X$ has countable fan-tightness then $X$ has countable tightness. But not conversely. The space $S_\mathfrak{c}$ has countable tightness, but does not have countable fan-tightness.

**Proposition 2.1.** *Let $X$ be a space. If every point of $X$ is tight then $X$ has countable fan-tightness.*

**Proof.** Fix $p \in X$ and subsets $A_n$ of $X$ for $n < \omega$ such that $p \in \bigcap_{n<\omega} \overline{A}_n$. We assume without loss of generality that for every $n$, $A_n$ is infinite and that $p \notin A_n$. Let $\mathcal{S}$ be the collection of all subsets $S = \{x_n\colon n < \omega\}$ of $X$ such that for every $n$, $x_n \in A_n$. We claim that $\mathcal{S}$ clusters at $p$. To see this, let $U$ be an arbitrary neighborhood of $p$. Since $U \cap A_n$ is infinite for every $n$, we may pick points $x_n$ in $X$ such that for every $n$,

$$x_n \in (U \cap A_n) \setminus \{x_0, \ldots, x_{n-1}\}.$$

Then $S = \{x_n\colon n < \omega\}$ belongs to $\mathcal{S}$, is infinite and is contained in $U$.

Since $p$ is a tight point of $X$, there are $S_k = \{x_n^k\colon n < \omega\} \in \mathcal{S}$ for $k < \omega$ such that $\{S_k\colon k < \omega\}$ clusters at $p$. For every $n$, define

$$K_n = \{x_n^0, x_n^1, \ldots, x_n^n\}$$

and $K = \bigcup_{n<\omega} K_n$. Let $U$ be an arbitrary neighborhood of $p$. Then for some $k$ we have that $U \cap S_k$ is infinite. Since $S_k \setminus K$ is finite we conclude that $U$ intersects $K$. This shows that $p \in \overline{K}$, which is as required. $\square$

Let $X$ be a space. If every point of $X$ is tight, then $X$ has countable fan-tightness as was shown in Proposition 2.1. But the converse is not true (at least, consistently). If $p \in \beta\omega \setminus \omega$ is a P-point then $\omega \cup \{p\}$ has countable fan-tightness (see [2, Proposition 2]). But $p$ is not a tight point of $\omega \cup \{p\}$ by our next result.

**Proposition 2.2.** *Let $p \in \beta\omega \setminus \omega$. Then $p$ is not a tight point of $\omega \cup \{p\}$.*

**Proof.** Suppose first that $p$ is a P-point and put $\mathcal{S} = \{\omega \setminus P\colon\ P \in p\}$. We claim that $\mathcal{S}$ clusters at $p$. To see this, let $P \in p$ and split $P$ into two infinite sets, say $P_0$ and $P_1$. Without loss of generality, assume that $P_0 \in p$. Then $\omega \setminus P_0 \in \mathcal{S}$ and $|(\omega \setminus P_0) \cap P| = \omega$.

Now suppose that $\mathcal{B}$ is a countable subfamily of $\mathcal{S}$ that clusters at $p$. Then $\{\omega \setminus B\colon\ B \in \mathcal{B}\}$ is a countable subfamily of $p$. Since $p$ is a P-point, there exists $P \in p$ such that for all $B \in \mathcal{B}$ we have $|P \setminus (\omega \setminus B)| < \omega$. But then $P \cap B$ is finite for every $B \in \mathcal{B}$, which is a contradiction.

If $p$ is not a P-point, then there is a partition $\{A_n\colon\ n < \omega\}$ of $\omega$ into infinite sets such that for every $n$, $A_n \notin p$, and if $P \in p$ then $|P \cap A_n| = \omega$ for infinitely many $n$. Now put

$$\mathcal{S} = \left\{ \bigcup_{n<\omega} F_n\colon\ F_n \subsetneq A_n \text{ finite},\ n < \omega \right\}.$$

It is clear that $\mathcal{S}$ clusters at $p$. Assume that some countable subfamily $\mathcal{T}$ of $\mathcal{S}$ clusters at $p$. Let $\mathcal{T} = \{T_n\colon\ n < \omega\}$ and for every $n < \omega$, let $T_n = \bigcup_{m<\omega} F_m^n$, with $F_m^n$ a finite subset of $A_m$ for every $m$. Put

$$K = F_0^0 \cup \left(F_1^0 \cup F_1^1\right) \cup \left(F_2^0 \cup F_2^1 \cup F_2^2\right) \cup \cdots.$$

Then $K \cap A_n$ is finite for every $n$, and as a consequence, $U = \omega \setminus K \in p$. But $|U \cap T_n| < \omega$ for every $n$, which is a contradiction. $\square$

We now show that the points of a regular countably compact spaces of countable tightness are tight.

**Theorem 2.3.** *Let $X$ be a subspace of a regular countably compact space $Y$. If $X$ has countable tightness in $Y$ then every point of $X$ is tight.*

**Proof.** Let $\mathcal{E}$ be a family of subsets of $X$ that clusters at a given point $p \in X$. For every neighborhood $U$ of $p$ in $Y$ first pick a neighborhood $V(U)$ of $p$ in $Y$ such that $\overline{V(U)} \subseteq U$ (here we use the fact that $Y$ is regular). Next pick an element $E(U) \in \mathcal{E}$ such that $F(U) = E(U) \cap V(U)$ is infinite. Passing to a subset of $F(U)$ if necessary, we may assume that $F(U)$ is countably infinite. Since $Y$ is countably compact, we may pick for every $U$ an accumulation point $p(U)$ of $F(U)$ in $Y$. Put $A = \{p(U)\colon\ U$ is a neighborhood of $p$ in $Y\}$. Then $p \in \overline{A}$ and since $X$ has countable tightness in $Y$, there is a countable family of neighborhoods $\mathcal{V}$ of $p$ such that

$$p \in \overline{\{p(V)\colon\ V \in \mathcal{V}\}}.$$

We claim that the family $\{E(V)\colon\ V \in \mathcal{V}\}$ clusters at $p$. To this end, let $U$ be a neighborhood of $p$ in $X$ and let $U'$ in $Y$ be open such that $U' \cap X = U$. There exists $V \in \mathcal{V}$ such that $p(V) \in U'$. Since $U'$ is a neighborhood of $p(V)$ and $p(V)$ is a cluster point of $F(V)$ we have that $U \cap F(V) = U' \cap F(V)$ is infinite. Since $F(V) \subseteq E(V)$ this proves that $U \cap E(V)$ is infinite, which is as required. $\square$

The following result is partly due to Arhangel'skiĭ and Bella [2].

**Corollary 2.4.** *Let $X$ be a regular countably compact space. Then $X$ has countable tightness if and only if $X$ has countable fan-tightness if and only if every point of $X$ is tight.*

**Proof.** If $X$ has countable tightness, then every point of $X$ is tight by Theorem 2.3 which in turn implies that $X$ has countable fan-tightness by Proposition 2.1. $\square$

The following result of which we present a new proof (for another proof, see Arhangel'skiĭ and Bella [2]) is due to Malykhin [3].

**Corollary 2.5.** *If $p \in \beta\omega \setminus \omega$ then $\omega \cup \{p\}$ cannot be embedded in a regular countably compact space with countable tightness.*

**Proof.** Striving for a contradiction, assume that $X$ is a regular countably compact space with countable tightness which contains $\omega \cup \{p\}$ as a subspace. Then every point of $X$ is tight by Corollary 2.5 and so $p$ is a tight point of $\omega \cup \{p\}$ which contradicts Proposition 2.2. $\square$

This result can be improved, as we will show in the remaining part of this section.

**Proposition 2.6.** *Let $X$ be a regular space with countable fan-tightness. If all closed separable subspaces of $X$ are Lindelöf with points $G_\delta$ then for any $A \subseteq X$ and any $p \in \overline{A} \setminus A$ there exists a countable set $B \subseteq A$ such that $p$ is the only accumulation point of $B$.*

**Proof.** Let $A \subseteq X$ and $p \in \overline{A} \setminus A$. Since $X$ has countable tightness, there exists a countable set $C \subseteq A$ such that $p \in \overline{C}$. For any $x \in \overline{C} \setminus \{p\}$ fix an open set $U_x$ satisfying $x \in U_x$ and $p \notin \overline{U_x}$ and let $\mathcal{U}$ be the family so obtained. Since $\overline{C}$ has the Lindelöf property and $p$ is a $G_\delta$ point in $\overline{C}$, it follows that even the subspace $\overline{C} \setminus \{p\}$ has the Lindelöf property. So $\mathcal{U}$ has an open countable refinement $\{V_n\colon\ n < \omega\}$ such that the family $\{V_n \cap C\colon\ n < \omega\}$ is locally finite in $\overline{C} \setminus \{p\}$. For any $n$ we have $p \in \overline{\bigcup\{V_m \cap C\colon\ m \geqslant n\}}$ and therefore we can select a finite set $K_n \subseteq \bigcup\{V_m \cap C\colon\ m \geqslant n\}$ such that $p \in \overline{\bigcup\{K_n\colon\ n < \omega\}}$. Putting $B = \bigcup\{K_n\colon\ n < \omega\}$ we get what we want. $\square$

Recall that, given a Tychonoff space $X$, a point $p \in \beta X \setminus X$ is said to be *far* if it is not in the closure of any closed discrete subset of $X$. If $p \in \beta X \setminus X$ is not far then it is called *near*.

**Corollary 2.7.** *Let $X$ be a Tychonoff space whose closed separable subspaces have the Lindelöf property and $p \in \beta X \setminus X$. If $X \cup \{p\}$ has countable fan-tightness then $p$ is near.*

Malykhin's result quoted above can now be generalized as follows:

**Theorem 2.8.** *If $X$ is a metrizable space and $p \in \beta X \setminus X$ then $X \cup \{p\}$ cannot be embedded into a countably compact regular space with countable tightness.*

**Proof.** Assuming the contrary, the space $X \cup \{p\}$ must have countable fan-tightness. Consequently, as stated in Corollary 2.7, there exists a countable closed discrete set $B \subseteq X$ for which $p \in \overline{B}$. But then, also $B \cup \{p\}$ can be embedded into a countably compact regular space with countable tightness—in contrast with Corollary 2.5. □

Recall that a space $X$ is said to be *bisequential* provided that for every filter $\xi$ and every point $p$ in the aderence of $\xi$ there exists a filter $\nu$ with a countable base which converges to $p$ and is syncronous with $\xi$. *Syncronous* means that for each $A \in \xi$ and each $B \in \nu$ the intersection $A \cap B$ is not empty.

Arhangel'skiĭ (see [1]) has shown that the product of a bisequential space with any space of countable tightness has still countable tightness. By Theorem 3.1 below, it then follows that each point of a bisequential space is tight. Here is a direct and easy proof of this assertion.

**Proposition 2.9.** *Every point of a bisequential space is tight.*

**Proof.** Let $X$ be a bisequential space and $\mathcal{E}$ a collection of subsets of $X$ which clusters at $p$. Define $\xi = \{\bigcup_{E \in \mathcal{E}} E \setminus F\colon\ F \subseteq E \text{ finite}\}$. It is clear that $\xi$ is a prefilter and $p$ is in the aderence of $\xi$. Let $\mathcal{U}$ be a countable base of a filter $\nu$ converging to $p$ and which is syncronous with $\xi$. Since $\nu$ and $\xi$ are syncronous, for every $U \in \mathcal{U}$ there must exist some $E(U) \in \mathcal{E}$ such that $|U \cap E(U)| \geqslant \omega$. The fact that $\nu$ converges to $p$ implies that every neighbourhood of $p$ contains some $U \in \mathcal{U}$ and consequently the subcollection $\{E(U)\colon\ U \in \mathcal{U}\}$ clusters at $p$. □

## 3. The main result

We now present our main result.

**Theorem 3.1.** *Let $X$ be a countable space. Then $X \times S_{\mathfrak{c}}$ has countable tightness if and only if every point of $X$ is tight.*

**Proof.** Assume that $X \times S_{\mathfrak{c}}$ has countable tightness and suppose that the family $\mathcal{E}$ clusters at $p \in X$. We may assume without loss of generality that every element $E \in \mathcal{E}$ is countably infinite and does not contain $p$. List $\mathcal{E}$ as $\{E_\alpha\colon\ \alpha < \mathfrak{c}\}$ (repetitions are permitted) and $E_\alpha$ as $\{e_n^\alpha\colon\ n < \omega\}$ (repetitions are NOT permitted). Let

$$A = \bigcup_{\alpha < \mathfrak{c}} \{\langle e_n^\alpha, z_n^\alpha \rangle\colon\ n < \omega\}.$$

We claim that $\langle p,0\rangle \in \overline{A}$. To see this, let $U$ and $V(f)$ be arbitrary neighborhoods of $p$ in $X$ and $0 \in S_{\mathfrak{c}}$, respectively. There exists $\alpha < \mathfrak{c}$ such that $|U \cap E_\alpha| = \omega$. Pick $n < \omega$ so large that $e_n^\alpha \in U \cap E_\alpha$ and $n \geqslant f(\alpha)$. Then

$$\langle e_n^\alpha, z_n^\alpha\rangle \in \big(U \times V(f)\big) \cap A,$$

as required.

Since $X \times S_{\mathfrak{c}}$ has countable tightness, there is a countable subset $F$ of $\mathfrak{c}$ such that if

$$B = \bigcup_{\alpha\in F} \big\{\langle e_n^\alpha, z_n^\alpha\rangle \colon\ n < \omega\big\}$$

then $\langle p,0\rangle \in \overline{B}$. Now put $\mathcal{F} = \{E_\alpha\colon\ \alpha \in F\}$. We claim that $\mathcal{F}$ clusters at $p$. To see this, let $U$ be an arbitrary neighborhood of $p$ in $X$. Striving for a contradiction, assume that for every $\alpha \in F$ we have $|U \cap E_\alpha| < \omega$. For every $\alpha \in F$ pick $n_\alpha < \omega$ such that if $n \geqslant n_\alpha$ then $e_n^\alpha \notin U$. Define $f \in \omega^{\mathfrak{c}}$ as follows: $f(\alpha) = n_\alpha$ if $\alpha \in F$ and $f(\alpha) = 0$ otherwise. Pick $\alpha \in F$ and $n < \omega$ such that $\langle e_n^\alpha, z_n^\alpha\rangle \in U \times V(f)$. Then $e_n^\alpha \in U \cap E_\alpha$ and so $n < n_\alpha$. However, $z_n^\alpha \in V(f)$ which implies $n \geqslant f(\alpha) = n_\alpha$. This is a contradiction.

Now assume that $p$ is a tight point of $X$. We will prove that the tightness of $X \times S_{\mathfrak{c}}$ at $\langle p,0\rangle$ is countable. Since $X$ is countable and every point of $S_{\mathfrak{c}}$ other than 0 is isolated, this clearly suffices to prove that $X \times S_{\mathfrak{c}}$ has countable tightness.

Let $A \subseteq (X \times S_{\mathfrak{c}}) \setminus \{\langle p,0\rangle\}$ be such that $\langle p,0\rangle \in \overline{A}$. Put

$$B = \big\{x \in X\colon\ \langle x,0\rangle \in \overline{(\{x\} \times S_{\mathfrak{c}}) \cap A}\big\}.$$

Assume first that $p \in \overline{B}$. Since $S_{\mathfrak{c}}$ has countable tightness, for every $x \in B$ there exists a countable $S_x \subseteq (\{x\} \times S_{\mathfrak{c}}) \cap A$ with $\langle x,0\rangle \in \overline{S_x}$. Then $\langle p,0\rangle \in \overline{\bigcup_{x\in B} S_x}$ and so we are done.

So we may assume without loss of generality that for every $x \in X$ we have

$$\langle x,0\rangle \notin \overline{(\{x\} \times S_{\mathfrak{c}}) \cap A}.$$

Now let $U$ be an arbitrary neighborhood of $p$ in $X$. We claim that there exists $\alpha(U) < \mathfrak{c}$ such that

$$\big\{n < \omega\colon\ (\exists x \in U)\big(\langle x, z_n^{\alpha(U)}\rangle \in A\big)\big\}$$

is infinite. If not, then for every $\alpha < \mathfrak{c}$ pick $n_\alpha < \omega$ such that if $n \geqslant n_\alpha$ and $x \in U$ then $\langle x, z_n^\alpha\rangle \notin A$. Put $f(\alpha) = n_\alpha$ for every $\alpha$. Then $(U \times V(f)) \cap A = \emptyset$, which is a contradiction. This proves the claim.

Pick an arbitrary $x \in U$ and observe that

$$\langle x,0\rangle \notin \overline{(\{x\} \times S_{\mathfrak{c}}) \cap A}.$$

Since $\langle x, z_n^{\alpha(U)}\rangle \to \langle x,0\rangle$ $(n \to \infty)$, this implies that there are only finitely many $n$ for which $\langle x, z_n^{\alpha(U)}\rangle \in A$. So by the above we may pick for every $n < \omega$ an element $x_n^{\alpha(U)} \in U$ and an integer $m(n)$ such that

(i) if $n \neq m$ then $x_n^{\alpha(U)} \neq x_m^{\alpha(U)}$;

(ii) $\langle x_n^{\alpha(U)}, z_{m(n)}^{\alpha(U)}\rangle \in A$ for every $n$;

(iii) $m(n) \to \infty$ $(n \to \infty)$.

Put $E(U) = \{x_n^{\alpha(U)}\colon n < \omega\}$.

The family $\{E(U)\colon U$ is a neighborhood of $p\}$ clusters at $p$. There consequently is a countable family $\mathcal{U}$ of neighborhoods of $p$ such that $\mathcal{F} = \{E(U)\colon U \in \mathcal{U}\}$ clusters at $p$. Put $F = \{\alpha(U)\colon U \in \mathcal{U}\}$ and

$$B = (X \times S_F) \cap A,$$

respectively. Then $B$ is countable and we claim that $\langle p, 0\rangle \in \overline{B}$. To see this, let $V$ and $V(f)$ be arbitrary neighborhoods of $p$ in $X$ and 0 in $S_{\mathfrak{c}}$, respectively. There exists $U \in \mathcal{U}$ such that $|E(U) \cap V| = \omega$. By (iii) we may pick $n < \omega$ so large that

$$x_n^{\alpha(U)} \in E(U) \cap V \quad \text{and} \quad m(n) \geqslant f\big(\alpha(U)\big).$$

We conclude that

$$\big\langle x_n^{\alpha(U)}, z_{m(n)}^{\alpha(U)}\big\rangle \in A \cap \big(V \times V(f)\big),$$

which is as required. $\square$

The following result answers a question in [2] in the affirmative.

**Corollary 3.2.** *Let $X$ be a space. If $X \times S_{\mathfrak{c}}$ has countable tightness then $X$ has countable fan-tightness.*

**Proof.** Suppose that $X \times S_{\mathfrak{c}}$ has countable tightness. Then $X$ has countable tightness, and so it suffices to prove that every countable subspace of $X$ has countable fan-tightness. So without loss of generality, assume that $X$ is countable. By Theorem 3.1, every point of $X$ is tight and so $X$ has countable fan-tightness by Proposition 2.1. $\square$

As mentioned in Section 2, in [2] it is shown that every countably compact regular space with countable tightness has countable fan-tightness. Corollary 3.2 can be used to present yet another proof of this fact. Indeed, it sufficies to take into account the well-known fact that the product of a countably compact regular space of countable tightness with a sequential space has still countable tightness.

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