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TWO POINT SET EXTENSIONS— A COUNTEREXAMPLE

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Abstract. We show that there exist Cantor sets in the circle that are not extendable to sets that meet every line in the plane in exactly two points. This result solves a problem that was formulated by R. D. Mauldin.

A planar set is called a *two point set* if every line intersects the set in exactly two points and a *partial two point set* if every line intersects the set in at most two points. Circles and their subsets are obvious examples of partial two point sets. In [1] and [3], Problem 1070, Dan Mauldin asks the question whether every compact zero-dimensional partial two point set can be extended to a two point set. We show that the answer is no. Let S^1 stand for the unit circle in the plane centered at the origin O.

Proposition. *There exists a Cantor set in S^1 that is not contained in a two point set.*

Proof. Let λ be the linear Lebesgue measure on smooth planar curves (lines and circles in our case). Select a dense open subset U of S^1 such that $\lambda(U) \leq 1$ and $C = S^1 \setminus U$ is a Cantor set. Let x be a point in the plane with norm $|x| \geq 2$. Consider as in Figure 1 the two tangent lines to the circle through x. The tangent points P and Q divide S^1 into two open arcs A and B. The open line segment L is perpendicular to the line through x and O. Since $|x| \geq 2$ we have $\lambda(L) \geq 2/\sqrt{3} > 1$. Let p_A and p_B be the radial projections with respect to x of A respectively B onto L. Note that both projections are contractions which implies that the image of each interval in A or B is an interval in L of shorter length. This means that

$$\begin{aligned}\lambda(p_A(U \cap A) \cup p_B(U \cap B)) &\leq \lambda(p_A(U \cap A)) + \lambda(p_B(U \cap B)) \\ &\leq \lambda(U \cap A) + \lambda(U \cap B) = \lambda(U) \leq 1 < \lambda(L).\end{aligned}$$

If we pick a y in $L \setminus (p_A(U \cap A) \cup p_B(U \cap B))$ then the line through x and y intersects C in two points.

Let ℓ be a line in the plane with distance at least 2 towards the origin. If C is contained in a two point set D then $D \cap \ell$ consists of two points. Pick an $x \in D \cap \ell$ and note that there is a line through x that intersects C in two points and hence it intersects D in three points. □

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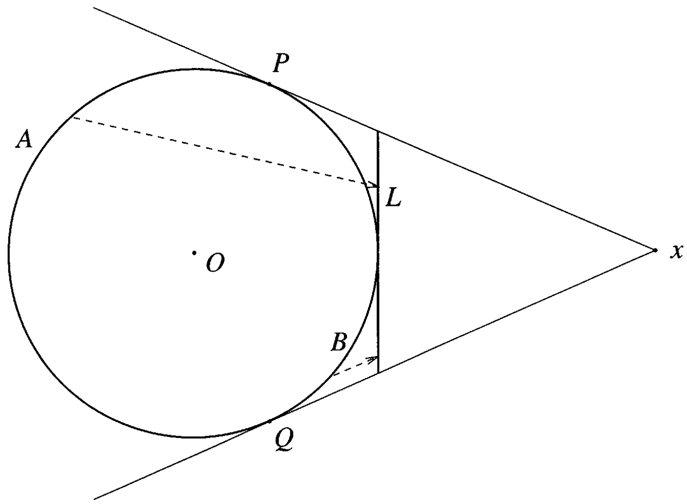

FIGURE 1

Remark. Observe that if $\lambda(U)$ approaches 0 then we can choose x closer to the circle. In fact, we can show that if C is a set in S^1 with $\lambda(S^1 \setminus C) \leq 1$ then each point whose distance towards S^1 is at least $\lambda(S^1 \setminus C)$ lies on a line that meets C in two points. (For points x inside the circle we apply a similar measure argument to the antipodal map $p_x : S^1 \to S^1$ with respect to x, which has the property $\lambda(p_x(U)) \leq \frac{1+|x|}{1-|x|}\lambda(U)$.)

The referee informed us that our result also follows from a theorem that was announced by Dan Mauldin at the 1995 BEST conference in Boise, Idaho. Mauldin's theorem [2], which was obtained independently, is more general than our proposition.

References

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