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Groups without convergent sequences

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Abstract

We investigate the question: which compact abelian groups have a dense (pseudocompact) subgroup without convergent sequences?

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1. Introduction

In recent years there has been a lot of interest in topological groups without convergent sequences that are pseudocompact or even countably compact. Sirota [8] constructed the first pseudocompact example in ZFC, and Hajnal and Juhász [3] constructed (under CH) the first countably compact example. For more recent developments see [2,5,6,9–11]. The aim of this paper is to investigate the following general question: which compact abelian groups have a dense (pseudocompact) subgroup without convergent sequences? We generalize some results in the literature – in particular results obtained for Boolean groups are extended to torsion groups. We also give an example of a compact metrizable abelian group no power of which has a dense subgroup without convergent sequences.

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2. Preliminaries

We use the standard representation for ordinals and cardinals: an ordinal is the set of all smaller ordinals and a cardinal is an ordinal that cannot be imbedded in any smaller ordinal.

If κ is an infinite cardinal then a subset of a topological space is called a $\mathcal{G}_\kappa$-set if it is the intersection of at most κ open sets. A subset X of Y is called κ-dense if every nonempty $\mathcal{G}_\kappa$-set of Y intersects X. The following result is useful: a subgroup of a compact group is dense and pseudocompact if and only if it is ω-dense (Comfort and Ross [1]).

If λ is a regular cardinal then $(x_\alpha)_{\alpha\in\lambda}$ is called a λ-sequence. A λ-sequence in a space is called (nontrivially) convergent if there is an $x \notin \{x_\alpha\colon \alpha \in \lambda\}$ such that for each neighbourhood U of x there is a $\beta \in \lambda$ with $x_\alpha \in U$ for each $\alpha \in \lambda \setminus \beta$.

A cardinal κ is a strong limit cardinal if for each cardinal $\tau < \kappa$ we have $2^\tau < \kappa$. We consider ω a strong limit cardinal. If κ and τ are two infinite cardinals such that $\kappa^\tau = \kappa$ then $\tau < \mathrm{cf}(\kappa)$. For strong limit cardinals κ the converse is also valid.

The weight of a space X is denoted by $w(X)$. If G is a topological group then $\mathrm{pc}(G)$ stands for the smallest infinite cardinal κ such that for every nonempty open subset U of G there is a set $D \subset G$ with $|D| < \kappa$ and $D \cdot U = G$. Note that G is precompact if $\mathrm{pc}(G) = \omega$.

If G is an abelian group and $m \in \mathbb{N}$ then mG stands for the subgroup $\{mx\colon x \in G\}$. Note that if G has finite order n then $mG = (m,n)G$ so in that case it suffices to consider m's that divide n.

If $n > 1$ then $\mathbb{Z}(n)$ stands for the cyclic group of order n, usually represented by $\mathbb{Z}$ with addition mod n. The subgroup generated by a subset A of a group is denoted by $\langle A\rangle$.

3. Dense subgroups without convergent sequences

The following theorem improves upon [5, Theorem 1] which states that under GCH every infinite precompact group G that satisfies $w(G)^\omega > w(G)$ has convergent sequences.

Theorem 1. *If G is an infinite topological group whose weight κ is a strong limit cardinal and such that $\mathrm{pc}(G) \leqslant \kappa$ then every dense subgroup of G has convergent $\mathrm{cf}(\kappa)$-sequences.*

Proof. Let H be a dense subgroup of G and put $\kappa = w(G)$. Then $|H| \geqslant \kappa$ because if $|H| < \kappa$ then $\kappa \leqslant 2^{|H|} < \kappa$. Put $\lambda = \mathrm{cf}(\kappa)$ and let $(\kappa_\beta)_{\beta<\lambda}$ be an increasing sequence of cardinals such that $\sup_{\beta<\lambda}\kappa_\beta = \kappa$. Let $\{B_\alpha\colon \alpha < \kappa\}$ be an open basis for G. We select for each ordinal $\alpha < \kappa$ a set $D_\alpha \subset G$ such that $|D_\alpha| < \kappa$ and $D_\alpha \cdot B_\alpha = G$.

Consider for each ordinal $\beta < \lambda$ the equivalence relation $\equiv_\beta$ on H defined by $x \equiv_\beta y$ if for each $\alpha < \kappa_\beta$ with $|D_\alpha| \leqslant \kappa_\beta$ and for each and $z \in D_\alpha$ we have $x \in z\cdot B_\alpha$ if and only if $y \in z \cdot B_\alpha$. Since the number of equivalence classes is at most $2^{\kappa_\beta \cdot \kappa_\beta} < \kappa \leqslant |H|$ we can find an equivalence class A_β of $\equiv_\beta$ with more than one element. Then $A_\beta^{-1} \cdot A_\beta \subset H$ contains an element x_β that is not equal to e.

Let U be a neighbourhood of e in G and select a $\alpha < \kappa$ such that $e \in B_\alpha$ and $B_\alpha^{-1} \cdot B_\alpha \subset U$. Let $\beta < \lambda$ be an arbitrary ordinal such that $\kappa_\beta > \alpha$ and $\kappa_\beta \geqslant |D_\alpha|$. Since $D_\alpha \cdot B_\alpha = G$ we have that A_β intersects $z \cdot B_\alpha$ for some $z \in D_\alpha$. Since A_β is an equivalence class of $\equiv_\beta$ this means that $A_\beta \subset z \cdot B_\alpha$ and hence that $A_\beta^{-1} \cdot A_\beta \subset B_\alpha^{-1} \cdot B_\alpha$. So $x_\beta \in U$ and we have shown that that $\lim_{\beta \to \lambda} x_\beta = e \in H$.

Corollary 2. *If G is a precompact abelian topological group such that for some natural number m the subgroup mG is infinite with a weight κ that is a strong limit cardinal then every dense subgroup of G has convergent* $\mathrm{cf}(\kappa)$*-sequences.*

Proof. Let H be a dense subgroup of G. Then mH is dense in mG and mG is precompact. According to Theorem 1, $mH \subset H$ contains a convergent $\mathrm{cf}(\kappa)$-sequence.

The following theorem generalizes a theorem of Sirota [8] which states that if $\kappa^\omega = \kappa$ then $\mathbb{Z}(2)^\kappa$ has a dense pseudocompact subgroup without convergent sequences.

Theorem 3. *Let κ and τ be infinite cardinals such that $\kappa^\tau = \kappa$ and let n be a natural number greater than* 1. *If $\{G_\alpha\colon \alpha \in \kappa\}$ is a collection of abelian topological groups of order n and weight at most κ then the group $\prod_{\alpha \in \kappa} G_\alpha$ contains a τ-dense subgroup H without convergent λ-sequences for every regular $\lambda \leqslant \tau$.*

Proof. Note that $\kappa^\tau = \kappa$ implies that $\tau < \kappa$. Pick in each G_α a z_α with order n. Since all groups are abelian we have $nx = 0$ for each element x of $G = \prod_{\alpha \in \kappa} G_\alpha$. Select in each G_α a τ-dense subset A_α. Since $w(G_\alpha) \leqslant \kappa$ we may assume that $|A_\alpha| \leqslant \kappa^\tau = \kappa$. Let $\pi_\alpha : G \to G_\alpha$ stand for the projection.

Let F consist of all functions of the form $x = (x_\alpha)_{\alpha \in D}$ where D is a subset of κ with cardinality τ such that $x_\alpha \in A_\alpha$ for every $\alpha \in D$. Obviously, we have $\kappa \leqslant |F| \leqslant (\kappa \cdot \kappa)^\tau = \kappa$. We shall extend every element x of F to an element $E(x)$ of G. Note that the resulting set $E(F)$ is automatically a τ-dense subset of G. Since $\kappa^\tau = \kappa$ we can find a enumeration $(S_\alpha, T_\alpha)_{\alpha \in \kappa}$ of all disjoint pairs of subsets of F with cardinality at most τ.

We will extend the elements of F to elements of G by transfinite induction. Let E_α denote the function that assigns to every element of F the extension obtained at stage $\alpha \leqslant \kappa$. Our induction hypothesis is

(1) $E_\beta(x) = E_\alpha(x) | \operatorname{Dom} E_\beta(x)$ for each $\beta < \alpha$ and $x \in F$,

(2) $|\operatorname{Dom} E_\alpha(x)| \leqslant \tau + |\alpha|$ for each $x \in F$.

Let E_0 be the identity on F. If $\alpha \leqslant \kappa$ is a limit ordinal then we put $E_\alpha(x) = \bigcup_{\beta < \alpha} E_\beta(x)$ for $x \in F$. Note that in both these cases the induction hypothesis is satisfied. Assume now that E_α has been defined for some $\alpha < \kappa$. By induction we have that the set

$$\Sigma = \bigcup \{ \operatorname{Dom} E_\alpha(x)\colon\ x \in S_\alpha \cup T_\alpha \}$$

has cardinality at most $\tau(\tau + |\alpha|)$ which is less than κ. So we can pick a $\gamma \in \kappa \setminus \Sigma$ and define for each $x \in F$:

$$E_{\alpha+1}(x) = \begin{cases} E_\alpha(x) \cup \{(\gamma, 0)\}, & \text{if } x \in S_\alpha \\ E_\alpha(x) \cup \{(\gamma, z_\gamma)\}, & \text{if } x \in T_\alpha \\ E_\alpha(x), & \text{if } x \in F \setminus (S_\alpha \cup T_\alpha). \end{cases}$$

Obviously, the induction hypothesis is also valid for $\alpha + 1$. Let $E(x)$ be the extension of $E_\kappa(x)$ that is obtained by assigning the value 0 to all unused indices.

Consider the subset $E(F)$ of G. If S and T are two disjoint subsets of $E(F)$ of cardinality at most τ then for some $\alpha < \kappa$ we have $S = E(S_\alpha)$ and $T = E(T_\alpha)$. Consequently there is a $\gamma < \kappa$ such that $\pi_\gamma(x) = 0$ for each $x \in S$ and $\pi_\gamma(x) = z_\gamma$ for each $x \in T$.

Let H be the subgroup of G that is generated by $E(F)$. Then H is like $E(F)$ τ-dense in G. Assume that $(x_\alpha)_{\alpha<\lambda}$ is a nontrivial convergent sequence in H for some regular cardinal $\lambda \leqslant \tau$. Since H is a group we may assume that the sequence converges to 0. Since λ is regular we may assume that all x_α's are distinct. Let Y be a subset of $E(F)$ such that $\{x_\alpha\colon \alpha < \lambda\} \subset \langle Y \rangle$ and $|Y| = \lambda$. We select inductively a subsequence $(x_{\xi(\alpha)})_{\alpha<\lambda}$ of $(x_\alpha)_{\alpha<\lambda}$ that lies outside a neighbourhood of 0, providing the desired contradiction. Along the way we also write every $x_{\xi(\alpha)}$ as a finite sum $\sum_{i=0}^{k_\alpha} n_i^\alpha y_i^\alpha$ of elements of Y and we define a nondecreasing sequence T_α of subsets of Y such that $T_\alpha \subset \{y_0^\beta\colon \beta \leqslant \alpha\}$.

Put $\xi(0) = 0$ and write $x_0 = \sum_{i=0}^{k_0} n_i^0 y_i^0$ where the y_i^0's are distinct elements of Y and $1 \leqslant n_i^0 < n$. Since $x_0 \neq 0$ the sum is nonempty and we can define $T_0 = \{y_0^0\}$. Let $\alpha < \lambda$ and consider the following subset of Y:

$$P_\alpha = \{y_i^\beta\colon \beta < \alpha,\ 0 \leqslant i \leqslant k_\beta\}.$$

Note that $|P_\alpha| < \lambda$ and hence $|\langle P_\alpha \rangle| < \lambda$ (for $\lambda = \omega$ we need the fact that G is abelian). Therefore there is a $\xi(\alpha)$ such that $x_{\xi(\alpha)} \notin \langle P_\alpha \rangle$ and $\xi(\alpha) > \sup_{\beta<\alpha} \xi(\beta)$ (λ is regular). Let $x_{\xi(\alpha)} = \sum_{i=0}^{k_\alpha} n_i^\alpha y_i^\alpha$ where the y_i^α's are distinct elements of Y and $1 \leqslant n_i^\alpha < n$. Since $x_{\xi(\alpha)} \notin \langle P_\alpha \rangle$ at least one of the y_i^α's, say y_0^α, is not in P_α. Put $\widetilde{T}_\alpha = \bigcup_{\beta<\alpha} T_\beta$ and define

$$T_\alpha = \begin{cases} \widetilde{T}_\alpha \cup \{y_0^\alpha\}, & \text{if } \sum\limits_{y_i^\alpha \in \widetilde{T}_\alpha} n_i^\alpha = 0 \bmod n \\ \widetilde{T}_\alpha, & \text{otherwise.} \end{cases}$$

Since $1 \leqslant n_0^\alpha < n$ this definition implies that

$$\sum_{y_i^\alpha \in T_\alpha} n_i^\alpha \neq 0 \bmod n.$$

Put $T = \bigcup_{\alpha<\lambda} T_\alpha$ and note that since every $y_0^\alpha \notin P_\alpha$ we have if $y_i^\alpha \in T$ then $y_i^\alpha \in T_\alpha$. Let $S = Y \setminus T$ and note that since $|Y| = \lambda \leqslant \tau$ there is a $\gamma < \kappa$ such that $\pi_\gamma(x) = 0$ for $x \in S$ and $\pi_\gamma(x) = z_\gamma$ for $x \in T$. Let $\alpha < \lambda$ and consider

$$\pi_\gamma(x_{\xi(\alpha)}) = \sum_{i=0}^{k_\alpha} n_i^\alpha \pi_\gamma(y_i^\alpha) = \sum_{y_i^\alpha \in T} n_i^\alpha z_\gamma = \Bigg(\sum_{y_i^\alpha \in T_\alpha} n_i^\alpha\Bigg) z_\gamma.$$

So every $\pi_\gamma(x_{\xi(\alpha)})$ is contained in the closed set $\{z_\gamma, 2z_\gamma, \ldots, (n-1)z_\gamma\}$ which does not contain 0. This proves that $(x_\alpha)_{\alpha<\lambda}$ does not converge to 0. The proof is complete.

If G is a nontrivial abelian group of finite order and $\kappa = w(G)^\omega$ then according to Theorem 3 G^κ has ω-dense subgroups without convergent sequences. The following proposition shows that finite order is essential.

Proposition 4. *There is a compact metric abelian group G such that for each cardinal $\kappa > 0$ every dense subgroup of G^κ has convergent sequences.*

Proof. Define the compact topological group

$$G = \prod_{n=1}^{\infty} \mathbb{Z}(2^n).$$

Let H be a dense subgroup of G^κ. Represent elements of G^κ by $x = (x_n^\alpha)$ where $x_n^\alpha \in \mathbb{Z}(2^n)$ for $n \in \mathbb{N}$ and $\alpha \in \kappa$. Select for each $n \in \mathbb{N}$ an element $x(n)$ of H such that $x(n)_{n+1}^0 = 1$. Observe that $2^n x(n)_{n+1}^0 = 2^n \neq 0 \in \mathbb{Z}(2^{n+1})$ and that $2^n x(n)_i^\alpha = 0$ for $0 \leqslant i \leqslant n$ and $\alpha \in \kappa$. This means that $2^n x(n)$ $(n = 1, 2, \ldots)$ is a nontrivial sequence in H that converges to 0.

Remark. If G is a compact abelian group of prime order p then G is isomorphic to $\mathbb{Z}(p)^\kappa$ for some cardinal κ. Theorem 3 then guarantees that G has dense pseudocompact subgroups without convergent sequences provided $\kappa^\omega = \kappa$. If n is composite then there exist compact abelian groups G of every weight such that $\operatorname{ord}(G) = n$ and every dense subgroup has convergent sequences. Let $n = ab$ with $a, b > 1$ and let κ be an infinite cardinal. Define

$$G = \mathbb{Z}(n)^\omega \times \mathbb{Z}(a)^\kappa.$$

Note that $aG = (a\mathbb{Z}(n))^\omega \times \{0\}$ is isomorphic to $\mathbb{Z}(b)^\omega$ and consequently has weight ω. So every dense subgroup of G has convergent sequences (cf. Corollary 2). These examples also show that in Theorem 3 we cannot replace the condition $\operatorname{ord}(G_\alpha) = n$ by for instance $1 < \operatorname{ord}(G_\alpha) \leqslant n$.

Theorem 5. *If τ is an infinite cardinal and G is a compact abelian torsion group such that for each natural number m (that divides $\operatorname{ord}(G)$) the group mG is finite or $w(mG)^\tau = w(mG)$ then G contains a τ-dense (and hence pseudocompact) subgroup without convergent λ-sequences for every regular $\lambda \leqslant \tau$.*

Proof. We say that a topological group G has the property $\mathfrak{P}_\tau$ if G has a τ-dense subgroup without convergent λ-sequences for every regular $\lambda \leqslant \tau$.

Assume that the theorem is false and let G be a counterexample of minimal order n for some cardinal τ (every compact abelian torsion group has finite order). According to [4, Theorem 25.9] we may assume that there exist natural numbers $\{c_1, \ldots, c_l\}$ and cardinals $\{\kappa_1, \ldots, \kappa_l\}$ such that

$$G = \prod_{i=1}^{l} \mathbb{Z}(c_i)^{\kappa_i}.$$

Since $\mathbb{Z}(a) \times \mathbb{Z}(b)$ is isomorphic to $\mathbb{Z}(ab)$ if a and b are relative prime we may assume that every c_i is a prime power. If G_1 and G_2 are topological groups with G_1 finite then obviously $G_1 \times G_2$ has $\mathfrak{P}_\tau$ if and only if G_2 has $\mathfrak{P}_\tau$. So we may assume that all κ_i's are infinite cardinals. The proof of the following observation is straightforward and left to the reader: if two groups G_1 and G_2 have $\mathfrak{P}_\tau$ then their product $G_1 \times G_2$ has $\mathfrak{P}_\tau$.

Assume now that n is not a prime power. Then there are natural numbers $a, b > 1$ such that $ab = n$ and $(a, b) = 1$. Write $G = G_a \times G_b$ where

$$G_a = \prod_{c_i | a} \mathbb{Z}(c_i)^{\kappa_i} \quad \text{and} \quad G_b = \prod_{c_i | b} \mathbb{Z}(c_i)^{\kappa_i}.$$

If d divides $a = \operatorname{ord}(G_a)$ then

$$dbG = (dbG_a) \times (dbG_b) = (dG_a) \times \{0\}.$$

So $w(dG_a) = w(dbG)$ and hence G_a satisfies the premise of the theorem. Since $\operatorname{ord}(G_a) < \operatorname{ord}(G)$ this means that G_a has the property $\mathfrak{P}_\tau$. The same goes for G_b and hence $G = G_a \times G_b$ has $\mathfrak{P}_\tau$. This contradicts our assumption so we may conclude that $n = p^k$ for some prime p and $k \in \mathbb{N}$.

One of the c_i's, say c_1, is then equal to p^k. Consider

$$p^{k-1}G = \left(p^{k-1}\mathbb{Z}(c_1)\right)^{\kappa_1} \times \{0\}$$

which is isomorphic to $\mathbb{Z}(p)^{\kappa_1}$ so its weight is κ_1. Consequently, we have $(\kappa_1)^\tau = \kappa_1$. Write $G = G_1 \times G_2$ where

$$G_1 = \prod_{\kappa_i \leqslant \kappa_1} \mathbb{Z}(c_i)^{\kappa_i} \quad \text{and} \quad G_2 = \prod_{\kappa_i > \kappa_1} \mathbb{Z}(c_i)^{\kappa_i}.$$

The group G_1 is isomorphic to

$$\mathbb{Z}(c_1)^{\kappa_1} \times \left(\mathbb{Z}(c_1) \times \prod_{\substack{i=2 \\ \kappa_i \leqslant \kappa_1}}^{l} \mathbb{Z}(c_i)^{\kappa_i} \right).$$

Note that this is a product of κ_1 groups each of which has order c_1 and weight at most κ_1. So according to Theorem 3 G_1 has the property $\mathfrak{P}_\tau$. This implies that G_1 is not isomorphic to G and hence $G_2 \neq \{0\}$. So there is a $\kappa_i > \kappa_1$. Let $c_j = \operatorname{ord}(G_2)$ and let d be a proper divisor of c_j. Consider $dG = (dG_1) \times (dG_2)$ and note that the weight of dG_1 is κ_1 where as the weight of dG_2 is at least $\kappa_j > \kappa_1$. So $w(dG_2) = w(dG)$ and since $\mathrm{ord}(G_2) < \mathrm{ord}(G)$ we have that G_2 has $\mathfrak{P}_\tau$. Consequently $G = G_1 \times G_2$ also has $\mathfrak{P}_\tau$ and we have arrived at the contradiction that proves the theorem.

Corollary 6 (GCH). *If G is a compact abelian torsion group and τ is an infinite cardinal then the following statements are equivalent:*

(1) *G has a τ-dense subgroup without convergent λ-sequences for every regular $\lambda \leqslant \tau$.*

(2) *For every natural number m the group mG is finite or* $\mathrm{cf}(w(mG)) > \tau$.

Proof. Under GCH $\mathrm{cf}(\kappa) > \tau$ if and only if $\kappa^\tau = \kappa$ and hence (2) $\Rightarrow$ (1) is Theorem 5.

To prove (1) $\Rightarrow$ (2) assume that there is an m such that mG is infinite and its weight κ has the property $\mathrm{cf}(\kappa) \leqslant \tau$. If H is a τ-dense subgroup of G then mH is τ-dense in mG. If κ is a limit cardinal then by GCH it is a strong limit and H has convergent $\mathrm{cf}(\kappa)$-sequences (Corollary 2). If κ is a successor then it is regular and $\kappa \leqslant \tau$. So every singleton in mG is a $\mathcal{G}_\tau$-set and hence $mH = mG$. Since mG is a compact group it has convergent sequences.

Since under GCH $\kappa^\omega > \kappa$ implies that κ is a strong limit of countable cofinality Corollary 2 and Theorem 5 combine to:

Theorem 7 (GCH). *If G is a compact abelian torsion group then the following statements are equivalent:*

(1) *Every dense subgroup of G contains convergent sequences,*

(2) *Every dense pseudocompact subgroup of G contains convergent sequences,*

(3) *There is a natural number m such that the group mG is infinite and* $\mathrm{cf}(w(mG)) = \omega$.

4. Remarks

Since under GCH every κ either has the property $\kappa^\omega = \kappa$ or it is a strong limit of countable cofinality Theorems 1 and 5 neatly combine to the criterion expressed by Theorem 7. In general, however, there may be many cardinals not covered by these theorems. Let us look at the cardinals below $\mathfrak{c}$. It was shown by Malykhin and Shapiro [5] that the statement that $\mathbb{Z}(2)^{\omega_1}$ has a dense pseudocompact subgroup without convergent sequences is consistent with ZFC and any possible assumption about the value of $\mathfrak{c}$. On the other hand we have:

Proposition 8 (MA). *Every infinite precompact group of weight less than $\mathfrak{c}$ contains a nontrivial convergent sequence.*

Proof. Let G be a precompact group with $w(G) < \mathfrak{c}$. We may assume that G is countably infinite. Since G is precompact we have that e is not an isolated point. Since $w(G) < \mathfrak{c}$ Martin's Axiom implies that e is the limit of some sequence in $G \setminus \{e\}$. This is well-known and for completeness sake we will include the simple proof. Let $\mathcal{U}$ be a neighbourhood basis at e with cardinality at most $w(G) < \mathfrak{c}$. The collection $\mathcal{U}$ obviously has the property that finite intersections of its elements are infinite and hence by MA there is an infinite subset E of G such that for every $U \in \mathcal{U}$ we have that $E \backslash U$ is finite (see [7, Corollary 8]). Clearly, $E \setminus \{e\}$ converges to e.

So the statement $\mathbb{Z}(2)^{\omega_1}$ has a dense (pseudocompact) subgroup without convergent sequences is independent of ZFC + $\neg$CH.

In light of Proposition 8, the question naturally arises whether every dense pseudocompact subgroup of $\mathbb{Z}(2)^{\omega_1}$ is countably compact under some additional axiom of set theory such as MA + $\neg$CH. But this is not even true in ZFC, as the following example shows. Let us think of $\mathbb{Z}(2)^{\omega_1}$ as the product $K = (\mathbb{Z}(2)^{\omega})^{\omega_1}$ and let Δ be the diagonal of this product. In addition, let Σ be the standard Σ-product in K and let G be any countably infinite subgroup of Δ. Since Σ is countably compact and dense, the group $H = G + \Sigma$ is dense and pseudocompact since it contains Σ. It is however not countably compact. For let $g_n \in G$ $(n < \omega)$ be any sequence in Δ converging to a point $x \in \Delta \setminus G$. We claim that $x \notin H$, which is clearly as required. Striving for a contradiction, assume that there are $g \in G$ and $\sigma \in \Sigma$ such that $x = g + \sigma$. Then $x + g = \sigma \in \Sigma$. But $x + g \neq 0$ because $x \notin G$. There consequently is a coordinate α for which $(x + g)(\alpha) = 1$. By the special choice of Δ there are consequently ω_1 coordinates with the same property. But then $x + g \notin \Sigma$, which is a contradiction.

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