The Homeomorphism Extension Theorem for $\beta\omega\setminus\omega$

ERIC K. VAN DOUWEN[a] AND JAN VAN MILL[b]

[b]*Faculteit Wiskunde en Informatica*
Vrije Universiteit
De Boelelaan 1081a
1081 HV Amsterdam
The Netherlands

ABSTRACT. CH implies that every homeomorphism of nowhere dense closed P-sets in $\beta\omega\setminus\omega$ can be extended to an autohomeomorphism of $\beta\omega\setminus\omega$. CH also implies that every closed P-set of weight at most c of a compact zero-dimensional F-space is a retract.

1. RESULTS

All spaces considered are Hausdorff. Here ω^* denotes $\beta\omega\setminus\omega$. A subset P of a space X is called a P-set if $\bigcap \mathscr{U}$ is a neighborhood of P whenever $\mathscr{U}$ is a countable family of neighborhoods of P and a point p is called a P-point if $\{p\}$ is a P-set.

Our main result is the following theorem.

THEOREM 1.1 (HOMEOMORPHISM EXTENSION THEOREM): The Continuum Hypothesis (CH) implies that every homeomorphism of one (nonempty) closed nowhere dense P-set of ω^* to another can be extended to a homeomorphism of ω^* onto itself.

A corollary is W. Rudin's Theorem that under CH for every two P-points p and q of ω^* there is a homeomorphism of ω^* onto itself that sends p to q [7].

A nontrivial corollary is also that under CH every nonempty closed nowhere dense P-set of ω^* is a retract of ω^*. We do not state this as a theorem since the following result is much more general.

THEOREM 1.2 (RETRACTION THEOREM): CH implies that if X is a zero-dimensional compact F-space, and if P is a nonempty nowhere dense closed P-set in X with $w(P) \leq c$, then P is a retract of X.

2. PROOF OF THE RETRACTION THEOREM

If X is a space, then $\mathscr{B}(X)$ denotes the Boolean algebra of clopen (= open and closed) subsets of X. If A and B are sets, then $A \,\Delta\, B$ denotes their symmetric difference, that is, $A \,\Delta\, B = (A\setminus B) \cup (B\setminus A)$. Also, "$\subset$" denotes *proper* inclusion. A

Mathematics Subject Classification. Primary 54D35.
Key words and phrases. CH, P-set, homeomorphism, retraction, $\beta\omega\setminus\omega$.
[a]Deceased.

compact space is an *F-space* if disjoint open F_σ-subsets have disjoint closures. Finally, a continuous function will be called a *map.*

Let Y be a compact zero-dimensional F-space, and let P be a nowhere dense closed P-set of Y with $w(P) \leq \omega_1$. We will construct an (Boolean algebra) embedding $f: \mathscr{B}(P) \to \mathscr{B}(Y)$ such that $f(A) \cap P = A$ for every $A \in \mathscr{B}(P)$. Such an embedding we call a *lifting.* By Stone duality this implies that P is a retract of Y.

Let $\mathscr{B}(P) = \{X_\alpha : \alpha < \omega_1\}$ and for $\alpha < \omega_1$ let $\mathscr{A}_\alpha$ be the subalgebra of $\mathscr{B}(P)$ generated by $\{X_\xi : \xi < \alpha\}$ (so $\mathscr{A}_0 = \{\varnothing\}$). We construct a partial lifting $f_\alpha : \mathscr{A}_\alpha \to \mathscr{B}(Y)$ for $\alpha < \omega_1$ such that $f_\alpha = f_\beta \upharpoonright \mathscr{A}_\alpha$ whenever $\alpha < \beta < \omega_1$. Then $f = \bigcup_{\alpha<\omega_1} f_\alpha$ is our lifting $\mathscr{B}(P) \to \mathscr{B}(Y)$. The definition of f_λ from $\{f_\alpha : \alpha < \lambda\}$ for $\lambda < \omega_1$ a limit ordinal (including $\lambda = 0$) is trivial. This reduces the proof to proving the following

LEMMA 2.1: Let $\mathscr{A}$ be a countable subalgebra of $\mathscr{B}(P)$. Let $A \in \mathscr{B}(P)$ and let $\mathscr{A}'$ be the subalgebra of $\mathscr{B}(P)$ generated by $\mathscr{A} \cup \{A\}$. For each partial lifting $f: \mathscr{A} \to \mathscr{B}(Y)$ there is a partial lifting $f' : \mathscr{A}' \to \mathscr{B}(Y)$ that extends f.

Proof: We claim that there is $B \in \mathscr{B}(Y)$ with $B \cap P = A$ such that

(1) $\forall X \in \mathscr{A} \, [X \subseteq A \Rightarrow f(X) \subseteq B]$,
(2) $\forall X \in \mathscr{A} \, [X \subseteq P \backslash A \Rightarrow f(X) \subseteq Y \backslash B]$.

Given such a B, one defines f' by

$$f'((X \cap A) \cup (Z \backslash A)) = (f(X) \cap B) \cup (f(Z) \backslash B) \qquad (X, Z \in \mathscr{A}).$$

It follows from Sikorski's Theorem (see, e.g., Koppelberg [3, corollary 5.8]) that f' is a well-defined embedding of $\mathscr{A}'$ into $\mathscr{B}(Y)$.

We now construct B. Define

$$\mathscr{D} = \{f(X) : X \in \mathscr{A}, X \subseteq A\} \qquad \text{and} \qquad \mathscr{E} = \{f(X) : X \in \mathscr{A}, X \subseteq P \backslash A\}.$$

Pick any $B' \in \mathscr{B}(Y)$ with $B' \cap P = A$. Then $\mathscr{E}' = \{E \cap B' : E \in \mathscr{E}\}$ and $\mathscr{D}' = \{D \backslash B' : D \in \mathscr{D}\}$ are countable families of clopen subsets of Y that miss P. Since P is a P-set there consequently exists a clopen neighborhood U of P such that

(3) $\bigcup \mathscr{D}' \cup \bigcup \mathscr{E}' \subseteq Y \backslash U$.

Since $\bigcup \mathscr{D}$ and $\bigcup \mathscr{E}$ are trivially disjoint and Y is a zero-dimensional compact F-space, there exists a clopen subset S of Y such that $S \subseteq Y \backslash U$ and

(4) $\bigcup \mathscr{D} \backslash U \subseteq S$ and $\bigcup \mathscr{E} \backslash U \subseteq Y \backslash S$.

Therefore we can define B satisfying (1) and (2) by

$$B = (B' \cap U) \cup S.$$

As $B' \cap P = A$ and $S \cap P = \varnothing$ we have $B \cap P = A$. $\square$

REMARK 2.2: CH is essential in Theorem 1.2 because under MA + $c = \omega_2$ there exists a compact zero-dimensional F-space Y (of weight c) with a P-set copy of $\beta\omega$ that is not a retract, van Douwen and van Mill [2].

3. PROOF OF THE HOMEOMORPHISM EXTENSION THEOREM

In the proof of the Homeomorphism Extension Theorem we will use two known results on constructing homeomorphisms.

The first one is Parovičenko's Theorem in [5] that under CH all compact zero-dimensional F-spaces of weight c in which every nonempty G_δ have infinite interior are homeomorphic (to ω^*) (see also [1, sec. 6], [4, corollary 1.2.4], and [9, p. 81]). This result can be proved as follows. Take two arbitrary compact zero-dimensional F-spaces of weight c in which nonempty G_δ have infinite interior, say X and Y. It can be shown that if $\mathscr{B}$ is a *countable* subalgebra of $\mathscr{B}(X)$ and if $A \in \mathscr{B}(X)$ and if $f: \mathscr{B} \to \mathscr{B}(Y)$ is an embedding (of Boolean algebras), then this embedding can be extended to an embedding from the subalgebra of $\mathscr{B}(X)$ that is generated by $\mathscr{B} \cup \{A\}$ into $\mathscr{B}(Y)$. Once this has been proved it is possible under CH to construct an isomorphism $\eta : \mathscr{B}(X) \to \mathscr{B}(Y)$ by a standard back-and-forth technique. As a consequence, $X \approx Y$ by Stone duality. So the proof of Parovičenko's Theorem shows that an arbitrary isomorphism between countable subalgebras of $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ can be extended to a full isomorphism from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$. It is this form of Parovičenko's Theorem that we will use in our proof.

The second one is a result due to M. E. Rudin [6] (see also Walker [9, p. 173]). Let $\mathscr{U} = \{U_n : n < \omega\}$ and $\mathscr{V} = \{V_n : n < \omega\}$ be pairwise disjoint families of clopen subsets of ω^*. In addition, for every $n < \omega$ let $h_n : \omega^* \to \omega^*$ be a homeomorphism such that $h_n[U_n] = V_n$. Then, as was shown by M. E. Rudin, under CH there exists a homeomorphism $h : \omega^* \to \omega^*$ such that for every n, $h \restriction U_n = h_n$. We use this result in the following form. Suppose that E and F are open F_σ subsets of ω^* and that $f: E \to F$ is a homeomorphism. We claim that f can be extended to a homeomorphism $\bar{f}: \omega^* \to \omega^*$. There clearly exists a pairwise disjoint clopen family $\{E_n : n < \omega\}$ in ω^* whose union is E (use that E is Lindelöf). Then $\{f[E_n] : n < \omega\}$ is a pairwise disjoint clopen family in ω^* whose union is F. Since all nonempty clopen subsets of ω^* are homeomorphic it is possible for every n to extend the homeomorphism $f \restriction E_n \to f[E_n]$ to a homeomorphism g_n of ω^* (let g_n be the union of f_n and an arbitrary homeomorphism from $\omega^* \backslash E_n \to \omega^* \backslash f[E_n]$). So M. E. Rudin's Theorem implies that under CH the desired extension $\bar{f}$ can be found.

We now turn to the proof of the Homeomorphism Extension Theorem. To this end, let A and B be nowhere dense closed P-sets in ω^* and let $h : A \to B$ be a homeomorphism. As to be expected, we will extend h in an inductive process of length ω_1.

By CH we can list $\{C \in \mathscr{B}(\omega^*) : C \cap A = \emptyset\}$, $\{C \in \mathscr{B}(\omega^*) : C \cap B = \emptyset\}$, and $\mathscr{B}(A)$ as $\{A_\alpha : \alpha < \omega_1\}$, $\{B_\alpha : \alpha < \omega_1\}$, and $\{X_\alpha : \alpha < \omega_1\}$, respectively. Without loss of generality, $A_0 = \emptyset = B_0$. The subalgebra of $\mathscr{B}(A)$ generated by $\{X_\beta : \beta < \alpha\}$ will be denoted by $\mathscr{A}_\alpha$. By Theorem 1.2 there exist retractions $r : \omega^* \to A$ and $\rho : \omega^* \to B$. If $X \subseteq A$, then we let X_r and X_ρ denote $r^{-1}[X]$ and $\rho^{-1}[h[X]]$, respectively.

By transfinite induction on $\alpha < \omega_1$ we now construct clopen subsets E_α and F_α of ω^* and a homeomorphism $f_\alpha : E_\alpha \to F_\alpha$ such that

(1) $A_\alpha \subseteq E_\alpha$ and $B_\alpha \subseteq F_\alpha$,
(2) $E_\alpha \subseteq \omega^* \backslash A$, $F_\alpha \subseteq \omega^* \backslash B$, and if $\beta < \alpha$, then $E_\beta \subset E_\alpha$ and $F_\beta \subset F_\alpha$,
(3) if $\beta < \alpha$, then $f_\alpha \restriction E_\beta = f_\beta$,

(4) if $\beta < \alpha$ and $X \in \mathscr{A}_\beta$, then

$$f_\alpha[X_r \cap (E_\alpha \backslash E_\beta)] = X_\rho \cap (F_\alpha \backslash F_\beta).$$

Assume that we defined E_β, F_β, and f_β for every $\beta < \alpha$, $\alpha < \omega_1$ (possibly $\alpha = 0$).

There are two cases to consider: α is a successor and α is a limit. We first show that the limit case can be reduced to a case similar to the successor case.

So assume that α is a limit and put $E = \bigcup_{\beta<\alpha} E_\beta$, $F = \bigcup_{\beta<\alpha} F_\beta$, and $f = \bigcup_{\beta<\alpha} f_\beta$. Observe that (2) implies that both E and F are noncompact open F_σ-subsets of ω^* and that $f : E \to F$ is a homeomorphism. By the preceding we can extend f to a homeomorphism $\bar{f} : \omega^* \to \omega^*$. Pick an arbitrary element $X \in \mathscr{A}_\alpha = \bigcup_{\beta<\alpha} \mathscr{A}_\beta$, say $X \in \mathscr{A}_\beta$. By (4),

$$X_r \cap (E \backslash E_\beta) = \bar{f}^{-1}[X_\rho] \cap (E \backslash E_\beta).$$

Consequently,

$$(X_r \,\Delta\, \bar{f}^{-1}[X_\rho]) \backslash E_\beta$$

is a clopen subset of ω^* that misses E. Since $\bigcup_{\beta<\alpha} \mathscr{A}_\beta$ is countable, and E is an open F_σ in ω^*, and ω^* is an F-space, there is a clopen subset S of ω^* such that

(5) $E \subseteq S$,

(6) $\forall \beta < \alpha \,\forall X \in \mathscr{A}_\beta \,[(X_r \,\Delta\, \bar{f}^{-1}[X_\rho]) \cap (S \backslash E_\beta) = \varnothing]$.

Since A and B are P-sets, and E is an F_σ-subset of ω^* that misses $A \cup \bar{f}^{-1}[B]$, we can additionally assume that

(7) $S \cap (A \cup \bar{f}^{-1}[B]) = \varnothing$.

So by putting $T = \bar{f}[S]$ we conclude that for the clopen sets $S \subseteq \omega^* \backslash A$ and $T \subseteq \omega^* \backslash B$ and for the homeomorphism $g = \bar{f} \restriction S : S \to T$ we have for every $\beta < \alpha$ and for every $X \in \mathscr{A}_\beta$ that

(8) $g[X_r \cap (S \backslash E_\beta)] = X_\rho \cap (T \backslash F_\beta)$.

So our task is now to enlarge S and T and to extend the homeomorphism g so that the new S contains A_α, the new T contains B_α, and the new g still satisfies (8) (with S replaced by the new S, etc.). In the case that α is a successor, we also have to achieve this so that without loss of generality we can now restrict ourselves to the case that α is a successor.

Assume that $\alpha = \beta + 1$ (possibly, $\beta = -1$). We will construct nonempty clopen subsets M and N of ω^* such that $M \subseteq \omega^* \backslash (A \cup E_\beta)$, $N \subseteq \omega^* \backslash (B \cup F_\beta)$ such that $E_\beta \cup M \supseteq A_\alpha$, $F_\beta \cup N \supseteq B_\alpha$, and a homeomorphism $\xi : M \to N$ such that for every $X \in \mathscr{A}_\beta$:

$$\xi[X_r \cap M] = X_\rho \cap N.$$

Then we put $E_\alpha = E_\beta \cup M$, $F_\alpha = F_\beta \cup N$, and $f_\alpha = f_\beta \cup \xi$. Then E_α, F_α, and f_α are clearly as required.

So in fact E_β, F_β, and f_β play no role of importance in the remaining part of the construction and for convenience we therefore assume that $E_\beta = \varnothing = F_\beta$.

We claim that there is a nonempty clopen subset M of ω^* disjoint from A such that the function $\mathscr{A}_\beta \to \mathscr{B}(M)$ defined by

$$X \mapsto X_r \cap M \qquad (X \in \mathscr{A}_\beta)$$

is an embedding of Boolean algebras. Indeed, for every nonempty $X \in \mathscr{A}_\beta$ pick a point in $X_r \setminus A$ (here we use that A is nowhere dense). The union of these points is a countable subset of $\omega^* \setminus A$. Since A is a P-set, there consequently exists a nonempty clopen set M in ω^* disjoint from A such that $\forall X \in \mathscr{A}_\beta \setminus \{\varnothing\}[X_r \cap M \neq \varnothing]$. Then M is clearly as required. Observe that without loss of generality we may assume that $A_\alpha \subseteq M$. There similarly exists a nonempty clopen subset N of ω^* disjoint from B such that the function $\mathscr{A}_\beta \to \mathscr{B}(N)$ defined by

$$X \mapsto X_\rho \cap N \qquad (X \in \mathscr{A}_\beta)$$

is an embedding (here we use that h is a homeomorphism). We also assume without loss of generality that $B_\alpha \subseteq N$. The function

$$\eta : X_r \cap M \mapsto X_\rho \cap N \qquad (X \in \mathscr{A}_\beta)$$

is an isomorphism of countable subalgebras of $\mathscr{B}(M)$ and $\mathscr{B}(N)$. By the remarks at the beginning of this section it follows that this isomorphism can be extended to an isomorphism

$$\hat{\eta} : \mathscr{B}(M) \to \mathscr{B}(N).$$

By Stone duality $\hat{\eta}$ corresponds to a homemorphism $\xi : M \to N$, which clearly has the property that

$$\forall X \in \mathscr{A}_\beta[\xi[X_r \cap M] = X_\rho \cap N]$$

(use that $\hat{\eta}$ extends η). This completes the transfinite construction.

The function $f = \bigcup_{\alpha<\omega_1} f_\alpha : \omega^* \setminus A \to \omega^* \setminus B$ is a homemorphism, and the function $\bar{h} = h \cup f$ is one-to-one. We claim that $\bar{h}$ is open at the points of A. By compactness of ω^* this then implies that $\bar{h}$ is a homeomorphism. To this end, pick an arbitrary clopen neighborhood C of an arbitrary point $x \in A$. Pick $\beta < \omega_1$ such that $C \cap A \in \mathscr{A}_\beta$. Observe that the family $\mathscr{C} = \{(C \cap A)_r \setminus E_\alpha : \alpha < \omega_1\}$ is decreasing and that $\bigcap \mathscr{C} = C \cap A$. Since C is a neighborhood of $C \cap A$, by compactness of ω^* there consequently exists $\alpha < \omega_1$ such that $\beta < \alpha$ and $P = (C \cap A)_r \setminus E_\alpha$ is contained in C. Observe that P is a clopen neighborhood of x. By (4) and the definition of $\bar{h}$ it follows that

$$\bar{h}[P] = (C \cap A)_\rho \setminus F_\alpha$$

and consequently $\bar{h}\,[P]$, and hence $\bar{h}\,[C]$, is a neighborhood of $\bar{h}\,(x)$.

REMARK 3.1: CH is essential in Theorem 1.1. It is consistent with MA + $\neg$CH that every autohomeomorphism of ω^* is trivial, Shelah and Steprāns [8]. In this model there are c autohomeomorphisms but 2^c P_c-points.

4. REMARKS BY JAN VAN MILL

Theorems 1.1 and 1.2 were proved in the Spring of 1980 (when Eric van Douwen visited me in Baton Rouge) by a complexification of W. Rudin's method from [7]. These results were not published because we derived Theorem 1.2 for the special case $X = \omega^*$ from Theorem 1.1 and the proof of Theorem 1.1 was technically very complicated: we felt that it was not in its final form yet. The simple proof of Theorem 1.2 presented here I found in Eric van Douwen's unpublished manuscripts and the simple proof of Theorem 1.1 was found recently by me. It is interesting to note that Theorem 1.2 was used here in the proof of Theorem 1.1. In our original proof we did everything in opposite order.

Remarks like these usually are not made in a mathematical paper. The reader is interested in the theorems only and not in their history. I felt, however, that I should offer the reader an explanation of why this joint paper was written four years after the death of the first-named author.

REFERENCES

1. COMFORT, W. W. & S. NEGREPONTIS. 1974. The theory of ultrafilters. Grundlehren der mathematischen Wissenschaften, Vol. 211. Springer-Verlag. Berlin.
2. VAN DOUWEN, E. K. & J. VAN MILL. 1980. Subspaces of basically disconnected spaces or quotients of countably complete Boolean Algebras. Trans. Am. Math. Soc. **259:** 121–127.
3. KOPPELBERG, S. 1989. Handbook of Boolean Algebras, Vol. I. North-Holland, Amsterdam, The Netherlands.
4. VAN MILL, J. 1984. An introduction to $\beta\omega$. *In* Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan, Eds.: 503–567. North-Holland, Amsterdam, The Netherlands.
5. PAROVIČENKO, I. I.1963. A universal bicompact of weight $\aleph$. Sov. Math. Dok. **4:** 592–592.
6. RUDIN, M. E. 1966. Types of ultrafilters. Paper presented at the Topology Seminar (Wisconsin, 1965). Annals of Mathematical Studies, No. 60, R. H. Bing and R. J. Bean, Eds.: 147–151. Princeton Univ. Press. Princeton, N.J.
7. RUDIN, W. 1956. Homogeneity problems in the theory of Čech compactifications. Duke Math. J. **23:** 409–419.
8. SHELAH, S. & J. STEPRĀNS. 1988. PFA implies all automorphisms are trivial. Proc. Am. Math. Soc. **104:** 1220–1225.
9. WALKER, R. C. 1974. The Stone-Čech Compactification. Springer-Verlag. Berlin.