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THE SPACE OF INFINITE-DIMENSIONAL COMPACTA AND OTHER TOPOLOGICAL COPIES OF $(l_f^2)^\omega$

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To Doug Curtis, on the occasion of his retirement

We show that there exists a homeomorphism from the hyperspace of the Hilbert cube Q onto the countable product of Hilbert cubes such that the $\geq k$-dimensional sets are mapped onto $B^k \times Q \times Q \times \cdots$, where B is the pseudoboundary of Q. In particular, the infinite-dimensional compacta are mapped onto B^ω, which is homeomorphic to the countably infinite product of l_f^2. In addition, we prove for $k \in \{1, 2, \ldots, \infty\}$ that the space of uniformly $\geq k$-dimensional sets in 2^Q is also homeomorphic to $(l_f^2)^\omega$.

1. Introduction. If X is a compact metric space then 2^X denotes the hyperspace of X equipped with the Hausdorff metric. According to Curtis and Schori [**6**] 2^X is homeomorphic to the Hilbert cube Q whenever X is a nontrivial Peano continuum.

Our primary interest is the subset of 2^Q consisting of all infinite-dimensional compacta. This space is an $F_{\sigma\delta}$-set in 2^Q and one may expect that it is homeomorphic to the countable product of the pre-Hilbert space

$$l_f^2 = \{x \in l^2 : x_i = 0 \text{ for all but finitely many } i\}.$$

We prove this conjecture. The space $(l_f^2)^\omega$ is in a sense maximal in the class $\mathscr{F}_{\sigma\delta}$ of absolute $F_{\sigma\delta}$-spaces and it has received a lot of attention in recent years because of its topological equivalence to numerous function spaces, see e.g. Dijkstra et al. [**7**].

For $k \in \{0, 1, 2, \ldots, \infty\}$ we let $\mathrm{Dim}_{\geq k}(X)$ denote the subspace consisting of all $\geq k$-dimensional elements of 2^X. We define $\mathrm{Dim}_k(X)$ and $\mathrm{Dim}_{\leq k}(X)$ in the same way. Let $\overline{\mathrm{Dim}}_{\geq k}(X)$ stand for all uniformly $\geq k$-dimensional compacta in 2^X, i.e. spaces such that every nonempty open subset is at least k-dimensional. The default value here is $X = Q$, i.e., $\mathrm{Dim}_{\geq k} = \mathrm{Dim}_{\geq k}(Q)$ etc.

Let I stand for the interval $[0, 1]$. The Hilbert cube is denoted by $Q = \prod_{i=1}^\infty I$ with metric $d(x, y) = \max\{2^{-i}|x_i - y_i| : i \in \mathbf{N}\}$. The pseudointerior of Q is $s = \prod_{i=1}^\infty (0, 1)$ and $B = Q \setminus s$ is the pseudoboundary.

THEOREM 1.1. (a) *There exists a homeomorphism* α *from* 2^Q *onto* $Q^{\mathbf{N}} = \prod_{i=1}^{\infty} Q$ *such that for every* $k \in \{0, 1, 2, \ldots\}$,

$$\alpha(\mathrm{Dim}_{\geq k}) = \underbrace{B \times \cdots \times B}_{k \text{ times}} \times Q \times Q \times \cdots .$$

This implies that $\alpha(\mathrm{Dim}_{\infty}) = B^{\mathbf{N}}$.

(b) *There exists a homeomorphism* β *from* 2^Q *onto* $Q^{\mathbf{N}}$ *such that for every* $k \in \{0, 1, 2, \ldots\}$,

$$\beta(\mathrm{Dim}_{\leq k}) = \underbrace{Q \times \cdots \times Q}_{k \text{ times}} \times s \times s \times \cdots .$$

The pseudoboundary B is an absorber for the collection of σ-compacta $\mathscr{F}_\sigma$. Furthermore, $B^{\mathbf{N}}$ is an absorber in $Q^{\mathbf{N}}$ for the collection $\mathscr{F}_{\sigma\delta}$. For definitions see §2 and §3. The space $B^{\mathbf{N}}$ is homeomorphic to $(l_f^2)^\omega$. If Y is an $\mathscr{F}_{\sigma\delta}$-absorber in Q, i.e., the pair (Q, Y) is homeomorphic to $(Q^{\mathbf{N}}, B^{\mathbf{N}})$, then we have the following:

THEOREM 1.2. *There exists a homeomorphism* α *from* 2^Q *onto* $Q^{\mathbf{N}}$ *such that for every* $k \in \{0, 1, 2, \ldots\}$,

$$\alpha(\overline{\mathrm{Dim}}_{\geq k}) = \underbrace{Y \times \cdots \times Y}_{k \text{ times}} \times Q \times Q \times \cdots .$$

This means that $\overline{\mathrm{Dim}}_{\geq k}$ *is homeomorphic to* $B^{\mathbf{N}}$ *and* $(l_f^2)^\omega$ *for* $k \in \{1, 2, \ldots, \infty\}$.

In the final section we illustrate the power of the technique that we developed to prove the main theorems by applying the method to function spaces $C_p(X)$.

For an explanation of undefined terminology see van Mill **[12]**.

2. Absorbing systems. Let Γ be an ordered set and let $\mathscr{M}_\gamma$ be a collection of spaces for each $\gamma \in \Gamma$. Each $\mathscr{M}_\gamma$ is assumed to be *topological* and *closed hereditary*. Let $\mathscr{M}$ stand for the whole system $(\mathscr{M}_\gamma)_{\gamma \in \Gamma}$. Let $X = (X_\gamma)_{\gamma \in \Gamma}$ be an order preserving indexed collection of subsets of a topological copy E of Q, i.e., $X_\gamma \subset X_{\gamma'}$ if and only if $\gamma \leq \gamma'$.

The system X is called *$\mathscr{M}$-universal* if for every order preserving system $(A_\gamma)_\gamma$ in Q such that $A_\gamma \in \mathscr{M}_\gamma$ for every $\gamma \in \Gamma$, there is an embedding $f: Q \to E$ with $f^{-1}(X_\gamma) = A_\gamma$. The system X is called *strongly $\mathscr{M}$-universal* if for every order preserving system $(A_\gamma)_\gamma$ in Q such that $A_\gamma \in \mathscr{M}_\gamma$ for every $\gamma \in \Gamma$, and for every map $f: Q \to E$

that restricts to a Z-embedding on some compact set K, there exists a Z-embedding $g: Q \to E$ that can be chosen arbitrarily close to f with the properties: $g|K = f|K$ and $g^{-1}(X_\gamma)\backslash K = A_\gamma \backslash K$ for every γ. The system X is called *reflexively universal* if for every map $f: E \to E$ that restricts to a Z-embedding on some compact set K, there exists a Z-embedding $g: E \to E$ that can be chosen arbitrarily close to f with the properties: $g|K = f|K$ and $g^{-1}(X_\gamma) \setminus K = X_\gamma \setminus K$ for every γ. Observe that X is strongly $\mathscr{M}$-universal whenever X is $\mathscr{M}$-universal and reflexively universal. If $X_\gamma \in \mathscr{M}_\gamma$ then the converse is also true.

The system X is called *$\mathscr{M}$-absorbing* if

(1) $X_\gamma \in \mathscr{M}_\gamma$ for every $\gamma \in \Gamma$,
(2) $\bigcup\{X_\gamma : \gamma \in \Gamma\}$ is contained in a σ Z-set of E, and
(3) X is strongly $\mathscr{M}$-universal.

This notion appears to be a successful synthesis of the Q-matrices technique of van Mill **[11]** and the generalized absorbers of Bestvina and Mogilski **[2]**. The power of the method we introduce here comes mainly from the relative ease of application.

As expected we have a uniqueness theorem for absorbing systems:

THEOREM 2.1. *If X and Y are both $\mathscr{M}$-absorbing systems in E respectively E' then (E, X) and (E', Y) are homeomorphic, i.e., there is a homeomorphism $h: E \to E'$ such that $h(X_\gamma) = Y_\gamma$ for all $\gamma \in \Gamma$. If $E = E'$ then the map h can be found arbitrarily close to the identity.*

Proof. This is a standard back and forth argument. Obviously, we may assume that $E = E' = Q$. Let $\bigcup_\gamma X_\gamma \subset \bigcup_i A_i$ and let $\bigcup_\gamma Y_\gamma \subset \bigcup_i B_i$, where $\varnothing = A_0 \subset A_1 \subset A_2 \subset \cdots$ and $\varnothing = B_0 \subset B_1 \subset B_2 \subset \cdots$ are sequences of Z-sets in Q. By induction we shall construct sequences of homeomorphisms $f_i: Q \to Q$ and $g_i = f_i \circ \cdots \circ f_0$ with the properties:

$$A_i \cap X_\gamma = A_i \cap g_i^{-1}(Y_\gamma), \qquad B_i \cap g_i(X_\gamma) = B_i \cap Y_\gamma,$$
$$f_i|(g_{i-1}(A_{i-1}) \cup B_{i-1}) = 1,$$

where 1 denotes the identity map. Put $f_0 = 1$.

Assume that f_i has been constructed. Since $X_\gamma \in \mathscr{M}_\gamma$ and $\mathscr{M}_\gamma$ is topological and closed hereditary we have $g_i(X_\gamma) \cap (g_i(A_{i+1}) \cup B_i) \in \mathscr{M}_\gamma$. Put $K = g_i(A_i) \cup B_i$ and observe that $g_i(X_\gamma) \cap K = Y_\gamma \cap K$. Since Y is strongly universal we can find a Z-embedding $\alpha: g_i(A_{i+1}) \cup B_i \to Q$ that fixes K and that has the property

$$\alpha^{-1}(Y_\gamma) \cap g_i(A_{i+1}) = g_i(X_\gamma \cap A_{i+1}).$$

Let $\tilde{\alpha}$ be an extension of α to a homeomorphism of Q. Since $\tilde{\alpha} \circ g_i(X)$ is just as X strongly universal we can find a Z-embedding $\beta\colon \alpha \circ g_i(A_{i+1}) \cup B_{i+1} \to Q$ that fixes $K' = \alpha \circ g_i(A_{i+1}) \cup B_i$ and that has the property

$$\beta^{-1}(\tilde{\alpha} \circ g_i(X_\gamma)) \cap B_{i+1} = Y_\gamma \cap B_{i+1}.$$

Let $\tilde{\beta}$ be an extension of β to a homeomorphism of Q. If we put $f_{i+1} = \tilde{\beta}^{-1} \circ \tilde{\alpha}$ then one can easily verify the induction hypothesis for $i+1$. Since $\tilde{\alpha}$ and $\tilde{\beta}$ and hence f_{i+1} can be chosen arbitrarily close to the identity we may assume that $h = \lim_{i\to\infty} g_i$ is a homeomorphism of Q. The function h maps each X_γ onto Y_γ.

3. Absorbing sequences in $Q^{\mathbf{N}}$. We shall now consider the special case that the system X is a decreasing sequence $Q \supset X_1 \supset X_2 \supset \cdots$. Formally, this corresponds to choosing $\Gamma = \mathbf{N}$ with an inverted ordering. As a further simplification we assume that all the $\mathscr{M}_\gamma$'s are equal to a fixed $\mathscr{M}$ and use the term $\mathscr{M}$-absorbing sequence. In addition, if Γ is a singleton then we call X an $\mathscr{M}$-absorber. Recall that the pseudoboundary B of Q is an $\mathscr{F}_\sigma$-absorber, where $\mathscr{F}_\sigma$ is the collection of σ-compact spaces. Observe that if X is an $\mathscr{M}$-absorbing sequence and $\mathscr{M}$ is closed under finite intersections then $X_\infty = \bigcap_{i=1}^\infty X_i$ is an $\mathscr{M}_\delta$-absorber, where $\mathscr{M}_\delta$ stands for the collection of countable intersections of elements of $\mathscr{M}$.

Let X be a subset of Q. We define three decreasing sequences of subsets of $Q^{\mathbf{N}}$:

$$S_n(X) = \underbrace{X \times \cdots \times X}_{n \text{ times}} \times Q \times Q \times \cdots,$$

$$S'_n(X) = \{x \in Q^{\mathbf{N}} : \text{at least } n \text{ of the } x_i\text{'s are in } X\},$$

$$S''_n(X) = \{x \in Q^{\mathbf{N}} : x_i \in X \text{ for some } i \geq n\}.$$

Note that $S_n(X) \subset S'_n(X) \subset S''_n(X)$ and that $S_\infty(X) = X^{\mathbf{N}}$ and $S'_\infty(X) = S''_\infty(X)$.

THEOREM 3.1. *If $X \subset Q$ is strongly $\mathscr{M}$-universal then the sequences $S(X)$, $S'(X)$ and $S''(X)$ are strongly $\mathscr{M}$-universal in $Q^{\mathbf{N}}$. If, in addition, $\mathscr{M}$ is closed under finite intersections then $X^{\mathbf{N}}$ and $S'_\infty(X)$ are strongly $\mathscr{M}_\delta$-universal.*

Proof. Let ρ_n be a metric on Q such that

$$\rho(x, y) = \max\{\rho_n(x_n, y_n) : n \in \mathbf{N}\}$$

is a metric on $Q^{\mathbf{N}}$. Consider a map $f\colon Q \to Q^{\mathbf{N}}$ that restricts to a Z-embedding on some compactum K and a sequence $Q \supset A_1 \supset A_2 \supset \cdots$ of elements of $\mathscr{M}$. We may assume that f is a Z-embedding. Write $Q \setminus K$ as a union of compacta $(F_i)_{i=0}^{\infty}$ with $F_i \subset \operatorname{int}(F_{i+1})$ and $F_0 = \varnothing$. Let $\varepsilon > 0$ and define the decreasing sequence $\varepsilon_i = \min\{2^{-i}\varepsilon, \frac{1}{2}\rho(f(K), f(F_i))\}$. Consider now the n-th component $f_n\colon Q \to Q$ of f. We shall construct a sequence $\alpha_0, \alpha_1, \ldots$ of functions from Q into Q with the following properties:

$$\begin{gathered}
\rho_n(\alpha_i, \alpha_{i-1}) < \varepsilon_{i+1}, \qquad \alpha_i|F_{i-1} = \alpha_{i-1}|F_{i-1}, \\
\alpha_i|Q \setminus F_{i+1} = f_n|Q \setminus F_{i+1}, \qquad \alpha_i|F_i \text{ is a Z-embedding}, \\
\alpha_i^{-1}(X) \cap F_i = A_n \cap F_i.
\end{gathered}$$

Put $\alpha_0 = f_n$ and assume that α_i has been constructed. Using the strong $\mathscr{M}$-universality of X we find a Z-embedding $\beta\colon F_{i+1} \to Q$, close to $\alpha_i|F_{i+1}$, with $\beta|F_i = \alpha_i|F_i$ and $\beta^{-1}(X) = A_n \cap F_{i+1}$. Extend β to a map $\alpha_{i+1}\colon Q \to Q$ that restricts to f on $Q \setminus F_{i+2}$.

The α_i's obviously form a Cauchy sequence and we can define the continuous map $g_n = \lim_{i\to\infty} \alpha_i$. One may verify that g_n has the following properties:

$$\begin{gathered}
\rho_n(g_n, f_n) < \varepsilon, \\
\text{if } x \in F_{i+1} \setminus F_i \text{ then } \rho_n(g_n(x), f_n(x)) < \rho(f(K), f(F_{i+1})), \\
g_n|K = f_n|K, \\
g_n|F_i \text{ is a Z-embedding for every } i, \\
g_n^{-1}(X) \setminus K = A_n \setminus K.
\end{gathered}$$

Define $g = (g_n)_n\colon Q \to Q^{\mathbf{N}}$. Note that g is one-to-one and hence an embedding. The set $g(Q)$ is contained in the σZ-set $f(K) \cup \bigcup_{i=0}^{\infty} g_1(F_i) \times Q \times Q \times \cdots$ and is therefore a Z-set. The maps f and g are ε-close and $f|K = g|K$. Let $x \in Q \setminus K$. If x is an element of A_n then $x \in \bigcap_{j=1}^{n} A_j$. Consequently, we have $g_j(x) \in X$ for $j = 1, 2, \ldots, n$. This means that $g(x) \in S_n(X) \subset S_n'(X) \subset S_n''(X)$. On the other hand, if $g(x)$ is an element of $S_n''(X)$ then $g_j(x) \in X$ for some $j \geq n$ and hence $x \in A_j \subset A_n$. This completes the proof.

Consider now the pseudoboundary B of the Hilbert cube. This is an $\mathscr{F}_\sigma$-absorber in Q. The conditions (1) and (2) of the definition of absorbing system are trivially satisfied by $S(B)$, $S'(B)$ and $S''(B)$, so we have:

COROLLARY 3.2. *The sequences* $S(B)$, $S'(B)$ *and* $S''(B)$ *are* $\mathscr{F}_\sigma$-*absorbing and hence they are homeomorphic in* $Q^{\mathbf{N}}$. *Moreover,* $B^{\mathbf{N}}$ *and* $S'_\infty(B)$ *are* $\mathscr{F}_{\sigma\delta}$-*absorbers.*

Consider the σ Z-set

$$\sigma = \{x \in Q : x_i = 0 \text{ for all but finitely many } i\}.$$

It is well known that σ is homeomorphic to l^2_f and that it is a so-called fd-capset in Q or, in our terminology, an absorber for the strongly countable dimensional σ-compacta. It is easily verified by juggling coordinates that the system $S(\sigma)$ is homeomorphic to $S(B)$ in $Q^{\mathbf{N}}$ and hence $\mathscr{F}_\sigma$-absorbing. Observe that the following systems are all homeomorphic: $S(\sigma)$ in $Q^{\mathbf{N}}$, $S(\sigma \times I)$ in $(Q \times I)^{\mathbf{N}}$, $S(\sigma) \times I^{\mathbf{N}}$ in $Q^{\mathbf{N}} \times I^{\mathbf{N}}$, $S(\sigma) \times Q^{\mathbf{N}}$ in $Q^{\mathbf{N}} \times Q^{\mathbf{N}}$, $S(\sigma \times Q)$ in $(Q \times Q)^{\mathbf{N}}$ and finally $S(B)$ in $Q^{\mathbf{N}}$.

We can take this one step further:

COROLLARY 3.3. *If* Y *is an* $\mathscr{F}_{\sigma\delta}$-*absorber in* Q *then the sequences* $S(Y)$, $S'(Y)$ *and* $S''(Y)$ *are* $\mathscr{F}_{\sigma\delta}$-*absorbing and hence they are homeomorphic in* $Q^{\mathbf{N}}$. *Moreover,* $Y^{\mathbf{N}}$ *and* $S'_\infty(Y)$ *are also* $\mathscr{F}_{\sigma\delta}$-*absorbers.*

4. The space of infinite-dimensional compacta. In this section we prove Theorem 1.1. The following lemma is easily verified.

LEMMA 4.1. *If* X *and* Y *are compact spaces and if* $F : X \to 2^Y$ *is continuous then* $G(A) = \bigcup\{F(a) : a \in A\}$ *defines a continuous map from* 2^X *into* 2^Y.

PROPOSITION 4.2. *The sequence* $(\mathrm{Dim}_{\geq k})_{k=1}^{\infty}$ *is reflexively universal in* 2^Q.

Proof. Let $F : 2^Q \to 2^Q$ be a map and let K be a closed subset of 2^Q such that $F|K$ is a Z-embedding. We may assume that F is a Z-embedding. Let $\varepsilon : 2^Q \to I$ be a map with the properties: $\varepsilon^{-1}(0) = F(K)$ and $\varepsilon(A) \leq d(A, F(K))/4$ for each $A \in 2^Q$. According to Curtis [5] the finite sets in 2^Q contain an fd-capset and hence there exists a deformation H_t of 2^Q such that $H_0 = 1$ and $H_t(A)$ is finite for $t > 0$ and $A \in 2^Q$. We may assume, moreover, that $d(H_t, 1) \leq 2t$ and that $H_t(A) \subset [0, 1-t]^{\mathbf{N}}$ for every t and A.

We shall use the vector addition and scalar multiplication operations that Q inherits from $\mathbf{R}^{\mathbf{N}}$. Define the homotopy $\alpha_t\colon 2^Q \to 2^Q$ by

$$\alpha_t(A) = \{0\} \cup \bigcup_{n=1}^{\infty} \{\tfrac{t}{n}\} \times \tfrac{t}{n}\vec{A},$$

where $\vec{A}$ is the subset of $\prod_{i=2}^{\infty} I$ that is obtained from A by a coordinate shift. Note that $\alpha_t(A) \subset [0, t]^{\mathbf{N}}$ and that $\alpha_0(A) = \{0\}$. The map $G\colon 2^Q \to 2^Q$ that approximates F is defined by

$$G(A) = H_{\varepsilon(F(A))}(F(A)) + \alpha_{\varepsilon(F(A))}(A).$$

The function G is continuous by Lemma 4.1 and the continuity of the homotopies H and α. Observe that $d(G(A), F(A)) \leq 3\varepsilon(F(A))$ for every $A \in 2^Q$. If $A \in K$ then $\varepsilon(F(A)) = 0$ and hence G restricts to F on K. Let A be an element of $2^Q \setminus K$. Then $t = \varepsilon(F(A)) > 0$ and hence $H_t(F(A))$ is finite. So $G(A)$ is a finite union of translates of $\alpha_t(A)$ and consequently a union of a finite set and a countable collection of copies of A. This means that G preserves dimension and

$$G^{-1}(\mathrm{Dim}_{\geq k}) \setminus K = \mathrm{Dim}_{\geq k} \setminus K.$$

We shall now show that G is one-to-one. The restriction of G to K is obviously one-to-one. If $A \in 2^Q \setminus K$ then $d(G(A), F(A)) \leq 3\varepsilon(F(A)) < d(F(K), F(A))$ and hence $G(A)$ is not in $G(K) = F(K)$. For the remaining case let $A, B \in 2^Q \setminus K$ such that $G(A) = G(B)$. Let $\pi\colon Q \to I$ be the projection onto the first coordinate and define the positive numbers $r = \varepsilon(F(A))$ and $t = \varepsilon(F(B))$. Select a point $y = (a, x) \in G(A) = G(B)$ such that $a = \min(\pi(G(A))) = \min(\pi(G(B)))$. Note that y is an element of both $H_r(F(A))$ and $H_t(F(B))$. Since the latter sets are finite we can define $\lambda > 0$ as one half of the distance of y towards the other points in $H_r(F(A)) \cup H_t(F(B))$.

Let m and n be the first numbers that satisfy $\frac{r}{m} \leq \lambda$ and $\frac{t}{n} \leq \lambda$. We now have:

$$\begin{aligned}(\{y\} + [0, \lambda]^{\mathbf{N}}) \cap G(A) &= \{y\} \cup \bigcup_{i=m}^{\infty} \{a + \tfrac{r}{i}\} \times (x + \tfrac{r}{i}\vec{A}) \\ = (\{y\} + [0, \lambda]^{\mathbf{N}}) \cap G(B) &= \{y\} \cup \bigcup_{i=n}^{\infty} \{a + \tfrac{t}{i}\} \times (x + \tfrac{t}{i}\vec{B}).\end{aligned}$$

This implies:

$$\{a + \tfrac{r}{m}\} \times (x + \tfrac{r}{m}\vec{A}) = \{a + \tfrac{t}{n}\} \times (x + \tfrac{t}{n}\vec{B}).$$

This means that $\frac{r}{m} = \frac{t}{n}$ and $\frac{r}{m}\vec{A} = \frac{t}{n}\vec{B}$ and hence that $A = B$. So G is one-to-one and therefore an embedding.

Observe that $\pi(G(A))$ is countable if $A \in 2^Q \setminus K$ so $G(A)$ is nowhere dense in Q. Since $D_t(A) = \{x \in Q : d(x, A) \leq t\}$ is a deformation of Q through the complement of $G(2^Q \setminus K)$, we have that $G(2^Q \setminus K)$ is a σ Z-set. Consequently, $G(2^Q) \subset F(K) \cup G(2^Q \setminus K)$ is a Z-set and G is a Z-embedding. This completes the proof.

Observing that G preserves many other properties we find for instance:

COROLLARY 4.3. *The sequence* $(\overline{\mathrm{Dim}_{\geq k}})_{k=1}^{\infty}$ *is reflexively universal in* 2^Q.

COROLLARY 4.4. *The sequence consisting of the collections of compacta of cohomological dimension not less than* k *is reflexively universal in* 2^Q.

COROLLARY 4.5. *The transfinite sequence* $\{A \in 2^Q : \mathrm{ind}(A) \geq \alpha\}_{\alpha < \omega_1}$ *is reflexively universal in* 2^Q.

THEOREM 4.6. *The sequence* $(\mathrm{Dim}_{\geq k})_{k=1}^{\infty}$ *is* $\mathscr{F}_\sigma$*-absorbing in* 2^Q. *Consequently,* Dim_∞ *is an* $\mathscr{F}_{\sigma\delta}$*-absorber.*

Proof. Let $k, n \in \mathbf{N}$ and define

$$\mathscr{G}_n = \{A \in 2^Q : \text{there is in } Q \text{ a finite open cover of } A \text{ with mesh} \leq 1/n \text{ and order} \leq k\}.$$

Obviously, $\mathscr{G}_n$ is an open subset of 2^Q. Note that $\mathrm{Dim}_{\geq k} = Q \setminus \bigcap_{n=1}^{\infty} \mathscr{G}_n$ is therefore an F_σ-set. According to Curtis [5] the finite sets in 2^Q contain an fd-capset and hence $\mathrm{Dim}_{\geq 1}$ is a σ Z-set.

In view of Proposition 4.2 it suffices to show that the system is $\mathscr{F}_\sigma$-universal. The space $\mathrm{Dim}_1(I)$ is an $\mathscr{F}_\sigma$-absorber in the Hilbert cube 2^I. This can be found essentially in Kroonenberg [10] if we note that $H_t(A) = \{x \in I : d(x, A) \leq t\}$ is a deformation of 2^I through $\mathrm{Dim}_1(I)$, see also [1]. So the pair $(2^I, \mathrm{Dim}_1(I))$ is homeomorphic to (Q, B). Corollary 3.2 now guarantees that $S'(\mathrm{Dim}_1(I))$ is an $\mathscr{F}_\sigma$-absorbing sequence in $(2^I)^{\mathbf{N}}$. Define the embedding $\alpha : (2^I)^{\mathbf{N}} \to 2^Q$ by $\alpha((P_i)_{i=1}^{\infty}) = \prod_{i=1}^{\infty} P_i$. Since $\prod_{i=1}^{\infty} P_i$ is k-dimensional if and only if precisely k of the P_i's are in $\mathrm{Dim}_1(I)$, we have

$$\alpha^{-1}(\mathrm{Dim}_{\geq k}) = S'_k(\mathrm{Dim}_1(I)).$$

The sequence $\mathrm{Dim}_{\geq k}$ is then $\mathscr{F}_\sigma$-universal because $S'(\mathrm{Dim}_1(I))$ is.

We find Theorem 1.1 by combining Theorem 2.1, Corollary 3.2 and Theorem 4.6. The fact that $(2^Q, (\mathrm{Dim}_{\geq k})_{k=1}^\infty)$ is homeomorphic to $(Q^{\mathbf{N}}, S(B))$ means that there exists a homeomorphism $\alpha: 2^Q \to Q^{\mathbf{N}}$ such that

$$\alpha(\mathrm{Dim}_{\geq k}) = \underbrace{B \times \cdots \times B}_{k \text{ times}} \times Q \times Q \times \cdots .$$

This implies that $\alpha(\mathrm{Dim}_\infty) = B^{\mathbf{N}}$, which space is homeomorphic to $(l^2_f)^\omega$. Observe that in view of the remark following Corollary 3.2 it is also possible to find an α' with

$$\alpha'(\mathrm{Dim}_{\geq k}) = \underbrace{\sigma \times \cdots \times \sigma}_{k \text{ times}} \times Q \times Q \times \cdots .$$

Comparing $(2^Q, \mathrm{Dim}_{\geq k})$ with $(Q^{\mathbf{N}}, S''(B))$ we find part (b) of Theorem 1.1. There exists a homeomorphism β from 2^Q onto $Q^{\mathbf{N}}$ such that for every $k \in \{0, 1, 2, \ldots\}$,

$$\beta(\mathrm{Dim}_{\leq k}) = \underbrace{Q \times \cdots \times Q}_{k \text{ times}} \times s \times s \times \cdots .$$

Note that

$$\beta(\mathrm{Dim}_k) = \underbrace{Q \times \cdots \times Q}_{k-1 \text{ times}} \times B \times s \times s \times \cdots$$

and hence the pair $(\mathrm{Dim}_{\leq k}, \mathrm{Dim}_k)$, $0 < k < \infty$, is homeomorphic to $(Q \times s, B \times s)$, i.e., Dim_k is a so-called Z-absorber in the topological Hilbert space $\mathrm{Dim}_{\leq k}$.

Let $\mathrm{cDim}_{\geq k}$ stand for all elements of 2^Q with cohomological dimension at least k with respect to for instance the group $\mathbf{Z}$.

Question. *Is* $\mathrm{cDim}_{\geq k}$ *σ-compact?*

Observe that it follows from the proof of Theorem 4.6 that the sequence $\mathrm{cDim}_{\geq k}$ is $\mathscr{F}_\sigma$-universal. If the answer to the question is yes then we have in view of Corollary 4.4 and the fact $\mathrm{cDim}_{\geq 1} = \mathrm{Dim}_{\geq 1}$ that $\mathrm{cDim}_{\geq k}$ is $\mathscr{F}_\sigma$-absorbing and cDim_∞ is homeomorphic to $B^{\mathbf{N}}$.

5. Uniformly $\geq k$-dimensional compacta in 2^Q. This section is devoted to the proof of Theorem 1.2. Consider the following decreasing sequence of subsets of $(2^Q)^{\mathbf{N}}$:

$$X_k = \{P \in (2^Q)^{\mathbf{N}} : P_i \in \mathrm{Dim}_{\geq k} \text{ for infinitely many } i\}.$$

LEMMA 5.1. *The sequence* $(X_k)_{k=1}^{\infty}$ *is* $\mathscr{F}_{\sigma\delta}$*-universal.*

Proof. Let $A_1 \supset A_2 \supset \cdots$ be a sequence of $F_{\sigma\delta}$-sets in Q. Choose σ-compact sets A_k^n such that $A_k^{n+1} \cup A_{k+1}^n \subset A_k^n$ and $A_k = \bigcap_{n=1}^{\infty} A_k^n$. Since $(\mathrm{Dim}_{\geq k})_{k=1}^{\infty}$ is $\mathscr{F}_\sigma$-universal, Theorem 4.6, there exist embeddings $f_n : Q \to 2^Q$ such that $f_n^{-1}(\mathrm{Dim}_{\geq k}) = A_k^n$. Put $f = (f_n)_n : Q \to (2^Q)^{\mathbf{N}}$. If $x \in A_k$ then $x \in A_k^n$ for all n. So $f_n(x) \in \mathrm{Dim}_{\geq k}$ for all n and hence $f(x) \in X_k$. If $x \notin A_k$ then $x \notin A_k^j$ for some j, so $x \notin A_k^n$ for all $n \geq j$. Consequently, $f_n(x) \notin \mathrm{Dim}_{\geq k}$ for all $n \geq j$ and $f(x) \notin X_k$.

REMARK. One may use the method of Theorem 3.1 to show that $(X_k)_k$ is in fact $\mathscr{F}_{\sigma\delta}$-absorbing in $(2^Q)^{\mathbf{N}}$.

PROPOSITION 5.2. *The sequence* $(\overline{\mathrm{Dim}}_{\geq k})_{k=1}^{\infty}$ *is strongly* $\mathscr{F}_{\sigma\delta}$*-universal.*

Proof. In view of Corollary 4.3 it suffices to show that the sequence is $\mathscr{F}_{\sigma\delta}$-universal. We shall prove that the system X_k can be embedded in $\overline{\mathrm{Dim}}_{\geq k}$.

Let G stand for the compact, multiplicative subspace $\{0\} \cup \{2^{-m} : m = 1, 2, \ldots\}$ of I. According to Curtis [5] there exists a deformation $H_t : 2^Q \to 2^Q$ such that $H_0 = 1$ and $H_t(A)$ is finite if $t > 0$. Let $P = (P_m)_{m=1}^{\infty}$ be an element of $(2^Q)^{\mathbf{N}}$. We define the continuous function $F : G \times (2^Q)^{\mathbf{N}} \to 2^Q$ by

$$F_0(P) = \{0\} \quad \text{and} \quad F_{2^{-m}}(P) = 2^{-m} P_m \cup \{0\}.$$

We shall define inductively a sequence of compacta $(A_n)_{n=1}^{\infty}$ such that

$$A_n \subset (G \times Q)^{n-1} \times G,$$

i.e., the n odd coordinates are in G and the $n-1$ even ones in Q. Put $A_1(P) = G$ and

$$A_{n+1}(P) = \bigcup \{\{(x, a)\} \times H_{ab}(F_a(P)) \times \{ab\} : (x, a) \in A_n(P) \text{ and } b \in G\}.$$

Here $(x, a) \in A_n$ means that $x \in (G \times Q)^{n-1}$ and $a \in G$. Note that since $ab < a$ the odd components of the points in A_n form a decreasing sequence. Applying Lemma 4.1 we find that every A_n is a compactum that depends continuously on P. We identify each A_n with its copy $A_n \times \{(0, 0, \ldots)\}$ in $(G \times Q)^{\mathbf{N}} \subset (I \times Q)^{\mathbf{N}}$. The Hilbert cube $Q' = (I \times Q)^{\mathbf{N}}$ is equipped with the metric $\rho = \max_{i \in \mathbf{N}} \rho_i$, where ρ_{2j-1} is a standard metric on I that is bounded by 2^{-2j+1} and ρ_{2j} is a standard metric on Q that is bounded by 2^{-2j}. Observe that $\pi_n(A_{n+1}) = A_n$, where π_n is the projection from Q' onto $(I \times Q)^{n-1} \times I$. This implies that $\rho(\pi_n, 1) \leq 2^{-2n}$ and $\rho(A_n, A_{n+1}) \leq 2^{-2n}$ so that $(A_n(P))_{n=1}^{\infty}$ is a Cauchy sequence of maps. So $\alpha(P) = \lim_{n\to\infty} A_n(P)$ defines a continuous map from $(2^Q)^{\mathbf{N}}$ into $2^{Q'}$. In addition, we find that $\alpha(P) = \bigcap_{n=1}^{\infty} \pi_n^{-1}(A_n)$. Since 0 is an element of every $F_t(P)$ we have $A_n \subset A_{n+1}$. This implies that $\alpha(P)$ is the closure of $Y = \bigcup_{n=1}^{\infty} A_n$ in Q'.

We show by induction that

$$A'_n = \{(x, a) \in A_n : a \neq 0\}$$

is countable. This is obviously true for A'_1. Let (x, a, p, ab) be an element of A'_{n+1}. So $ab \neq 0$, $(x, a) \in A_n$ and $p \in H_{ab}(F_a(P))$. This implies $a \neq 0$ and $(x, a) \in A'_n$ and hence we have:

$$A'_{n+1} = \bigcup \{\{(x, a)\} \times H_{ab}(F_a(P)) \times \{ab\} : \\ (x, a) \in A'_n \text{ and } b \in G \setminus \{0\}\}.$$

This is a countable union of finite sets because $H_{ab}(F_a(P))$ is finite if $ab \neq 0$. Consequently, the set A'_{n+1} is countable.

Assume that $P \notin X_k$. We shall prove that 0 has a neighbourhood in $\alpha(P)$ with dimension less than k. Since $P \notin X_k$ there exists an m such that $\dim(P_i) < k$ for all $i \geq m$. So if we put $c = 2^{-m}$ then $\dim(F_a(P)) < k$ for $a \leq c$. Let C consist of all points in Q' whose first component is less than or equal to c. We shall prove inductively that $\dim(A_n \cap C) < k$. Obviously, we have $\dim(A_1 \cap C) = 0$. Assume that $\dim(A_n \cap C) < k$ and consider

$$A_{n+1} \cap C = \bigcup \{\{(x, a)\} \times H_{ab}(F_a(P)) \times \{ab\} : \\ (x, a) \in A_n \cap C \text{ and } b \in G\}.$$

If $a = 0$ then $ab = 0$ and $H_{ab}(F_a(P)) = \{0\}$. Consequently, we have:

$$A_{n+1} \cap C = (A_n \cap C) \cup \bigcup \{\{(x, a)\} \times H_{ab}(F_a(P)) \times \{ab\} : \\ (x, a) \in A'_n \cap C \text{ and } b \in G\}.$$

Note that the $H_{ab}(F_a(P))$ in this expression is either finite or homeomorphic to $F_a(P)$. Since the odd components of points form a decreasing sequence in G we have that $a \leq c$ whenever (x, a) is a point in $A_n \cap C$. So every $F_a(P)$ is less than k-dimensional. Since A'_n is countable, the set $A_{n+1} \cap C$ is a countable union of $< k$-dimensional compacta and therefore $\dim(A_{n+1} \cap C) < k$. Note that $\alpha(P) \cap C = \bigcap_{n=1}^{\infty} \pi_n^{-1}(A_n \cap C)$. Since $\pi_n^{-1}(A_n \cap C)$ is the product of a $< k$-dimensional compactum and a Hilbert cube of diameter $\leq 2^{-2n}$, there is for every n an open cover of $\pi_n^{-1}(A_n \cap C)$ (and hence of $\alpha(P) \cap C$) with mesh $\leq 2^{-2n}$ and order $\leq k$. Consequently, we have $\dim(\alpha(P) \cap C) < k$ and

$$\alpha(P) \notin \overline{\mathrm{Dim}}_{\geq k}(Q').$$

Consider now the case $P \in X_k$. This means that $\dim(F_a(P)) \geq k$ for infinitely many $a \in G$. Let $(x, 0) \in A_n$. We show by induction that A_{n+1} is at least k-dimensional at this point, i.e., every neighbourhood of the point in A_{n+1} has dimension no less than k. First, consider $0 \in A_1$. We have:

$$A_2 = \bigcup_{a, b \in G} \{a\} \times H_{ab}(F_a(P)) \times \{ab\}.$$

Selecting $b = 0$ we find

$$\lim_{a \to 0} \{a\} \times H_0(F_a(P)) \times \{0\} = \lim_{a \to 0} \{a\} \times F_a(P) \times \{0\} = \{0\}$$

and hence A_2 is $\geq k$-dimensional at 0.

Assume that the induction hypothesis is valid for points $(x, 0)$ in A_n. If $(y, 0) \in A_{n+1}$ then $y = (x, a, p)$, where $(x, a) \in A_n$ and $p \in H_0(F_a(P)) = F_a(P)$. If $a = 0$ then $F_a(P) = \{0\}$ and $p = 0$. This means that $(y, 0) = (x, 0, 0, 0) \in A_n$ and by induction A_{n+1} and therefore A_{n+2} are $\geq k$-dimensional at the point. If $a \neq 0$ then for $b, c \in G$ we have:

$$\{(x, a)\} \times H_{ab}(F_a) \times \{ab\} \times H_{abc}(F_{ab}) \times \{abc\} \subset A_{n+2},$$

where we denote $F_a(P)$ simply by F_a. Since $\lim_{b \to 0} H_{ab}(F_a) = H_0(F_a) = F_a$ in 2^Q we can find points $p_b \in H_{ab}(F_a)$ such that $\lim_{b \to 0} p_b = p$. Selecting $c = 0$ we find

$$\lim_{b \to 0} \{(x, a, p_b, ab)\} \times F_{ab} \times \{0\} = \{(x, a, p, 0, 0, 0)\}.$$

Since F_{ab} is $\geq k$-dimensional for infinitely many b's we have that A_{n+2} is $\geq k$-dimensional at $(y, 0, 0, \ldots) = (x, a, p, 0, 0, \ldots)$. This completes the induction.

If x is an element of A_n then $(x, 0, 0)$ is in A_{n+1} and hence A_{n+2} is $\geq k$-dimensional at x. Consequently, the set $Y = \bigcup_{n=1}^{\infty} A_n$

is $\geq k$-dimensional at each of its points. So its closure $\alpha(P)$ is an element of $\overline{\mathrm{Dim}}_{\geq k}(Q')$ and we have:

$$\alpha^{-1}(\overline{\mathrm{Dim}}_{\geq k}(Q')) = X_k.$$

This does not quite complete the proof of Proposition 5.2 since α is not one-to-one. This can easily be fixed, however. Define the map β from $(2^Q)^{\mathbf{N}}$ into the hyperspace of $Q'' = I \times Q' \times \prod_{i=1}^{\infty} Q$ by

$$\beta(P) = (\{0\} \times \alpha(P) \times \{(0, 0, \ldots)\}) \cup (\{1\} \times Q' \times \prod_{i=1}^{\infty} P_i).$$

The map β is obviously one-to-one and hence an embedding. Note that $\beta(P)$ is a topological sum of a copy of $\alpha(P)$ and a uniformly infinite-dimensional space, so we retain the property

$$\beta^{-1}(\overline{\mathrm{Dim}}_{\geq k}(Q'')) = X_k.$$

We may conclude that $(\overline{\mathrm{Dim}}_{\geq k}(Q''))_{k=1}^{\infty}$ is $\mathscr{F}_{\sigma\delta}$-universal just as $(X_k)_{k=1}^{\infty}$.

THEOREM 5.3. *The sequence* $(\overline{\mathrm{Dim}}_{\geq k})_{k=1}^{\infty}$ *is* $\mathscr{F}_{\sigma\delta}$*-absorbing and* $\overline{\mathrm{Dim}}_{\infty}$ *is an* $\mathscr{F}_{\sigma\delta}$*-absorber in* 2^Q.

Proof. Note that $\overline{\mathrm{Dim}}_{\geq 1}$ is contained in the σ Z-set $\mathrm{Dim}_{\geq 1}$. It remains to be shown that every $\overline{\mathrm{Dim}}_{\geq k}$ is in $\mathscr{F}_{\sigma\delta}$. Let $\{O_i : i \in \mathbf{N}\}$ be a countable open basis for the topology of Q and let $k \in \mathbf{N}$. Write every O_i as a countable union of compacta $F_i^1 \subset F_i^2 \subset \cdots$. Define the collections

$$\mathscr{G}_i^j = \{A \in 2^Q : \text{there is in } Q \text{ an finite open cover } \mathscr{U} \text{ of } A \cap F_i^j \text{ with mesh} \leq 1/j \text{ and order} \leq k\}.$$

If $A \in \mathscr{G}_i^j$ and $\mathscr{U}$ is such a cover then put $\varepsilon = \rho(A, F_i^j \setminus \bigcup \mathscr{U})$. Observe that if $\rho(A, B) < \varepsilon$ then $B \cap F_i^j$ is also covered by $\mathscr{U}$ and hence $\mathscr{G}_i^j$ is open in 2^Q. So $\mathscr{G}_i = \bigcap_{j=1}^{\infty} \mathscr{G}_i^j$ is a G_δ-set. Since a countable union of $< k$-dimensional compacta is again $< k$-dimensional one easily verifies that an element A of 2^Q is in $\mathscr{G}_i$ if and only if $\dim(A \cap O_i) < k$. The collection $\mathscr{G}_i' = \mathscr{G}_i \setminus \{A \in 2^Q : A \cap O_i = \varnothing\}$ is obviously also G_δ. Observe that $\bigcup_{i=1}^{\infty} \mathscr{G}_i'$ is precisely the complement of $\overline{\mathrm{Dim}}_{\geq k}$ in 2^Q. This shows that $\overline{\mathrm{Dim}}_{\geq k}$ is in $\mathscr{F}_{\sigma\delta}$.

We find Theorem 1.2 by combining Theorem 2.1, Corollary 3.3 and Theorem 5.3. If Y is an $\mathscr{F}_{\sigma\delta}$-absorber in Q then there exists a homeomorphism α from 2^Q onto $Q^{\mathbf{N}}$ such that for every $k \in \{0, 1, 2, \ldots\}$,

$$\alpha(\overline{\mathrm{Dim}}_{\geq k}) = \underbrace{Y \times \cdots \times Y}_{k \text{ times}} \times Q \times Q \times \cdots .$$

Note that $\overline{\mathrm{Dim}}_{\geq k}$, $0 < k \leq \infty$, is an $\mathscr{F}_{\sigma\delta}$-absorber and hence homeomorphic to $B^{\mathbf{N}}$ and $(l_f^2)^\omega$.

6. Function spaces in the topology of pointwise convergence. In this section the Hilbert cube Q is represented by $\widehat{\mathbf{R}}^{\mathbf{N}}$, where $\widehat{\mathbf{R}}$ stands for the compactification $[-\infty, \infty]$. Consequently, $\mathbf{R}^{\mathbf{N}}$ is the pseudointerior of Q. If X is countable metric space then $C_p(X)$ denotes the space of continuous, realvalued functions on X endowed with the topology of pointwise convergence. Define the following subspaces of $\mathbf{R}^{\mathbf{N}}$:

$$c_0 = \left\{x \in \mathbf{R}^{\mathbf{N}} : \lim_{i \to \infty} x_i = 0\right\}$$

and for $n \in \mathbf{N}$

$$\Sigma_n = \{x \in \mathbf{R}^{\mathbf{N}} : |x_i| \leq 2^{-n} \text{ for all but finitely many } i\}.$$

Observe that $\Sigma = (\Sigma_n)_n$ is a decreasing sequence of σ Z-sets in Q with the property that its intersection is c_0. The aim of this section is to show that c_0 and $C_p(X)$ are $\mathscr{F}_{\sigma\delta}$-absorbers in the Hilbert cubes $\widehat{\mathbf{R}}^{\mathbf{N}}$ respectively $\widehat{\mathbf{R}}^X$. This is an improvement over the result of Dobrowolski, Gul'ko and Mogilski **[8]** and, independently, Cauty **[3]** that c_0 and $C_p(X)$ are homeomorphic to $(l_f^2)^\omega$.

PROPOSITION 6.1. *The system Σ is $\mathscr{F}_\sigma$-universal in Q.*

Proof. We shall use the following fact: if A is an $\mathscr{F}_\sigma$-absorber in Q and A' is a σ Z-set then for every σ-compactum C in Q there is an embedding $f\colon Q \to Q$ such that $f^{-1}(A) = C$ and $f(Q \setminus C) \cap A' = \varnothing$. This can be seen as follows. The proof of Theorem 2.1 shows that if $A_1 \supset A_2$ is an $\mathscr{F}_\sigma$-absorbing system in Q then there is a homeomorphism $h\colon Q \to Q$ such that $h(A) = A_2$ and $h(A') \subset A_1$. Such a system exists by Corollary 3.2 and it has the required property.

Let $A_1 \supset A_2 \supset \cdots$ be a sequence of σ-compacta in Q. Let α be a bijection from $\mathbf{N} \times \mathbf{N}$ onto $\mathbf{N}$ and define $N_i = \{\alpha(i, j) : j \in \mathbf{N}\}$. For every $i \in \mathbf{N}$ define the Hilbert cube $Q_i = [-2^{-i+1}, 2^{-i+1}]^{N_i}$. It is easily verified with the capset characterization theorem in Curtis **[4]** that

$$C_i = \{x \in Q_i : |x_{\alpha(i,j)}| \leq 2^{k-j} \text{ for some } k\}$$

is an $\mathscr{F}_\sigma$-absorber in Q_i. Observe that for every $x \in C_i$ we have $\lim_{j\to\infty} x_{\alpha(i,j)} = 0$. Define in Q_i the σ Z-set

$$D_i = \{x \in Q_i : |x_{\alpha(i,j)}| \leq 2^{-i} \text{ for all but finitely many } j\}.$$

Let $f_i \colon Q \to Q_i$ be an embedding such that $f_i^{-1}(C_i) = A_i$ and $f_i(Q \setminus A_i)$ does not meet D_i. Consider the embedding $f = (f_i)_{i\in\mathbf{N}} \colon Q \to \prod_{i=1}^{\infty} Q_i \subset Q$. Let $x \in A_n$. If $i > n$ then we have $f_i(x) \in Q_i$ and hence all components of $f_i(x)$ are in $[-2^{-n}, 2^{-n}]$. If $i \leq n$ then we have $x \in A_i$ and hence $f_i(x) \in C_i$. Note that only finitely many components of $f_i(x)$ are outside $[-2^{-n}, 2^{-n}]$ and hence only finitely many components of $f(x)$ are outside this interval. This means that $f(x)$ is an element of Σ_n. If $x \notin A_n$ then we have $f_n(x) \notin D_n$. This means that infinitely many components of $f_n(x)$ have absolute value greater than 2^{-n} and hence $f(x) \notin \Sigma_n$. So we may conclude that $f^{-1}(\Sigma_n) = A_n$.

A subset A is *locally homotopy negligible in* X if for every map $f \colon M \to X$ from an absolute neighbourhood retract M and for every open cover $\mathscr{U}$ of X there exists a homotopy $h \colon M \times [0, 1] \to X$ such that $\{h(\{x\} \times [0, 1])\}_{x\in M}$ refines $\mathscr{U}$, $h(x, 0) = f(x)$ and $h(M \times (0, 1]) \subset X \setminus A$. According to Theorem 2.4 in Toruńczyk **[13]** A is locally homotopy negligible if the above condition is satisfied for $M = Q$.

For a space X and $* \in X$ we define the weak cartesian product

$$W(X, *) = \{x \in X^{\mathbf{N}} : x_i = * \text{ for all but finitely many } i\}.$$

Let Γ be an ordered set. The following lemma is an adaptation to our needs of Proposition 3.2 in Dobrowolski, Gul′ko and Mogilski **[8]**.

LEMMA 6.2. *Let* $X = (X_\gamma)_{\gamma\in\Gamma}$ *be an order preserving system in* Q *such that* $Q \setminus \bigcap_{\gamma\in\Gamma} X_\gamma$ *is locally homotopy negligible in* Q *and let* $* \in \bigcap_{\gamma\in\Gamma} X_\gamma$. *Assume that there exists a homeomorphism* $\Phi \colon Q \to Q^{\mathbf{N}}$ *satisfying*

$$W(X_\gamma, *) \subset \Phi(X_\gamma) \subset X_\gamma^{\mathbf{N}}$$

for all $\gamma \in \Gamma$. *Then* X *is reflexively universal.*

Proof. Let $f \colon Q \to Q$ be a map that restricts to a Z-embedding on some compact set K and let $\varepsilon \colon Q \to (0, 1)$ be a continuous function. We can assume that $f(Q \setminus K) \subset \bigcap_{\gamma\in\Gamma} X_\gamma \setminus f(K)$. We choose a metric d on $Q^{\mathbf{N}}$ so that $d(x, x') \leq 2^{-k-2}$ if x and x' agree on the first k

coordinates. Let $\varepsilon'\colon Q^{\mathbf{N}} \to (0, 1)$ be a Lipschitz function such that if maps $f_1, f_2\colon Q \to Q^{\mathbf{N}}$ are ε'-close, then $\Phi^{-1}\circ f_1$ and $\Phi^{-1}\circ f_2$ are ε-close. Define $\delta\colon Q^{\mathbf{N}} \to [0, 1)$ by $\delta(x) = \min\{\varepsilon(x), d(x, \Phi\circ f(K))\}$. Let ϕ_i be the i-th component of the map $\Phi\circ f$. By local homotopy negligibility of $Q\backslash\bigcap_{\gamma\in\Gamma}X_\gamma$ there exists a homotopy $h\colon [0, 1]\times Q \to Q$ with $h(0, x) = x$, $h((0, 1]\times Q) \subset \bigcap_{\gamma\in\Gamma}X_\gamma$ and $h(1, x) = *$. Define a homotopy $H_k\colon [0, 1]\times Q \to Q$ by

$$H_k(t, x) = \begin{cases} h(2-2t, x), & \text{if } \frac{1}{2} \le t \le 1, \\ h_k(2t, x), & \text{if } 0 \le t \le \frac{1}{2}, \end{cases}$$

where $h_k\colon [0, 1]\times Q \to Q$ is a homotopy such that $h_k((0, 1]\times Q) \subset \bigcap_{\gamma\in\Gamma}X_\gamma$, $h_k(0, x) = \phi_k(x)$ and $h_k(1, x) = *$. For $x \in \{y \in Q : 2^{-k-1} \le \delta(\Phi\circ f(y)) \le 2^{-k}\}$, $k = 1, 2, \ldots$, define

$$\begin{aligned} f'(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_k(x), H_{k+1}(-k-\log_2\delta(\Phi\circ f(x)), x), \\ x, x, h(-k-\log_2\delta(\Phi\circ f(x)), x), *, *, \ldots) \end{aligned}$$

and extend f' on K by $f'|K = \Phi\circ f|K$. By the construction $f'\colon Q \to Q^{\mathbf{N}}$ is a continuous, one-to-one map which is ε'-close to $\Phi\circ f$. Moreover, $(f')^{-1}(X_\gamma^{\mathbf{N}})\backslash K = X_\gamma\backslash K$ and $f'(X_\gamma\backslash K) \subset W(X_\gamma, *)$. Hence, the map $g = \Phi^{-1}\circ f'$ is a Z-embedding which is ε-close to f and satisfies $g^{-1}(X_\gamma)\setminus K = X_\gamma\setminus K$.

Let $\Phi\colon \widehat{\mathbf{R}}^{\mathbf{N}} \to (\widehat{\mathbf{R}}^{\mathbf{N}})^{\mathbf{N}}$ be any map that simply rearranges coordinates. It is easily seen that with this map the system Σ satisfies the conditions of Lemma 6.2. So we have:

THEOREM 6.3. *The system Σ is $\mathscr{F}_\sigma$-absorbing and c_0 is an $\mathscr{F}_{\sigma\delta}$-absorber in Q.*

The space $\mathbf{R}_f^{\mathbf{N}}$ is defined as $W(\mathbf{R}, 0)$. This space is homeomorphic to l_f^2 and furthermore the pair $(\widehat{\mathbf{R}}^{\mathbf{N}}, \mathbf{R}_f^{\mathbf{N}})$ is homeomorphic to $(I^{\mathbf{N}}, \sigma)$. This means, according to §3 that there exists a homeomorphism $\alpha\colon Q \to Q^{\mathbf{N}}$ such that for every $k \in \mathbf{N}$,

$$\alpha(\Sigma_k) = \underbrace{\mathbf{R}_f^{\mathbf{N}} \times \cdots \times \mathbf{R}_f^{\mathbf{N}}}_{k \text{ times}} \times Q \times Q \times \cdots.$$

Consequently, c_0 is mapped by α onto $(\mathbf{R}_f^{\mathbf{N}})^{\mathbf{N}}$. In [**9**, Question 6.11] the following problem is posed. Does there exist a homeomorphism from $\mathbf{R}^{\mathbf{N}}$ onto $(\mathbf{R}^{\mathbf{N}})^{\mathbf{N}}$ that maps c_0 onto $(\mathbf{R}_f^{\mathbf{N}})^{\mathbf{N}}$? Such a homeomorphism cannot exist because c_0 is contained in the σ-compactum consisting of bounded sequences where as $(\mathbf{R}_f^{\mathbf{N}})^{\mathbf{N}}$ contains a copy of $\mathbf{R}^{\mathbf{N}}$ that is closed in $(\mathbf{R}^{\mathbf{N}})^{\mathbf{N}}$.

LEMMA 6.4. *If A is strongly $\mathscr{M}$-universal in Q and X is locally homotopy negligible in a compact absolute retract M then $A \times (M \setminus X)$ is strongly $\mathscr{M}$-universal in $Q \times M$.*

Proof. This is similar to the proof of Theorem 3.1. Let $f = (f_1, f_2)$ be a Z-embedding of Q in $Q \times M$. Let K and C be subsets of Q such that K is closed and C is an element of $\mathscr{M}$. Select a map $\varepsilon: Q \to I$ such that $\varepsilon^{-1}(0) = K$ and $\varepsilon(x) \leq \rho(f(x), f(K))$ for each $x \in Q$. Just as in the proof of Theorem 3.1 we can find a map $g_1: Q \to Q$ such that f_1 and g_1 are ε-close, $g_1^{-1}(A) \setminus K = C \setminus K$, $g_1|Q \setminus K$ is a one-to-one map whose range is a σ Z-set. Since X is locally homotopy negligible we can find a map $g_2: Q \to M$ such that f_2 and g_2 are ε-close and $g_2(Q \setminus K) \subset M \setminus X$. The map $g = (g_1, g_2)$ is a Z-embedding of Q into $Q \times M$ with $g|K = f|K$ and $g^{-1}(A \times (M \setminus X)) \setminus K = C \setminus K$.

THEOREM 6.5. *If X is a countable, nondiscrete metric space then $C_p(X)$ is an $\mathscr{F}_{\sigma\delta}$-absorber in $\widehat{\mathbf{R}}^X$.*

This means that there exists a homeomorphism $\beta: \widehat{\mathbf{R}}^X \to Q^{\mathbf{N}}$ such that $\beta(C_p(X)) = (\mathbf{R}_f^{\mathbf{N}})^{\mathbf{N}}$.

Proof. It is well known (and easily verified) that $C_p(X)$ is an element of $\mathscr{F}_{\sigma\delta}$. Let A be a convergent sequence in X. Observe that $\bigcup_{n=1}^{\infty}\{f \in \widehat{\mathbf{R}}^X : |f(a)| \leq n \text{ for every } a \in A\}$ is a σ Z-set that contains $C_p(X)$. It remains to be shown that $C_p(X)$ is strongly $\mathscr{F}_{\sigma\delta}$-universal.

We first prove this for the convergent sequence $\widehat{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. In $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{N}}$ we use the following arithmetic: $1/0 = \infty$ and $\infty + a = \infty$ if a is finite. Define the following continuous function from $\widehat{\mathbf{R}}$ into $\widehat{\mathbf{R}}^{\widehat{\mathbf{N}}}$:

$$\Psi(r)(n) = \operatorname{sign}(r) \min\{|r|, n\}.$$

Note that $\Psi(r)(n)$ is finite if $n \neq \infty$ and $\lim_{n\to\infty} \Psi(r)(n) = \Psi(r)(\infty) = r$. This means that $\Psi(\mathbf{R})$ is a subset of $C_p(\widehat{\mathbf{N}})$. If $f \in \widehat{\mathbf{R}}^{\mathbf{N}}$ then $\hat{f}$ is the extension of f over $\widehat{\mathbf{N}}$ that assigns 0 to ∞. It is easily seen that $\Phi(f, r) = \hat{f} + \Psi(r)$ is a well-defined map from $\widehat{\mathbf{R}}^{\mathbf{N}} \times \widehat{\mathbf{R}}$ onto $\widehat{\mathbf{R}}^{\widehat{\mathbf{N}}}$. Observing that $\Phi^{-1}(h) = (h - \Psi(h(\infty))|\mathbf{N}, h(\infty))$ we find that Φ is a homeomorphism. Note that $\Phi(c_0 \times \mathbf{R}) = C_p(\widehat{\mathbf{N}})$. According to Lemma 6.4 $c_0 \times \mathbf{R}$ is strongly $\mathscr{F}_{\sigma\delta}$-universal in $Q \times \widehat{\mathbf{R}}$ and hence $C_p(\widehat{\mathbf{N}})$ is strongly $\mathscr{F}_{\sigma\delta}$-universal in $\widehat{\mathbf{R}}^{\widehat{\mathbf{N}}}$.

We use a similar argument to reduce the problem for $C_p(X)$ to $C_p(\widehat{\mathbf{N}})$. Let d be a metric on X and let A be a convergent sequence in X. We may assume that $C_p(A)$ is strongly $\mathscr{F}_{\sigma\delta}$-universal in $\widehat{\mathbf{R}}^A$. Choose a retraction r from X onto A. The formula

$$\Psi(g)(x) = \operatorname{sign}(g(r(x))) \min\{|g(r(x))|, 1/d(x, r(x))\}$$

defines a continuous selection that extends every $g \in \widehat{\mathbf{R}}^A$ to an element of $\widehat{\mathbf{R}}^X$. The map Ψ has the following properties: $\Psi(g)|A = g$, $\Psi(g)|X \setminus A$ has its values in $\mathbf{R}$ and $\Psi(C_p(A)) \subset C_p(X)$. If $f \in \widehat{\mathbf{R}}^{X\setminus A}$ then $\hat{f}$ is the extension of f over X with zeros. As above it is easily seen that $\Phi(f, g) = \hat{f} + \Psi(g)$ is a well-defined map from $\widehat{\mathbf{R}}^{X\setminus A} \times \widehat{\mathbf{R}}^A$ onto $\widehat{\mathbf{R}}^X$ and a homeomorphism. Let $C_p(X, A)$ stand for $\{f|X \setminus A : f \in C_p(X) \text{ and } f|A = 0\}$ and note that $\Phi(C_p(X, A) \times C_p(A)) = C_p(X)$. It is easily seen that the complement of $C_p(X, A)$ in $\widehat{\mathbf{R}}^{X\setminus A}$ is locally homotopy negligible and hence Lemma 6.4 implies that $C_p(X)$ is strongly $\mathscr{F}_{\sigma\delta}$-universal in $\widehat{\mathbf{R}}^X$. This completes the proof of Theorem 6.5.

REFERENCES

[1] R. D. Anderson, D. W. Curtis and J. van Mill, *A fake topological Hilbert space*, Trans. Amer. Math. Soc., **272** (1982), 311–321.

[2] M. Bestvina and J. Mogilski, *Characterizing certain incomplete infinite-dimensional absolute retracts*, Michigan Math. J., **33** (1986), 291–313.

[3] R. Cauty, *L'espace des functions continues d'un espace métrique dénombrable*, preprint.

[4] D. W. Curtis, *Boundary sets in the Hilbert cube*, Topology Appl., **20** (1985), 201–221.

[5] ——, *Hyperspaces of finite sets as boundary sets*, Topology Appl., **22** (1986), 97–107.

[6] D. W. Curtis and R. M. Schori, *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math., **101** (1978), 19–38.

[7] J. J. Dijkstra, T. Dobrowolski, W. Marciszewski, J. van Mill and J. Mogilski, *Recent classification and characterization results in geometric topology*, Bull. Amer. Math. Soc. (N.S.), **22** (1990), 277–283.

[8] T. Dobrowolski, S. P. Gul'ko and J. Mogilski, *Function spaces homeomorphic to the countable product of l_2^f*, Topology Appl., **34** (1990), 153–160.

[9] T. Dobrowolski and J. Mogilski, *Problems on topological classification of incomplete metric spaces*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland, Amsterdam, 1990.

[10] N. Kroonenberg, *Pseudo-interiors of hyperspaces*, Compositio Math., **32** (1976), 113–131.

[11] J. van Mill, *Topological equivalence of certain function spaces*, Compositio Math., **63** (1987), 159-188.
[12] ——, *Infinite-Dimensional Topology, Prerequisites and Introduction*, North-Holland, Amsterdam, 1989.
[13] H. Toruńczyk, *Concerning locally homotopy negligible sets and characterization of l_2-manifolds*, Fund. Math., **101** (1978), 93–110.

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