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EVERY TOTALLY DISCONNECTED SEPARABLE METRIZABLE TOPOLOGICAL GROUP IS AN AUTOHOMEOMORPHISM GROUP

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De Groot proved that every group is the autohomeomorphism group of some metrizable space. A space is totally disconnected if every connected subset of it contains at most one point. We prove that every separable metrizable totally disconnected topological group is topologically isomorphic to the autohomeomorphism group of some separable metrizable space, when given the compact-open topology. It is known that, for example, the circle group cannot be realized in this way.

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Introduction

For every space X let $\mathscr{H}(X)$ be the group of all homeomorphisms of X. It is well known that $\mathscr{H}(X)$ can be topologized in various natural ways. We shall always endow $\mathscr{H}(X)$ with the compact-open topology, i.e., the topology having the collection

$$[K, U]=\{\phi \in \mathscr{H}(X): \phi(K) \subseteq U\},$$

with $K \subseteq X$ compact and $U \subseteq X$ open, as an (open) subbase.

It was shown by de Groot in [6] that every group is algebraically isomorphic to a group of the form $\mathscr{H}(X)$. This result suggests the obvious question which *topological* groups are *topologically* isomorphic to a group of the form $\mathscr{H}(X)$ (it can be shown that not all topological groups are representable in this way). It is clear that this is a difficult problem.

A space X is called *totally disconnected* if every connected subset contains at most one point. Clearly, every zero-dimensional space is totally disconnected. The converse is not true, consider e.g. the subspace E of Hilbert space consisting of all points all coordinates of which are rational [5, 1.2.15].

The aim of this paper is to prove the following:

0.1. Theorem. *For every totally disconnected separable metrizable topological group G there exists a separable metrizable space X such that G is topologically isomorphic to $\mathcal{H}(X)$.*

Since by van Mill [9] there exist n-dimensional separable metrizable totally disconnected groups for every $0 \leq n < \infty$, we obtain in particular for every $1 \leq n < \infty$ an example of a topological space X_n whose homeomorphism group $\mathcal{H}(X_n)$ is n-dimensional. Such examples were first constructed by Keesling and Wilson [7].

Theorem 0.1 is of interest because homeomorphism groups of various interesting spaces, such as the universal Menger curve, are totally disconnected (Anderson [2]).

Our construction is based on ideas in [8] and [10].

1. Preliminaries

All spaces under consideration are separable and metrizable. It is well known that every topological group G is (topologically) isomorphic to a dense subgroup of a topologically complete group $\bar{G}$.

Let X be a topologically complete space containing more than one point and let M_X denote the topological space of equivalence classes of measurable functions from $[0, 1]$ into X with the topology of convergence in measure. The topology of M_X is determined by the metric

$$\rho(f, g) = \sqrt{\int_0^1 (d(f(t), g(t)))^2 \, dt},$$

where d is any bounded metric compatible with the topology of X. Bessaga and Pełczyński [1, VI.7, Theorem 7.1] showed that M_X is homeomorphic to Hilbert space l^2. It is easily seen that the set of constant functions is closed in M_X and isometric to X. We identify X and this isometric copy of X in M_X.

Now let G be a topological group and consider $M_{\bar{G}}$. As remarked by Bessaga and Pełczyński [1, VI.7, Corollary 7.1], pointwise multiplication defines a (topological) group structure on $M_{\bar{G}}$ which extends the group structure on $\bar{G}$.

The domain and range of a function f shall be denoted by dom(f) and range(f), respectively. Q denotes the Hilbert cube. Let X be a space. A closed subset $A \subseteq X$ is called a *Z-set* provided that for every continuous function $f: Q \to X$ and for every $\varepsilon > 0$ there exists a continuous function $g: Q \to X \setminus A$ such that $d(f, g) < \varepsilon$ (here d denotes any admissible metric on X and $d(f, g) = \sup\{d(f(x), g(x)) : x \in Q\}$); for basic properties of Z-sets, see Bessaga and Pełczyński [1] and Chapman [3]. The collection of all Z-sets in X is denoted by $\mathcal{Z}(X)$. A σ-Z-set is a countable union of Z-sets and the collection of all σ-Z-sets in X is denoted by $\mathscr{Z}_\sigma(X)$. It is easy to prove that if $A \in \mathscr{Z}_\sigma(X)$ and A is closed, then $A \in \mathscr{Z}(X)$. Observe that $\mathscr{Z}(X)$ is invariant under the homeomorphisms of X, i.e., for every $Z \in \mathscr{Z}(X)$ and for every $h \in \mathscr{H}(X)$, $h(Z) \in \mathscr{Z}(X)$.

We need some very simple properties of Z-sets in l^2. It is easy to see that every singleton in l^2 is a Z-set. In addition, since l^2 is path-connected, it follows easily that the complement in l^2 of any σ-Z-set is connected (in fact, the complement of any σ-Z-set in l^2 is homeomorphic to l^2; this result is due to Anderson, for details see [1, V.6, Theorem 6.4]). Observe that every Z-set in l^2 is nowhere dense. We shall also need the following triviality: if $A \subseteq l^2$ is closed and $l^2 \backslash A$ is the union of two disjoint nonempty open sets, then A is not a Z-set. Indeed, simply observe that the complement of every Z-set in l^2 is connected.

Now again let G be a topological group and consider $M_{\bar{G}}$. As remarked above, $M_{\bar{G}}$ is homeomorphic to l^2. It can be shown that $\bar{G}$ is a Z-set in $M_{\bar{G}}$. This needs some verification. However, it is not necessary to do that. Since we only need a topological group which is homeomorphic to l^2 and which contains $\bar{G}$ as a Z-set subgroup, we can replace $M_{\bar{G}}$ by $M_{\bar{G}} \times l^2$ (which is clearly homeomorphic to l^2) and identify $\bar{G}$ with $\bar{G} \times \{(0, 0, \ldots)\}$. Because $\{(0, 0, \ldots)\}$ is a Z-set in l^2 it follows easily that $\bar{G} \times \{(0, 0, \ldots)\}$ is a Z-set in $M_{\bar{G}} \times l^2$. For this reason we shall assume in the sequel that $\bar{G}$ is a Z-set in $M_{\bar{G}}$.

2. Essential Cantor sets

Throughout this section, G denotes a fixed topological group and A a fixed closed subgroup of G. A subset B of G is called *essential modulo A* if for all distinct $x, y \in B$,

$$x \notin y \cdot A. \qquad (1)$$

Observe that (1) is equivalent to

$$y \notin x \cdot A \Leftrightarrow x^{-1} \cdot y \notin A \Leftrightarrow y^{-1} \cdot x \notin A. \qquad (2)$$

The following lemma is trivial; nonetheless it will be quite useful for establishing essentialness of a set which is constructed by "approximation".

2.1. Lemma. *If $\{x_1, \ldots, x_n\} \subseteq G$ is essential modulo A, then there exists $\varepsilon > 0$ such that if $d(x_i, y_i) < \varepsilon$ for each i, then $\{y_1, \ldots, y_n\}$ is essential modulo A.*

Proof. Observe that A is closed. □

Let $B \subseteq G$ and let g be a function from B into G. A subset P of B is said to be (g, A)-*essential* if the following conditions are satisfied:

(1) $g|P$ is injective;

(2) $P \cap g(P) = \emptyset$;

(3) $P \cup g(P)$ is essential modulo A.

2.2. Lemma. *Let $B \subseteq G$ and let g be a continuous function from B into G. If $\{x_1, \ldots, x_n\} \subseteq G$ is (g, A)-essential, then there is $\varepsilon > 0$ such that if $y_i \in G$ and $d(x_i, y_i) < \varepsilon$ for every i, then $\{y_1, \ldots, y_n\}$ is (g, A)-essential.*

Proof. Use that g is continuous and apply Lemma 2.1. □

2.3. Proposition. *Let G be a topological group, let $A \subseteq G$ be a closed subgroup, and $g: B \to G$ be a continuous function defined on a topologically complete subset of G. If B contains an uncountable (g, A)-essential subset, then B contains a (g, A)-essential Cantor set.*

Proof. Let d be a metric on G, and choose a complete metric ρ on B. For each $x \in B$ and $\varepsilon > 0$, let $B(x, \varepsilon) = \{b \in B: \rho(x, b) \leq \varepsilon\}$. Since each (separable metric) space is the union of a countable set and a perfect set (= dense in itself), the hypothesis implies that B contains a perfect (g, A)-essential set P. Using finite disjoint unions of balls about points of P, we may construct a Cantor set K in the complete space B by the standard procedure; a little extra care will ensure that K is (g, A)-essential. It suffices to describe the first two steps in the inductive construction.

Pick any $p_1 \in P$. Since $g(p_1) \neq p_1$, there exists $0 < \varepsilon_1 < 1$ such that $B(p_1, \varepsilon_1) \cap g(B(p_1, \varepsilon_1)) = \emptyset$. Let $B_1 = B(p_1, \varepsilon_1)$. Since the set $\{p_1, g(p_1)\}$ is essential modulo A, by Lemma 2.2 we may assume that ε_1 is so small that for any $F \subseteq B_1 \cup g(B_1)$ such that F contains at most a single point from each of B_1 and $g(B_1)$, F is essential modulo A. Let $K_1 = B_1$.

Since P is perfect, there exist distinct points $p_{1,0}$ and $p_{1,1}$ in $P \cap B_1$. Choose $0 < \varepsilon_2 < \frac{1}{2}$ such that, for $B_{1,0} = B(p_{1,0}, \varepsilon_2)$ and $B_{1,1} = B(p_{1,1}, \varepsilon_2)$, we have $B_{1,0} \cup B_{1,1} \subseteq B_1$, $B_{1,0} \cap B_{1,1} = \emptyset$, and $g(B_{1,0}) \cap g(B_{1,1}) = \emptyset$. Since the set $\{p_{1,0}, p_{1,1}, g(p_{1,0}), g(p_{1,1})\}$ is essential modulo A, as above we may also assume that ε_2 is small enough so that, for any $F \subseteq B_{1,0} \cup B_{1,1} \cup g(B_{1,0}) \cup g(B_{1,1})$ such that F contains at most a single point from each of $B_{1,0}$, $B_{1,1}$, $g(B_{1,0})$, and $g(B_{1,1})$, F is essential modulo A. Let $K_2 = B_{1,0} \cup B_{1,1}$.

Continuing with this procedure in the standard manner, we obtain a nested sequence (K_n) of closed sets in B. Let $K = \bigcap_{n=1}^{\infty} K_n$. It is clear that K is the required Cantor set, cf. the proof of [10, Proposition 3.4]. □

Remark. If we assume only that B contains an uncountable essential subset modulo A, then the above construction shows that B contains an essential Cantor set modulo A.

Now let G be a topological group and let A be a closed subgroup of G. Let $\mathcal{H}(G, A)$ denote the collection of all homeomorphisms $h: K_1 \to K_2$ between disjoint Cantor sets in G such that $K_1 \cup K_2$ is essential modulo A.

2.4. Proposition. *For every topological group G and every closed subgroup A of G there is a subset B of G having the following properties:*

(1) *B is essential modulo A;*

(2) *for each $h \in \mathscr{H}(G, A)$, there exists $x \in \text{dom}(h)$ such that $x \in B$ but $h(x) \notin B \cdot A$.*

Proof. It is clear that $\mathscr{H}(G, A)$ has size at most $\mathfrak{c}$. Let $<$ be a well-ordering of $\mathscr{H} = \mathscr{H}(G, A)$ such that for each $h \in \mathscr{H}$ the section $\{g \in \mathscr{H}: g < h\}$ has size less than $\mathfrak{c}$.

By transfinite induction we shall construct for every $h \in \mathscr{H}$ a point $b(h) \in \text{dom}(h)$ such that

(1) $\{b(g): g \leq h\}$ is essential modulo A;

(2) $(\{b(g): g \leq h\} \cdot A) \cap \{g(b(g)): g \leq h\} = \emptyset$.

For the first element $f \in \mathscr{H}$ we may take any point $b(f) \in \text{dom}(f)$. For $h \in \mathscr{H}$ suppose that the points $b(g)$ have been constructed for $g < h$. Put $K = \text{dom}(h)$.

Claim 1. *The set $E = \{x \in K: \{x\} \cup \{b(g): g < h\}$ is not essential modulo $A\}$ has cardinality less than $\mathfrak{c}$.*

Observe that $|\{b(g): g < h\}| < \mathfrak{c}$. For every $x \in E$ there exists $g_x \in \{b(g): g < h\}$ such that $x \in g_x \cdot A$. Since K is essential modulo A, the function $x \to g_x$ is one-to-one. We are done.

Claim 2. *The set $F = \{x \in K: h(x) \in \{b(g): g < h\} \cdot A\}$ has cardinality less than $\mathfrak{c}$.*

For every $x \in F$ pick $h_x \in \{b(g): g < h\}$ such that $h(x) \in h_x \cdot A$. Then the function $x \to h_x$ is one-to-one because $h(K)$ is essential modulo A.

Now pick an arbitrary element $x \in K \setminus (E \cup F)$. Observe that $h(x) \notin x \cdot A$ since $\{x, h(x)\}$ is essential modulo A. Consequently, by Claims 1 and 2 we see that $b(h) = x$ is as desired.

This completes the transfinite induction.

Now put $B = \{b(h): h \in \mathscr{H}\}$. □

2.5. Corollary. *Let G be a topologically complete topological group and let A be a closed nowhere dense subgroup of G. Finally, let $B \subseteq G$ have the properties* (1) *and* (2) *of Proposition* 2.4. *Then*

(1) *if S is a G_δ-subset of G such that $B \cap S = \emptyset$, then S can be covered by countably many translates of A;*

(2) *B is dense in G;*

(3) *if $B \subseteq B' \subseteq G$, then B' is a Baire space.*

Proof. Let $\mathscr{E}$ be the collection of all G_δ-subsets of G which cannot be covered by countably many translates of A. We shall prove that B intersects every element of $\mathscr{E}$. To this end, let $E \in \mathscr{E}$. We claim that E contains an uncountable subset which is essential modulo A. Let $F \subseteq E$ be maximal with respect to being essential modulo A and assume that F is countable. Pick an arbitrary $x \in E \setminus F$. Then $F \cup \{x\}$ is not essential modulo E, so there exists an element $y \in F$ such that $x \in y \cdot A$. Consequently, E can be covered by countably many translates of A; contradiction. The remark following Proposition 2.3 therefore implies that E contains a Cantor set K which is essential modulo A. Let K_1 and K_2 be two disjoint Cantor sets in K and consider any homeomorphism $h: K_1 \to K_2$. Then $h \in \mathscr{H}(G, A)$ which implies that $\emptyset \neq B \cap K_1 \subseteq B \cap E$.

Now observe that every nonempty open set in G belongs to $\mathscr{E}$. This implies that B is dense. In addition, every dense G_δ-subset of G belongs to $\mathscr{E}$. This implies that for every $B' \subseteq G$ with $B \subseteq B'$ we have that B' is a Baire space. □

3. Countable A-type for maps

Let G be a topological group and let $A \subseteq G$ be a subgroup. If $X \subseteq G$ and $f: X \to G$ is continuous, then f has *countable A-type* provided that there exists a countable subset F of X such that for every $x \in X$,

$$x \in F \cdot A \quad \text{or} \quad f(x) \in F \cdot A \quad \text{or} \quad f(x) \in x \cdot A. \tag{3}$$

3.1. Proposition. *Let G be a topologically complete topological group and let A be a closed nowhere dense subgroup of G. Let $B \subseteq G$ have the properties* (1) *and* (2) *of Proposition* 2.4 *and let $f: B \to B \cdot A$ be continuous. If S is a G_δ-subset of G containing B such that f can be extended to a continuous function $\bar{f}: S \to S$, then $\bar{f}$ has countable A-type.*

Proof. Suppose that S contains an uncountable $(\bar{f}, A)$-essential subset. By Proposition 2.3, S contains an $(\bar{f}, A)$-essential Cantor set K. Then $\bar{f} \mid K \in \mathscr{H}(G, A)$, so by hypothesis there exists $x \in B \cap K$ such that $\bar{f}(x) \notin B \cdot A$. Since $\bar{f}$ extends f this is a contradiction.

Now let $H \subseteq S$ be a maximal $(\bar{f}, A)$-essential subset. By what we just proved, H is countable. Put $F = H \cup \bar{f}(H)$ and observe that F is countable.

Take any $x \in S$ such that $x \notin F \cdot A$ and $\bar{f}(x) \notin F \cdot A$. Since $H \subseteq F \cdot A$, $x \notin H$. Consequently, $H \cup \{x\}$ is not $(\bar{f}, A)$-essential, so one of the following three cases holds:

Case 1: *$\bar{f} \mid H \cup \{x\}$ is not injective.* This implies that there is a point $y \in H$ such that $\bar{f}(x) = \bar{f}(y)$, i.e., $\bar{f}(x) \in \bar{f}(H) \subseteq F \subseteq F \cdot A$; contradiction.

Case 2: $(H \cup \{x\}) \cap (\bar{f}(H \cup \{x\})) \neq \emptyset$. Since $H \cap \bar{f}(H) = \emptyset$ there are three possibilities: $x \in \bar{f}(H)$, $\bar{f}(x) = x$, or $\bar{f}(x) \in H$. The first and the third case are impossible and the second case implies that $\bar{f}(x) = x \cdot e \in x \cdot A$.

Case 3: *Not Case* 2 *and* $(H \cup \{x\}) \cup (\bar{f}(H) \cup \{\bar{f}(x)\})$ *is not essential modulo A.* Since $x \notin F \cdot A$ and $\bar{f}(x) \notin F \cdot A$ and F is essential modulo A, $\{x, \bar{f}(x)\}$ is not essential modulo A, i.e., $\bar{f}(x) \in x \cdot A$. □

4. The construction

Throughout this section, G denotes a fixed totally disconnected topological group. We consider its completion $\bar{G}$ and the topological group $H = M_{\bar{G}}$ defined in Section

1. Let B be such as in Proposition 2.4 (for $G=H$ and $A=\bar{G}$). We shall prove that for $X=B\cdot G$ the groups $\mathscr{H}(X)$ and G are topologically isomorphic.

4.1. Lemma. *Let $T\in\mathscr{Z}_\sigma(H)$. Then $B\backslash T$ is connected and dense in H.*

Proof. To the contrary, assume that $B\backslash T$ is not connected. Then there is a closed subset E in H such that $H\backslash E$ can be written as $U\cup V$, where U and V are disjoint open subsets of H, $B\backslash T\subseteq U\cup V$, $(B\backslash T)\cap U\neq\emptyset$ and $(B\backslash T)\cap V\neq\emptyset$. Put $F=E\backslash T$. Observe that F is a G_δ-subset of H that misses B. Consequently, F can be covered by countably many translates of $\bar{G}$ (Proposition 2.5(1)). Since $\bar{G}\in\mathscr{Z}(H)$ and the collection of all Z-sets of H is invariant under homeomorphisms, it follows that E can be covered by countably many Z-sets. Since E is closed this implies that $E\in\mathscr{Z}_\sigma(H)$ and consequently, $E\in\mathscr{Z}(H)$. This implies that, $H\backslash E$ is connected, contradiction.

That $B\backslash T$ is dense in H follows from Corollary 2.5(2) and (3). □

4.2. Lemma. *Let $S\subseteq H$ be a G_δ-subset containing B. Then $H\backslash S\in\mathscr{Z}_\sigma(H)$.*

Proof. There is a family $\{A_n: n\in\mathbb{N}\}$ of closed subsets of H such that $\bigcup_{n=1}^{\infty}A_n=H\backslash S$. Since $A_n\cap B=\emptyset$, by Corollary 2.5(1) it follows that $A_n\in\mathscr{Z}_\sigma(H)$. Consequently, $H\backslash S\in\mathscr{Z}_\sigma(H)$. □

4.3. Corollary. *Let $S\subseteq H$ be a G_δ-subset containing B. If $Z\in\mathscr{Z}(S)$, then the closure of Z in H belongs to $\mathscr{Z}(H)$.*

Proof. By Lemma 4.2, $H\backslash S\in\mathscr{Z}_\sigma(H)$. Consequently, each map $f:Q\to H$ can be approximated by a map $g:Q\to S$ [1, the proof of V.2, Proposition 2.2(d)]. Now since $Z\in\mathscr{Z}(S)$, g can be approximated by a map $h:Q\to S\backslash Z$. Clearly, $h(Q)$ misses the closure of Z in H. □

4.4. Theorem. *$\mathscr{H}(X)$ and G are topologically isomorphic.*

Proof. Let $f:X\to X$ be a homeomorphism. There is a G_δ-subset S of H that contains X while moreover f can be extended to a homeomorphism $\bar{f}:S\to S$ (Engelking [4, 4.3.21]). By Proposition 3.1 there exists a countable subset $F\subseteq S$ such that for any $x\in S$,

$$x\in F\cdot\bar{G}\quad\text{or}\quad\bar{f}(x)\in F\cdot\bar{G}\quad\text{or}\quad\bar{f}(x)\in x\cdot\bar{G}.\tag{4}$$

Put $E=\{x\in S:\bar{f}(x)\in x\cdot\bar{G}\}$.

Claim 1. *$H\backslash E$ is contained in an element of $\mathscr{Z}_\sigma(H)$.*

Put $T=((F\cdot\bar{G})\cap S)\cup\bar{f}^{-1}((F\cdot\bar{G})\cap S)$. Observe that for every $x\in F$, $x\cdot\bar{G}\cap S\in\mathscr{Z}(S)$ (Lemma 4.2). Consequently, since $\mathscr{Z}(S)$ is invariant under homeomorphisms of S, $\bar{f}^{-1}((F\cdot\bar{G})\cap S)\in\mathscr{Z}_\sigma(S)$. The desired result now follows from (4), Lemma 4.2 and Corollary 4.3.

Claim 2. *If $x \in E \cap B$, then $\bar{f}(x) = f(x) \in x \cdot G$.*

Indeed, $\bar{f}(x) = f(x) \in x \cdot \bar{G} \cap X = x \cdot \bar{G} \cap B \cdot G$. We conclude that there exist $b \in B$, $y \in G$ and $z \in \bar{G}$ such that

$$f(x) = x \cdot z = b \cdot y.$$

Since B is essential modulo $\bar{G}$ this implies that $x = b$ and in turn that $z = y$.

Now define a function $\phi : E \cap B \to G$ by $\phi(x) = x^{-1} \cdot f(x)$. Then by Claim 2, ϕ is well defined. In addition, ϕ is clearly continuous. By Claim 1 and Lemma 4.1, $E \cap B$ contains a connected and dense subset of H. Since G is totally disconnected, we obtain that ϕ is constant. Consequently there exists a fixed $p \in G$ such that for every $x \in E \cap B$, $f(x) = x \cdot p$. Since $E \cap B$ is dense in H, and therefore in S this implies that for *every* $x \in S$, $\bar{f}(x) = x \cdot p$. Consequently, $\bar{f}$ is the restriction of the translation $x \to x \cdot p$ to S.

We conclude that every homeomorphism of X is the restriction to X of a (right) translation by an element of G. Since X is a union of cosets of G, every (right) translation by an element of G induces a homeomorphism of X. We conclude that G and $\mathscr{H}(X)$ are algebraically isomorphic. We shall prove that the isomorphism is also topological.

It will be convenient to introduce some notation. For every $a \in G$ we let $f_a : X \to X$ denote the restriction of the translation $x \to x \cdot a$ to X. As shown above, the function $\xi : G \to \mathscr{H}(X)$ defined by $\xi(a) = f_a$ is an algebraic isomorphism between G and $\mathscr{H}(X)$. We shall prove that ξ is in fact a topological isomorphism.

To this end, consider a basic open set in $\mathscr{H}(X)$. That is a set of the form

$$L = [K_1, U_1] \cap \cdots \cap [K_n, U_n],$$

where the $K_1, \ldots, K_n \subseteq X$ are compact and the $U_1, \ldots, U_n \subseteq X$ are open. Then

$$\xi^{-1}(L) = \{a \in G: \xi(a) \in L\} = \{a \in G: (\forall i \leq n)(K_i \cdot a \subseteq U_i)\}$$

$$= \{a \in G: (\forall i \leq n)(a \in K_i^{-1} \cdot U_i)\} = \bigcap_{i=1}^{n} K_i^{-1} \cdot U_i \cap G,$$

which is open in G. We conclude that ξ is continuous.

Now let $U \subseteq G$ be open. We shall prove that $\xi(U)$ is open in $\mathscr{H}(X)$. To this end, take an arbitrary $f_a \in \xi(U)$. Then $a \in U$. We claim that $f_a \in [\{e\}, U] \subseteq \xi(U)$. That $f_a \in [\{e\}, U]$ is clear since $f_a(e) = e \cdot a = a \in U$. Now take an arbitrary element $f_b \in [\{e\}, U]$. Then $f_b(e) = e \cdot b = b \in U$, i.e., $f_b \in \xi(U)$. We conclude that $\xi(U)$ is open, and hence that ξ is a homeomorphism. □

Remark. It is clear that the above proof also shows that G and $\mathscr{H}(X)$ are also topologically isomorphic when $\mathscr{H}(X)$ is endowed with the topology of pointwise convergence.

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