Topology and its Applications 22 (1986) 267-282
North-Holland

THE BAIRE CATEGORY THEOREM IN PRODUCTS OF LINEAR SPACES AND TOPOLOGICAL GROUPS

Jan van MILL
Subfaculteit Wiskunde, Vrije Universiteit, Amsterdam, The Netherlands

Roman POL*
Department of Mathematics, Warsaw University, 00-901 Warsaw, Poland

Received 16 October 1984
Revised 13 May 1985

A space is a Baire space if the intersection of countably many dense open sets is dense. We show that if X is a non-separable completely metrizable linear space (pathconnected abelian topological group) then X contains two linear subspaces (subgroups) E and F such that both E and F are Baire but $E\times F$ is not. If X is a completely metrizable linear space of weight $\aleph_1$ then X is the direct sum $E\oplus F$ of two linear subspaces E and F such that both E and F are Baire but $E\times F$ is not.

AMS(MOS)Subj. Class: 54E52, 22A99, 46A06

Baire space
connected abelian topological group
complete metrizable linear space

Introduction

All spaces under discussion are metrizable.

The fact that in complete metric spaces the Baire Category Theorem holds, i.e. that the intersection of countably many dense open sets is dense, is a well-known and important tool in functional analysis. A space for which the Baire Category Theorem holds will be called a *Baire space* from now on (this is standard terminology). The question whether the product of two Baire spaces is a Baire space was raised in [3, 10, 13]. Using forcing techniques, this question was answered in the negative by Cohen [4]. Earlier, it was known from the work by Oxtoby [10] and Krom [7] that the Continuum Hypothesis implies the existence of two Baire spaces X and Y whose product in not Baire. In Fleissner and Kunen [6] direct constructions of such spaces were obtained and in addition they constructed an example of a single Baire space X whose square is not Baire.

* This paper was written while the second author was visiting the Department of Mathematics & Computer Science of the Free University in Amsterdam.

It is natural to ask whether the product of two normed Baire spaces is Baire. Arias de Reyna [1] claims to have answered this question in the negative. However, it seems that there is a gap in his reasoning—we express our thanks to D.J. Lutzer and M. Valdivia for this information. The main result of this paper is that every non-separable topologically complete linear space (pathconnected abelian topological group) L contains two linear subspaces (sub-groups) A and B such that both A and B are Baire but $A\times B$ is not. Independently, Valdivia [14] has also constructed two normed Baire spaces whose product is not Baire. We will also show that every topologically complete linear space L of weight $\aleph_1$ is the direct sum $E\oplus F$ of two linear subspaces E and F such that both E and F are Baire but $E\times F$ is not.

The essence of our construction, as well as Valdivia's and Arias de Reyna's method, is the construction of Fleissner and Kunen [6]. However, instead of using their ideas directly, we apply rather a general variant of their construction described in [12], which allows us to get rid of 'coordinates' in the spaces and which simplifies the argumentation.

1. Preliminaries

As noted in the introduction, all spaces under discussion are metrizable. For every space X we let ρ denote a compatible metric on X. This is done for simplicity and will very likely cause no confusion. If X is a space and $\varepsilon>0$ then $B(x,\varepsilon)=\{y\in X\colon \rho(x,y)<\varepsilon\}$.

If X is a set and κ is a cardinal then

$$[X]^{<\kappa}=\{A\subseteq X\colon |A|<\kappa\}.$$

$\mathfrak{c}$ denotes $2^{\aleph_0}$.

As usual, ω_1 denotes the first uncountable ordinal. A subset $C\subseteq\omega_1$ is called *closed* if for every strictly increasing sequence $\alpha_1<\alpha_2<\cdots$ of elements of C we have $\sup_{n\in\mathbb{N}}\alpha_n\in C$. In addition, a subset $S\subseteq\omega_1$ is called *stationary* if $S\cap C\neq\emptyset$ for every closed unbounded subset $C\subseteq\omega_1$. A function $f\colon A\to\omega_1$, where A is a subset of ω_1, is called *regressive* if $f(\alpha)<\alpha$ for every $\alpha\in A$. If $S\subseteq\omega_1$ is stationary, and if $f\colon S\to\omega_1$ is regressive then $f^{-1}(\{\alpha\})$ is stationary for some $\alpha\in\omega_1$ (in particular, $f^{-1}(\{\alpha\})$ is uncountable). This is called the "Pressing Down Lemma" (abbreviated PDL). In addition, if $S\subseteq\omega_1$ is stationary then S can be split into ω_1 disjoint stationary sets. For proofs of these facts see Fleissner [5].

2. Independent Cantor sets

Our basic tool is that in topologically complete linear spaces, or topologically complete topological groups, very special 'independent' Cantor sets exist. In this section we will construct these Cantor sets. Our results were inspired by the work of Mycielski [9] and also by Mauldin [8].

We deal with the linear case first. Let L be a linear space. If $A \subseteq L$ then $\operatorname{lin}(A)$ denotes the linear hull of A. If $E \subseteq L$ is a linear subspace then $A \subseteq L$ is said to be *linearly independent over E* provided that for every $x \in A$ we have $x \notin \operatorname{lin}(A \setminus \{x\}) + E$. If $A \subseteq L$ is not linearly independent over E then we say that A is linearly dependent over E.

2.1. Proposition. *Let Z be a topologically complete linear space and let $Z_1 \subseteq Z_2 \subseteq \cdots$ be a sequence of closed linear subspaces of Z each of infinite codimension in Z. If G is a dense G_δ in Z then there exists a Cantor set $C \subseteq G$ such that C is linearly independent over $\bigcup_{i=1}^{\infty} Z_i$.*

Proof. For every $n, m \in \mathbb{N}$ define $R_{nm} = \{(x_1, \ldots, x_n) \in Z^n \colon \{x_1, \ldots, x_n\}$ is linearly dependent over $Z_m\}$. Since Z_m is closed it follows that R_{nm} is closed. We claim that R_{nm} is nowhere dense in Z^n, for every $m \in \mathbb{N}$. We prove this by induction on n. Given $\varepsilon > 0$ and a point $(x_1, \ldots, x_{n+1}) \in Z^{n+1}$, use the induction hypothesis to find a point $(y_1, \ldots, y_n) \in Z^n \setminus R_{nm}$ such that $\rho(x_i, y_i) < \varepsilon$ for every $i \leqslant n$. Then choose $y_{n+1} \in B(x_{n+1}, \varepsilon) \setminus \operatorname{lin}(\{y_1, \ldots, y_n\}) + Z_m$.

It is possible to choose y_{n+1} since Z_m has infinite codimension which implies that $\operatorname{lin}(\{y_1, \ldots, y_n\}) + Z_m$ is nowhere dense in Z. It is clear that the point $(y_1, \ldots, y_{n+1}) \notin R_{n+1,m}$.

Since G is dense in Z the existence of C now immediately follows from Mycielski [9, Theorem 1]. □

We now let G be a connected abelian topological group. Our aim is to prove a version of Proposition 2.1 for G. If A is a closed subgroup of G, then either the elements of G of infinite rank over A form a dense G_δ-set in G (Lemma 2.3) or for some m, each element of G has rank $\leqslant m$ over A (Lemma 2.4) and accordingly, we have to deal with different notions of independence over A, considering the factor group G/A either as a module over the integers or as a module over the integers modulo m, for some natural number m (see the definition after Lemma 2.5). This causes some additional complications in comparison to the linear case, see Sections 3 and 5.

Throughout, let G be a topologically complete connected abelian topological group. If $A \subseteq G$ then $\langle A \rangle$ is the subgroup of G generated by A. As usual, $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$. Let $A \subseteq G$ be a subgroup. For every $n \in \mathbb{Z}'$ define

$$n^{-1}A = \{x \in G \colon nx \in A\}.$$

2.2. Lemma. *If A is a closed subgroup of G and* $\operatorname{Int} n^{-1}A \neq \emptyset$ *for some $n \in \mathbb{Z}$ then $n^{-1}A = G$.*

Proof. It is clear that $n^{-1}A$ is a closed subgroup of G. Since $\operatorname{Int} n^{-1}A \neq \emptyset$ it therefore follows that $n^{-1}A$ is open, whence clopen. The connectivity of G now implies that $n^{-1}A = G$. □

Let $A \subseteq G$ be a closed subgroup. We call A *thin* provided that there is an $x \in G$ with $\mathbb{Z}' \cdot x \cap A = \emptyset$.

2.3. Lemma. *Let $A \subseteq G$ be a closed thin subgroup. The set $A^* = \{x \in G: \mathbb{Z}' \cdot x \cap A = \emptyset\}$ is a dense G_δ in G.*

Proof. Obviously, $A^* = G \setminus \bigcup_{n \in \mathbb{Z}'} n^{-1}A$. Since each $n^{-1}A$ is closed, A^* is a G_δ. If $\operatorname{Int} n^{-1}A \neq \emptyset$ for some $n \in \mathbb{Z}'$ then $n^{-1}A = G$ (Lemma 2.2) so then A is not thin. We conclude that each $n^{-1}A$ is nowhere dense. From this and the completeness of G it follows that A^* is dense. □

2.4. Lemma. *Let $A \subseteq G$ be a closed subgroup which is not thin. Then there exists an $n \in \mathbb{Z}$ such that $n^{-1}A = G$.*

Proof. A is not thin so $G = \bigcup_{n \in \mathbb{Z}'} n^{-1}A$. Since each $n^{-1}A$ is closed, the Baire Category Theorem implies that $\operatorname{Int} n^{-1}A \neq \emptyset$ for some $n \in \mathbb{Z}'$. Observe that $n^{-1}A = (-n)^{-1}A$ and apply Lemma 2.2. □

Let $A \subseteq G$ be a closed subgroup. Define $\kappa(A) \in \mathbb{N} \cup \{\infty\}$ by

$$\kappa(A) = \infty \quad \text{iff } A \text{ is thin,}$$

$$\kappa(A) = \min\{n \in \mathbb{N}: n^{-1}A = G\} \quad \text{if } A \text{ is not thin.}$$

By Lemma 2.4, $\kappa(A)$ is well-defined.

2.5. Lemma. *Let $A \subseteq G$ be a closed subgroup for which $\kappa(A) < \infty$. The sets $m^{-1}A$ are closed and nowhere dense for all $0 < m < \kappa(A)$.*

Proof. If $m^{-1}A$ has nonempty interior for some $0 < m < \kappa(A)$, apply Lemma 2.2 to conclude that $\kappa(A) \leqslant m < \kappa(A)$. □

Let $A \subseteq G$ be a closed subgroup. We call $B \subseteq G$ *independent over* A provided that for all $n \in \mathbb{N}$, distinct $x_1, \ldots, x_n \in B$ and $m_1, \ldots, m_n \in \mathbb{Z}$ we have

$$m_1 x_1 + \cdots + m_n x_n \in A \Rightarrow |m_i| = 0 \pmod{\kappa(A)} \qquad \text{for all } i \leqslant n,$$

where $|m_i| = 0 \pmod{\infty}$ means $m_i = 0$.

2.6. Lemma. *Let $A \subseteq G$ be a proper closed subgroup. If $x_1, \ldots, x_n \in G$ are distinct and $\varepsilon > 0$ then there are distinct $y_1, \ldots, y_n \in G \setminus A$ such that $\rho(x_i, y_i) < \varepsilon$ for all $1 \leqslant i \leqslant n$ while moreover $\{y_1, \ldots, y_n\}$ is independent over A.*

Proof. We have to distinguish several cases.

Case 1. $n = 1$ and $\kappa(A) = \infty$. Apply Lemma 2.3 to find $y_1 \in G$ with $\rho(x_1, y_1) < \varepsilon$ and $\mathbb{Z}' \cdot y_1 \cap A = \emptyset$. It is clear that y_1 is as required.

Case 2. $n=1$ and $\kappa(A)<\infty$. Since A is a proper closed subgroup, A is nowhere dense, so $\kappa(A)>1$. By Lemma 2.5 $\bigcup_{1\leqslant m<\kappa(A)} m^{-1}A$ is nowhere dense, so there is point $y_1\in G$ with $\rho(x_1,y_1)<\varepsilon$ and $y_1\notin\bigcup_{1\leqslant m<\kappa(A)} m^{-1}A$. Then y_1 is clearly as required (use that A is a subgroup).

Now suppose that the Lemma is true for m and let us try to prove the Lemma for $m+1$. To this end, take distinct $x_1,\ldots,x_{m+1}\in G$. Let $V_1,\ldots,V_{m+1}$ be pairwise disjoint closed neighborhoods of the points $x_1,\ldots,x_{m+1}$, each of diameter less than ε. By induction hypothesis there exist $y_i\in V_i$ $(1\leqslant i\leqslant m)$ such that $\{y_1,\ldots,y_m\}$ is independent over A. For all $(n_1,\ldots,n_{m+1})\in\mathbb{Z}^{m+1}$ with $|n_{m+1}|\neq 0 \pmod{\kappa(A)}$, define

$$C_{(n_1,\ldots,n_{m+1})}=\{x\in G\colon n_{m+1}x\in n_1y_1+\cdots+n_my_m+A\}.$$

Then $C_{(n_1,\ldots,n_{m+1})}$ is closed. Suppose that the union of the $C_{(n_1,\ldots,n_{m+1})}$'s covers V_{m+1}. Since G is topologically complete and $\operatorname{Int} V_{m+1}\neq\emptyset$, we can find a sequence $(n_1,\ldots,n_{m+1})\in\mathbb{Z}^{m+1}$, $|n_{m+1}|\neq 0 \pmod{\kappa(A)}$, such that

$$U=\operatorname{int} C_{(n_1,\ldots,n_{m+1})}\neq\emptyset.$$

Take $x\in U$ and $y\in C_{(n_1,\ldots,n_{m+1})}$. Then $y\in V$, where $V=(y-x)+U$, and V is a neighborhood of y. Take $u\in U$ arbitrarily. Then

$$n_{m+1}y=n_1y_1+\cdots+n_my_m+a_0,$$

$$n_{m+1}x=n_1y_1+\cdots+n_my_m+a_1,$$

$$n_{m+1}u=n_1y_1+\cdots+n_my_m+a_2,$$

for certain a_0, a_1, $a_2\in A$. We conclude that

$$n_{m+1}((y-x)+u)=n_1y_1+\cdots+n_my_m+((a_0-a_1)+a_2).$$

We conclude that $(y-x)+u\in C_{(n_1,\ldots,n_{m+1})}$, whence $V\subseteq C_{(n_1,\ldots,n_{m+1})}$. Therefore $C_{(n_1,\ldots,n_{m+1})}$ is open, whence clopen, and hence equals G. Consequently,

$$n_{m+1}y\in n_1y_1+\cdots+n_my_m+A,\quad \forall y\in G,$$

in particular, $0=n_{m+1}\cdot 0\in n_1y_1+\cdots+n_my_m+A$ from which, by induction hypothesis, follows that $|n_1|=|n_2|=\cdots=|n_m|=0 \pmod{\kappa(A)}$. Consequently,

$$n_{m+1}y\in A,\quad \forall y\in G. \tag{$*$}$$

Case 3. $\kappa(A)=\infty$. Since $|n_{m+1}|\neq 0 \pmod{\infty}$, which is equivalent to $n_{m+1}\neq 0$, $(*)$ contradicts A being thin.

Case 4. $\kappa(A)<\infty$. Without loss of generality, $n_{m+1}\in\mathbb{N}$. Choose p, $l\in\mathbb{N}\cup\{0\}$ such that $0<p<\kappa(A)$ and $n_{m+1}=l\cdot\kappa(A)+p$. Then

$$py=n_{m+1}y-l\cdot\kappa(A)y\in A,\quad \forall y\in G,$$

since by $(*)$ $n_{m+1}y\in A$ and by definition $\kappa(A)y\in A$. Since $p<\kappa(A)$ we conclude that $p=0$. From this it follows that $n_{m+1}=l\cdot\kappa(A)=0 \pmod{\kappa(A)}$, which is a contradiction.

We conclude that there is a point

$$y \in V_{m+1} \setminus \bigcup \{C_{(n_1,\dots,n_{m+1})} : (n_1, \dots, n_{m+1}) \in \mathbb{Z}^{m+1}, |n_{m+1}| = 0 \ (\mathrm{mod}\ \kappa(A))\}.$$

Now suppose that $n_1 y_1 + \cdots + n_{m+1} y \in A$ for certain $n_1, \dots, n_{m+1} \in \mathbb{Z}$.

By our choice of y we have that $|n_{m+1}| = 0 \ (\mathrm{mod}\ \kappa(A))$, consequently,

$$n_1 y_1 + \cdots + n_m y_m \in A.$$

Since the $y_1, \dots, y_m$ are independent over A it follows that $|n_1| = \cdots = |n_m| = 0 \ (\mathrm{mod}\ \kappa(A))$. We conclude that $y_{m+1} = y$ is as required. □

2.7. Lemma. *Let $A \subsetneq G$ be a proper closed subgroup. For each $n \in \mathbb{N}$ the set $R_n = \{(x_1, \dots, x_n) \in G^n : \{x_1, \dots, x_n\}$ is dependent over $A\}$ is of first category in G^n.*

Proof. For every $(m_1, \dots, m_n) \in \mathbb{Z}^n$ define

$$C_{(m_1,\dots,m_n)} = \{(x_1, \dots, x_n) \in G^n : m_1 x_1 + \cdots + m_n x_n \in A\}.$$

Let $\mathscr{C} = \{C_{(m_1,\dots,m_n)} : (m_1, \dots, m_n) \in \mathbb{Z}^n\}$. Observe that every $C \in \mathscr{C}$ is closed and that there is a (countable) subfamily $\mathscr{F} \subseteq \mathscr{C}$ such that

$$R_n = \bigcup \mathscr{F}.$$

That each $F \in \mathscr{F}$ is nowhere dense follows from Lemma 2.6. □

We can now prove a result similar to Proposition 2.1.

2.8. Proposition. *Let G be a connected abelian topological group which is topologically complete. In addition, let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of proper closed subgroups of G, and let $E \subseteq G$ be a dense G_δ. There is a Cantor set $K \subseteq E$ such that K is independent over A_n, for every $n \in \mathbb{N}$.*

Proof. This directly follows from Lemma 2.7. and Mycielski [9, Theorem 1]. □

3. Applications of independent Cantor sets

In this section we apply the results obtained in Section 2 to get certain 'decompositions' of small linear spaces and topological groups. Again, we deal with the linear case first.

3.1. Theorem. *Let Z be a topologically complete metrizable linear space of weight at most continuum and let $Z_1 \subseteq Z_2 \subseteq \cdots$ be a sequence of closed linear subspaces of Z each of infinite codimension. Then there exists a linear subspace $V \subseteq Z$ of second category in Z such that $Z = V \oplus (\bigcup_{i=1}^{\infty} Z_i)$.*

Proof. Since Z has weight at most $\mathfrak{c}$, Z has cardinality at most $\mathfrak{c}$, whence Z has at most $\mathfrak{c}$ separable closed subspaces. The family $\mathscr{C}=\{C\subseteq Z$: C is a Cantor set which is linearly independent over $\bigcup_{i=1}^{\infty} Z_i\}$ has cardinality at most $\mathfrak{c}$. Let $\{C_\alpha: \alpha<\mathfrak{c}\}$ enumerate $\mathscr{C}$. Inductively, choose $x_\alpha\in C_\alpha$ such that

$$\operatorname{lin}\{x_\xi: \xi\leqslant\alpha\}\cap\bigcup_{i=1}^{\infty} Z_i=\{0\}$$

for every $\alpha<\mathfrak{c}$. Since the algebraic dimension over $\bigcup_{i=1}^{\infty} Z_i$ of the set $E=\operatorname{lin}\{x_\xi: \xi<\alpha\}+\bigcup_{i=1}^{\infty} Z_i$ is less than $\mathfrak{c}$, it is possible to pick a point $x\in C_\alpha\backslash E$. It is easily seen that $x_\alpha=x$ as required.

Put $W=\operatorname{lin}\{x_\alpha: \alpha<\mathfrak{c}\}$. Clearly, $W\cap\bigcup_{i=1}^{\infty} Z_i=\{0\}$ and W is of second category in Z by Proposition 2.1. Now let $V\subseteq Z$ be any linear subspace containing W such that $V\cap\bigcup_{i=1}^{\infty} Z_i=\{0\}$ and $V+\bigcup_{i=1}^{\infty} Z_i=Z$. Then V is clearly as required. $\square$

We now want to prove a generalization of Theorem 3.1. for topological groups. Unfortunately we cannot obtain a result quite as strong as Theorem 3.1., see the example following Theorem 3.3.

We will now prove a simple but important lemma.

3.2. Lemma. *Let G be a topologically complete connected abelian topological group and let $\mathscr{A}$ be a collection proper closed subgroups of G. Then there exists an $A\in\mathscr{A}$ such that $\kappa(A)=\kappa(B)$ for every $B\in\mathscr{A}$ containing A.*

Proof. If $\kappa(A)=\infty$ for every $A\in\mathscr{A}$ then there is nothing to prove. Suppose therefore that $\kappa(A)<\infty$ for certain $A\in\mathscr{A}$. Let

$$n=\min\{\kappa(A): A\in\mathscr{A} \text{ and } \kappa(A)<\infty\}$$

and choose $A\in\mathscr{A}$ for which $\kappa(A)=n$. We claim that A is as required. To this end, take $B\in\mathscr{A}$ such that $A\subseteq B$. Then

$$nx\in A\subseteq B$$

for every $x\in G$. We conclude that $\kappa(B)\leqslant\kappa(A)$, whence $\kappa(B)=\kappa(A)$ by our choice of $\kappa(A)$. $\square$

3.3. Theorem. *Let G be a topologically complete metrizable connected abelian topological group of weight at most continuum and let $A_1\subseteq A_2\subseteq\cdots$ be a sequence of closed proper subgroups of G. There exists a subgroup $E\subseteq G$ of second category in G and an $n\in\mathbb{N}$ such that $E\cap\bigcup_{i=1}^{\infty} A_i\subseteq A_n$. If $\kappa(A_i)=\infty$ for every $i\in\mathbb{N}$ then we can construct E such that $E\cap\bigcup_{i=1}^{\infty} A_i=(0)$.*

Proof. By Lemma 3.2. we can find an $n\in\mathbb{N}$ such that $\kappa(A_i)=\kappa(A_n)$ for every $i\geqslant n$. We claim that there is a subgroup $E\subseteq G$ of second category in G such that $E\cap\bigcup_{i=1}^{\infty} A_i\subseteq A_n$.

As in the proof of Theorem 3.1, it follows that G contains at most $\mathfrak{c}$ Cantor sets. Let $\{K_\alpha: \alpha < \mathfrak{c}\}$ enumerate all Cantor sets in G which are independent over A_i, for every $i \in \mathbb{N}$. We will construct the required subgroup E such that $E \cap K_\alpha \neq \emptyset$ for every $\alpha < \mathfrak{c}$. By Proposition 2.8, E then intersects every dense G_δ in G, whence E is of second category in G.

By transfinite induction on $\alpha < \mathfrak{c}$ we will pick $x_\alpha \in K_\alpha$ such that for every $\alpha < \mathfrak{c}$ the set $\{x_\beta: \beta \leqslant \alpha\}$ is independent over A_i, for every $i \in \mathbb{N}$. We will then show that $E = \langle\{x_\alpha : \alpha < \mathfrak{c}\}\rangle$ is as required.

Let x_0 be any point of K_0 and suppose that x_β has been defined for every $\beta < \alpha$. Suppose that for every $x \in K_\alpha$ there exist $n_x \in \mathbb{Z}$, $m(x) \in \mathbb{N}$ and $F_x \in [\{x_\beta: \beta < \alpha\}]^{<\omega}$ such that $|n_x| \neq 0 \pmod{\kappa(A_{m(x)})}$ while moreover

$$n_x x \in A_{m(x)} + \langle F_x \rangle.$$

Since $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$ and $|[\{x_\beta: \beta < \alpha\}]^{<\omega}| < \mathfrak{c}$, we can find an uncountable set $H \subseteq K_\alpha$, a fixed $\bar{n} \in \mathbb{Z}$, a fixed $\bar{m} \in \mathbb{N}$ and a fixed finite subset $F \subseteq \{x_\beta: \beta < \alpha\}$ such that

$$\bar{n}x \in A_{\bar{m}} + \langle F \rangle, \quad \bar{n} = n_x \quad \text{and} \quad \bar{m} = m(x),$$

for every $x \in H$. Since $|\langle F \rangle| \leqslant \aleph_0$ we can find an uncountable subset $H' \subseteq H$ and a point $p \in \langle F \rangle$ such that

$$\bar{n}x \in A_{\bar{m}} + p$$

for every $x \in H'$. Now take distinct $x, y \in H'$. Then

$$\bar{n}x - \bar{n}y \in A_{\bar{m}},$$

whence $\bar{n} = 0 \pmod{\kappa(A_{\bar{m}})}$. This is a contradiction.

We conclude that we can find a point $x \in K_\alpha$ such that for every $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that $|n| \neq 0 \pmod{\kappa(A_m)}$, and every finite $F \subseteq \{x_\beta: \beta < \alpha\}$ we have that

$$nx \notin A_m + \langle F \rangle.$$

Define $x_\alpha = x$. It is clear that x_α is as required.

Now let $E = \langle\{x_\alpha: \alpha < \mathfrak{c}\}\rangle$. We claim that $E \cap \bigcup_{i=1}^\infty \subseteq A_n$. Take $x \in E \cap \bigcup_{i=1}^\infty A_i$ arbitrarily. There exist $\alpha_1 < \cdots < \alpha_m < \mathfrak{c}$ and $n_1, \ldots, n_m \in \mathbb{Z}$ such that

$$x = n_1 x_{\alpha_1} + \cdots + n_m x_{\alpha_m}.$$

There is an $m_0 \geqslant n$ with $x \in A_{m_0}$. Since $\{x_{\alpha_1}, \ldots, x_{\alpha_m}\}$ is independent over A_{m_0} we conclude that $|n_1| = \cdots = |n_m| = 0 \pmod{\kappa(A_{m_0})}$. Since $\kappa(A_{m_0}) = \kappa(A_n)$ this implies that $x \in A_n$.

Observe that if $\kappa(A_i) = \infty$ for every $i \in \mathbb{N}$ then the same proof shows that $E \cap \bigcup_{i=1}^\infty A_i = \{0\}$. $\square$

Observe that in the above theorem in general it is impossible to construct E with the additional property that is connected. Simply observe that the only connected subgroup of $\mathbb{R}$ is $\mathbb{R}$ itself.

We will now construct an example showing that Theorem 3.3. cannot be improved.

Let I denote the unit interval $[0, 1]$ and, as usual, let λ denote Lebesgue measure on I. If X is a space then a function $f: I \to X$ is called *measurable* if $f^{-1}(U)$ is a Borel subset of I for every open $U \subseteq X$. Two measurable functions $f, g: I \to X$ are said to be *equivalent* if

$$\lambda(\{x \in I: f(x) \neq g(x)\} = 0.$$

Let M_X denote the topological space of equivalence classes of measurable functions from I into X with the topology of convergence in measure. Bessaga and Pelczyński [2, p. 194] have shown that M_X is homeomorphic to l_2, the separable Hilbert space, if and only if X is a topologically complete separable space containing more than one point. They also show that if G is a topological group then pointwise multiplication defines a compatible topological group structure on M_G, i.e. if $f, g \in M_G$ then $f \cdot g \in M_G$ and $f^{-1} \in M_G$ are defined by

$$\begin{cases} (f \cdot g)(x) = f(x) \cdot g(x), \\ (f^{-1})(x) = f(f(x))^{-1}. \end{cases}$$

Let $\bar{G} = \mathbb{Z}_4$ with the discrete topology (of course) and put $H = M_{\bar{G}}$. As remarked above, $H \approx l_2$, whence H is a connected topological group.[1] Put $A = \{2f: f \in H\}$. Then A is a subgroup of H and we first claim that A is closed. Let $g_n \in A$, $n \in \mathbb{N}$, be a sequence converging to a point $g \in H$. Then $\lim_{n\to\infty} 2g_n = 2g$. Since $g_n = 2f_n$, for certain $f_n \in H$, we have $2g_n = 4f_n = 0$ for every $n \in \mathbb{N}$. We conclude that $2g = 0$. We therefore find that $g(I) \subseteq \{0, 2\}$. Define $g': I \to G$ by $g'(x) = 0$ iff $g(x) = 0$ and $g'(x) = 1$ iff $g(x) = 2$. It is trivial that g' is measurable. Since clearly $2g' = g$, it follows that $g \in A$. Whence A is closed. We next claim that A is a proper subgroup of H. This is a triviality of course since the function $f \in H$ with constant value 1 obviously does not belong to A.

We now let E be any subgroup of H such that $E \neq \{0\}$. We claim that $E \cap A \neq \{0\}$. Indeed, take $x \in E \setminus \{0\}$ arbitrarily. Then $2x \in E \cap A$. If $2x \neq 0$ then we are done. If $2x = 0$ then, as above, we find that $x \in A$. So then $x \in E \cap A$, and in this case we are also done. From this we conclude that the subgroup E in Theorem 3.3. in general cannot be chosen such that $E \cap \bigcup_{i=1}^{\infty} A_i = \{0\}$. The question naturally arises whether we can always choose E in such a way that $E + \bigcup_{i=1}^{\infty} A_i = G$ (and $E \cap \bigcup_{i=1}^{\infty} A_i \subseteq A_n$ for some $n \in \mathbb{N}$ of course). We do not know the answer to this question.

4. A Lemma

The aim of this section is to prove Lemma 4.3., which is needed in Section 5 to prove all results which were announced in the introduction.

[1] (For this conclusion we do not need the Bessaga–Pelczyński result of course, it is routine to verify directly that $M_{\bar{G}}$ is path-connected.)

Throughout, let X be a space and assume that $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where

(1) X_α is closed, separable, and nowhere dense in X for every $\alpha < \omega_1$,

(2) $X_\alpha \subseteq X_\beta$ if $\alpha < \beta < \omega_1$,

(3) $X_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$ if $\alpha < \omega_1$ is a limit ordinal.

For every $\alpha < \omega_1$ define $P_\alpha = X_\alpha \setminus \bigcup_{\beta<\alpha} X_\beta$. Given a subset $\Sigma \subseteq \omega_1$ put

$$X(\Sigma) = \bigcup_{\alpha\in\Sigma} P_\alpha.$$

4.1. Lemma. *For every pair Σ_0, Σ_1 of disjoint subsets of ω_1 we have that the set $X(\Sigma_0) \times X(\Sigma_1)$ is of first category in the product $X \times X$.*

Proof. See [12, the proof of Lemma 1]. $\square$

4.2. Lemma. *Let $\Sigma \subseteq \omega_1$ be stationary and let $x_\xi \in P_\xi$ for $\xi \in \Sigma$. Then there is a point $x \in X$ such that each neighborhood of x contains uncountably many x_ξ's.*

Proof. See [11, the proof of (ii)→(iii) in Theorem 1]; cf. also [5, p. 181–182]. $\square$

We will now formulate and prove the main result in this section.

4.3. Lemma. *Let $\Sigma \subseteq \omega_1$ be stationary and let $E_\xi \subseteq P_\xi$ be a set of second category in X_ξ for $\xi \in \Sigma$. Then the set $E = \bigcup_{\xi\in\Sigma} E_\xi$ is of second category in X.*

Proof. Let F_n, $n \in \mathbb{N}$, be a sequence of closed subsets of X such that $E \subseteq \bigcup_{n\in\mathbb{N}} F_n$. We have to show that at least one F_i has nonempty interior in X. Fix $\xi \in \Sigma$. Since E_ξ is of second category in X_ξ, there exist a point $a_\xi \in E_\xi$ and i_ξ, $n_\xi \in \mathbb{N}$ such that

$$B(a_\xi, 1/i_\xi) \cap X_\xi \subseteq F_{n_\xi}.$$

There exist a stationary set $\Lambda \subseteq \Sigma$ and a pair i, n of natural numbers such that for each $\xi \in \Lambda$ we have that $i_\xi = i$ and $n_\xi = n$. By Lemma 4.2 there exists a point $a \in X$ such that the set

$$\Gamma = \{\xi \in \Lambda : \rho(a_\xi, a) < 1/2i\}$$

is uncountable. We claim that $B(a, 1/2i) \subseteq F_n$. Indeed, if $x \in B(a, 1/2i)$ choose $\xi < \omega_1$ with $x \in X_\xi$ and take $\gamma \in \Gamma$ with $\xi < \gamma$.

Then $\rho(x, a_\gamma) < 1/i$, so $x \in X_\xi \cap B(a_\gamma, 1/i) \subseteq X_\gamma \cap B(a_\gamma, 1/i) \subseteq F_n$ (since $\gamma \in \Gamma$). $\square$

5. The construction

After the preparatory work in the previous sections we are now in a position to formulate and prove our main results. Again, we deal with the linear case first.

5.1. Theorem. *Let L be a topologically complete metrizable linear space of weight $\aleph_1$. Then X is the direct sum $E \oplus F$ of two linear subspaces E and F such that both E and F are second category subsets of L whose product $E \times F$ is first category (in itself). In particular, both E and F are Baire spaces but $E \times F$ is not.*

Proof. We will construct a sequence

$$\{0\} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\alpha \subseteq \cdots \subseteq L, \quad \alpha < \omega_1,$$

of closed linear subspaces of L satisfying (1), (2) and (3) of Section 4 and, in addition, having the following property:

(1) X_α is of infinite codimension in $X_{\alpha+1}$, $\alpha < \omega_1$.

Let $\{x_\alpha : \alpha < \omega_1\}$ be dense in L, and assume that X_β has been defined for every $\beta < \alpha$, $\alpha < \omega_1$, having the additional property that if β is not a limit then

$$\{x_\xi : \xi \leqslant \beta\} \subseteq X_\beta.$$

If α is a limit put $X_\alpha = \overline{(\bigcup_{\beta<\alpha} X_\beta)}$. If α is not a limit, observe that X_α, being separable, has infinite codimension in L (use that L is not separable). So there exists a countably infinite linearly independent set $A \subseteq L$ such that $\operatorname{Lin} A \cap X_\alpha = \{0\}$. Put

$$X_{\alpha+1} = \overline{\operatorname{lin}(X_\alpha \cup A \cup \{x_\beta : \beta \leqslant \alpha\})}.$$

It is clear that $X_{\alpha+1}$ is as required.

Since L is metrizable, $\bigcup_{\alpha<\omega_1} X_\alpha$ is closed in L and since $x_\alpha \in X_{\alpha+1}$ for every $\alpha < \omega_1$, it follows that $\bigcup_{\alpha<\omega_1} X_\alpha = L$.

Throughout the rest of this proof we adopt the notation of Section 4.

Claim 1. If $\alpha_n < \cdots < \alpha_0 < \omega_1$ and $x_i \in P_{\alpha_i}$, then

$$x_0 + \cdots + x_n \in P_{\alpha_0}.$$

Observe that $x_1 + \cdots + x_n \in X_{\alpha_1}$. Suppose that $x_0 + \cdots + x_n \in X_\xi$ for certain $\xi < \alpha_0$. Then, for $\beta = \max(\alpha_1, \xi)$, we have that

$$x_0 = (x_0 + \cdots + x_n) - (x_1 + \cdots + x_n) \in X_\beta.$$

Since $x_0 \in P_{\alpha_0}$ and $P_{\alpha_0} \cap X_\beta = \emptyset$, this is a contradiction.

We are now in a position to construct the required linear subspaces E and F. Fix an ordinal $\alpha < \omega_1$ and let $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$ be a sequence of ordinals such that α is the least ordinal greater than every α_i. Using Theorem 3.1. with $Z = X_\alpha$ and $Z_i = X_{\alpha_i}$, one can find a linear subspace V_α of X_α of second category in X_α such that

$$(2) \qquad X_\alpha = V_\alpha \oplus \Big(\bigcup_{\xi<\alpha} X_\xi\Big),$$

so in particular,

$$(3) \qquad V_\alpha \setminus \{0\} \subseteq P_\alpha.$$

Split ω_1 into two disjoint stationary sets Σ_0, Σ_1 and let

$$V_i = \mathrm{Lin}\Big(\bigcup_{\alpha\in\Sigma_i} V_\alpha\Big), \quad i=0,1.$$

We claim that $E = V_0$ and $F = V_1$ are as required.

Claim 2. $V_i\setminus\{0\} \subseteq L(\Sigma_i) = \bigcup_{\alpha\in\Sigma_i} P_\alpha, \quad i=0,1.$

Take $x\in V_i\setminus\{0\}$ arbitrarily. Find ordinals $\alpha_j\in\Sigma_i$, $0\leq j\leq n$, and points $x_j\in V_{\alpha_j}\setminus\{0\}\subseteq P_{\alpha_j}$ (by (3)), such that $\alpha_n<\cdots<\alpha_0$ and $x=x_0+\cdots+x_n$. By Claim 1 it follows that $x = P_{\alpha_0}\subseteq L(\Sigma_i)$.

By Claim 2 and Lemma 4.1. it now follows that $V_0\times V_1$ is of first category in the product $L\times L$. However, since each V_α is a linear subspace of second category in X_α, and hence dense in X_α, it easily follows that both V_0 and V_1 are dense in L, whence $V_0\times V_1$ is dense in $L\times L$. Since $V_0\times V_1$ is of first category in $L\times L$ we conclude that $V_0\times V_1$ is of first category in itself.

Since $V_i\cap P_\alpha\supseteq V_\alpha\setminus\{0\}$ for every $\alpha\in\Sigma_i$, and since V_α is of second category in X_α, it follows that $V_i\cap P_\alpha$ is of second category in X_α for every $\alpha\in\Sigma_i$. By Lemma 4.3. we conclude that both V_0 and V_1 are of second category in L.

It remains to check that $L=V_0\oplus V_1$. Since by Claim 2 we have that $V_0\cap V_1=\{0\}$, it suffices to show that $L=V_0+V_1$. We will check by induction that $X_\alpha\subseteq V_0+V_1$. Clearly $X_0\subseteq V_0+V_1$. Assume that $X_\xi\subseteq V_0+V_1$ for every $\xi<\alpha$, $\alpha<\omega_1$. Without loss of generality we may assume that $\alpha\in\Sigma_0$. By (2),

$$X_\alpha = V_\alpha\oplus\Big(\bigcup_{\xi<\alpha} X_\xi\Big),$$

so $X_\alpha\subseteq V_\alpha+V_0+V_1$. But $V_\alpha\subseteq V_0$, i.e. $V_\alpha+V_0=V_0$. □

5.2. Corollary. *Let L be a topologically complete metrizable non-separable linear space. Then L contains two linear subspaces E and F such that both E and F are Baire but $E\times F$ is not.*

Proof. Observe that L contains a closed linear subspace of weight $\aleph_1$ and apply Theorem 5.1. □

We will now prove the announced results for topological groups.

5.3. Theorem. *Let G be a topologically complete metrizable abelian topological group of weight $\aleph_1$ and assume that G has a dense pathcomponent. Then G contains two subgroups E and F such that both E and F are second category subsets of G whose product is of first category (in itself). In particular, both E and F are Baire spaces but $E\times F$ is not. The subgroups E and F have the additional property that $E\cap F$ is separable (hence small).*

Proof. Our strategy is similar to the one in the proof of Theorem 5.1. Let $K \subseteq G$ be a dense pathcomponent. We will construct a sequence

$$\{0\} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\alpha \subseteq \cdots \subseteq G, \quad \alpha < \omega_1,$$

of closed and connected subgroups of G satisfying (1), (2) and (3) of Section 4 and, in addition, having the following property:

(1) X_α is a proper subgroup of $X_{\alpha+1}$, $\alpha < \omega_1$.

Let $\{x_\alpha: \alpha < \omega_1\}$ be dense in G, and assume that X_β has been defined for every $\beta < \alpha$, $\alpha < \omega_1$, having the additional property that if β is not a limit then

$$\{x_\xi: \xi \leqslant \beta\} \subseteq X_\beta.$$

If α is a limit put $X_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$. Now suppose that α is not a limit, say $\alpha = \beta+1$. Put $A = X_\alpha \cup \{x_\beta, x_\alpha\}$ and observe that A is closed and separable, whence nowhere dense in G (since G is connected, every separable subspace of G is nowhere dense). Let D be a countable dense subset of A and for every $d \in D$ choose a sequence in $K \backslash A$ converging to d. The union of these sequences yields a countable set $H \subseteq K \backslash A$ such that $A \subseteq \bar{H}$. For every $x, y \in H$ let $I(x, y) \subseteq G$ be a path connecting x and y. Then $H' = \bigcup \{I(x, y): x, y \in H\}$ is a separable connected set such that $A \subseteq \bar{H}'$. Now let $B = \langle H' \rangle$. Then B is connected since H' is connected. Similarly, B is separable. It is easily seen that $X_\alpha = \bar{B}$ satisfies our requirements. As in the proof of Theorem 5.1. it follows that $\bigcup_{\alpha<\omega_1} X_\alpha = G$. Again, we adopt the notation of Section 4.

For every $\alpha < \omega_1$ let $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$ be a sequence of ordinals such that α is the least ordinal greater than every α_i. By Theorem 3.3. we can find a subgroup $V_\alpha \subseteq X_\alpha$ and an integer $n(\alpha) \in \mathbb{N}$ such that

(a) V_α is of second category in X_α,

(b) $V_\alpha \cap \bigcup_{i=1}^\infty X_{\alpha_i} \subseteq X_{\alpha_{n(\alpha)}}$.

We conclude that for every $\alpha < \omega_1$ there is an $f(\alpha) < \alpha$ such that

(c) $V_\alpha \cap \bigcup_{\beta<\alpha} X_\beta \subseteq X_{f(\alpha)}$.

By PDL there is a stationary set $S \subseteq \omega_1$ and a $\beta < \omega_1$ such that for all $\alpha, \alpha' \in S$ we have $f(\alpha) = f(\alpha') = \beta < \alpha$.

Claim 1. If $\alpha_m < \cdots < \alpha_0 < \omega_1$, $\alpha_i \in S$ for every i, and if $x_i \in V_{\alpha_i}$, $n_i \in \mathbb{Z}$, then $n_0x_0 + \cdots + n_mx_m \in X_\beta \cup \bigcup_{i=0}^m P_{\alpha_i}$.

We prove the Claim by induction on m. If $m = 0$ then there is nothing to prove since then $n_0x_0 \in V_{\alpha_0} \subseteq X_\beta \cup P_{\alpha_0}$. Suppose therefore that $m > 0$.

By induction hypothesis we have

$$n_1x_1 + \cdots + n_mx_m \in X_\beta \cup \bigcup_{i=1}^m P_{\alpha_i}.$$

Case 1. $n_0x_0 \in X_\beta$. Choose $1 \leqslant i \leqslant m$ such that $n_1x_1 + \cdots + n_mx_m \in X_\beta \cup P_{\alpha_i}$. If $n_1x_1 + \cdots + n_mx_m \in X_\beta$ then there is nothing to prove. Suppose therefore that $y = n_1x_1 + \cdots + n_mx_m \in P_{\alpha_i}$. Observe that $n_0x_0, y \in X_{\alpha_i}$, so $n_0x_0 + y \in X_{\alpha_i}$. If $n_0x_0 + y \in X_\eta$ for certain $\beta \leqslant \eta < \alpha_i$ then $y \in X_\eta$ since X_η is a subgroup and $n_0x_0 \in X_\eta$, contradiction. We conclude that $n_0x_0 + y \in P_{\alpha_i}$.

Case 2. $n_0x_0 \in P_{\alpha_0}$. Again, choose $1 \leq i \leq m$ such that $y = n_1x_1 + \cdots + n_mx_m \in X_\beta \cup P_{\alpha_i}$. If $n_0x_0 + y \notin P_{\alpha_0}$ then $n_0x_0 + y \in X_\eta$ for certain $\alpha_i \leq \eta < \alpha_0$. Since X_η is a subgroup and $y \in X_\eta$ this would imply that $n_0x_0 \in X_\eta$, which is a contradiction since $P_{\alpha_0} \cap X_\eta = \emptyset$.

Split S into two disjoint stationary sets Σ_0, Σ_1 and let

$$V_i = \langle \bigcup_{\alpha \in \Sigma_i} V_\alpha \rangle, \quad i = 0, 1.$$

We claim that $E = V_0$ and $F = V_1$ are as required.

Claim 2. $V_i \subseteq X_\beta \cup G(\Sigma_i)$, whence $V_0 \cap V_1 \subseteq X_\beta$ (recall that $G(\Sigma_i) = \bigcup_{\alpha \in \Sigma_i} P_\alpha$).

Take $x \in V_i$ arbitrarily. There exist $\alpha_m < \cdots < \alpha_0 < \omega_1$, $\alpha_j \in \Sigma_i$ for every j, and $x_j \in V_{\alpha_j}$, $n_j \in \mathbb{Z}$ such that $x = n_0x_0 + \cdots + n_mx_m$. By Claim 1 it now follows that

$$x \in X_\beta \cup \bigcup_{j=0}^{m} P_{\alpha_j} \subseteq X_\beta \cup G(\Sigma_i).$$

By Claim 2 and Lemma 4 we may conclude that $(V_0 \backslash X_\beta) \times (V_1 \backslash X_\beta)$ is of first category in the product $G \times G$. Since

$$V_0 \times V_1 \subseteq (V_0 \backslash X_\beta) \times (V_1 \backslash X_\beta) \cup (X_\beta \times G) \cup (G \times X_\beta),$$

it also follows that $V_0 \times V_1$ is of first category in the product $G \times G$ (observe that X_β is nowhere dense in G). Since each V_α is a subgroup of second category of the *connected* group X_α, it follows that V_α is dense in X_α, and from this it easily follows that both V_0 and V_1 are dense in G. Consequently, $V_0 \times V_1$ is a dense first category subset of $G \times G$, whence $V_0 \times V_1$ is of first category in itself.

Since $V_i \cap P_\alpha \supseteq V_\alpha \backslash \{0\}$ for every $\alpha \in \Sigma_i$, and since V_α is of second category in X_α, it follows that $V_i \cap P_\alpha$ of second category in X_α for every $\alpha \in \Sigma_i$. From Lemma 4.3. we therefore conclude that $V_i \backslash X_\beta$ is of second category in G, for every $i \in \{0, 1\}$. Since X_β is closed and nowhere dense in G it follows that V_i is of second category in G, $i = 0, 1$. □

5.4. Corollary. *Let G be a topologically complete metrizable non-separable abelian topological group and assume that G has a dense pathcomponent. Then G contains two subgroups E and F such that both E and F are Baire but E × F is not.*

Proof. We will show that G contains a closed subgroup H of weight $\aleph_1$ having a dense pathcomponent. Then we can apply Theorem 5.3. to obtain the desired result.

Let $K \subseteq G$ be a dense pathcomponent. Since G is a topological group we may assume that $0 \in K$. By transfinite induction, we will construct for every $\alpha < \omega_1$ a separable closed subgroup $H_\alpha \subseteq G$ and a countable set $D_\alpha \subseteq K$ such that

(1) if $\beta < \alpha$ then H_β is a proper subgroup of H_α,

(2) D_α is dense in H_α,

(3) if $\beta \leq \alpha$, $x \in D_\beta$ and $y \in D_\alpha$ then x and y can be joined by path in H_α.

Then $H=\bigcup_{\alpha<\omega_1} H_\alpha$ is clearly as required (observe that by (1) H has weight $\aleph_1$).

Put $H_0=\{0\}$ and $D_0=\{0\}$. Assume that H_β and D_β have been defined for every $\beta<\alpha$, $\alpha<\omega_1$. If α is a limit, put $H_\alpha=\bigcup_{\beta<\alpha} H_\beta$ and $D_\alpha=\bigcup_{\beta<\alpha} D_\beta$. Suppose therefore that $\alpha=\beta+1$. Let $D=\bigcup_{\xi\leqslant\beta} D_\xi$ and for every $d\in D$ choose a sequence from $K\setminus H_\beta$ converging to d. Let H be the union of these sequences and for every $x, y\in H\cup D$ let $I(x,y)\subseteq G$ be a path connecting x and y. Put $B=\bigcup\{I(x,y): x, y\in H\cup D\}$. Observe that B is separable and that $B\subseteq K$. Put $H_\alpha=\langle B\rangle$ and let $D_\alpha\subseteq\langle B\rangle$ be any countable dense set. Since B is pathconnected it also follows that $\langle B\rangle$ is pathconnected (observe that $\langle B\rangle$ is a continuous image of a topological sum of countably many finite powers of B). We conclude that $\langle B\rangle\subseteq K$ from which it follows that $D_\alpha\subseteq K$, as required. □

5.5. Remark. Observe that in the proofs of Theorem 5.3. and Corollary 5.4. we 'only' used that G contains a dense set D such that for all $x, y\in D$ there is a separable connected subset of G that contains both x and y. We do not know whether every topologically complete connected abelian topological group has this property.

6. Remarks

It can be shown that Theorem 5.1. is also true for linear spaces of weight $\aleph_2$. Under the Generalized Continuum Hypothesis we can show that every non-separable Banach space admits a bounded linear operator onto a Banach space of weight $\aleph_1$ or $\aleph_2$. Since this operator is open, applying Theorem 5.1. and its generalization, it follows that every non-separable Banach space B is the sum of two linear subspaces E and F such that both E and F are Baire but $E\times F$ is not. This result suggests the following question.

6.1. Question. Let B be a non-separable Banach space. Is B the direct sum $E\oplus F$ of two linear subspaces E and F such that both E and F are Baire but $E\times F$ is not?

The following question seems particularly hard and interesting.

6.2. Question. Is there a normed linear space X such that X is Baire but $X\times X$ is not?

In the light of Theorem 5.1. our results for topological groups are not satisfying. In Section 3 by means of an example we demonstrated that our construction for topological groups is inadequate for getting a direct sum decomposition such as in Theorem 5.1. For this reason we pose the following

6.3. Question. Let G be a pathconnected topologically complete topological group of weight $\aleph_1$. Is G the sum $E+F$ of two subgroups E and F such that both E

and F are Baire but $E \times F$ is not? When is it possible to write G as the direct sum $E \oplus F$ of two subgroups E and F such that both E and F are Baire but $E \times F$ is not?

If the answer to Question 6.3. is in the affirmative then the same question should be posed for topological groups of arbitrary uncountable weight.

References

[1] J. Arias de Reyna, Normed barely Baire spaces, Isr. J. Math. 42 (1982) 33-36.
[2] C. Bessaga and A. Pelczyński, Selected topics in infinite-dimensional topology (PWN, Warsaw, 1975).
[3] G. Choquet, Lectures on analysis, Vol. I (Benjamin, New York, 1969) 105-126.
[4] P.E. Cohen, Products of Baire spaces, Proc. Am. Math. Soc. 55 (1976) 119-124.
[5] W.G. Fleissner, Applications of stationary sets in topology (Academic Press, New York, 1980) 163-193.
[6] W.G. Fleissner and K. Kunen, Barely Baire spaces, Fund. Math. 101 (1978) 229-240.
[7] M.R. Krom, Cartesian products of metric Baire spaces, Proc. Am. Math. Soc. 42 (1974) 588-594.
[8] R.D. Mauldin, On the Borel subspaces of algebraic structures, Indiana Univ. Math. J. 29 (1980) 261-265.
[9] J. Mycielski, Independent sets in topological algebras, Fund. Math. 55 (1964) 139-147.
[10] J.C. Oxtoby, Cartesian products of Baire spaces, Fund. Math. 49 (1961) 157-166.
[11] R. Pol, Note on decompositions of metrizable spaces, Fund. Math. 95 (1977) 95-103.
[12] R. Pol, Note on category in Cartesian products of metrizable spaces, Fund. Math. 102 (1979) 55-59.
[13] R. Sikorski, On the Cartesian product of metric spaces, Fund. Math. 34 (1977) 288-292.
[14] M. Valdivia, Products of Baire topological vector spaces, Fund. Math. 125 (1985) 71-80.