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A SEPARABLE NORMAL TOPOLOGICAL GROUP WHICH IS NOT LINDELÖF

Klaas Pieter HART and Jan van MILL
Subfaculteit Wiskunde en Informatica, Vrije Universiteit, Amsterdam, The Netherlands

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We construct in ZFC the example mentioned in the title.

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topological group separable normal non-Lindelöf

0. Introduction

In 1968 Wilansky asked whether a separable normal topological group must be Lindelöf. In [7] this question was answered in the negative, assuming CH, by constructing a hereditarily separable, normal group which is not Lindelöf. We give an example of a separable normal group which contains a closed subspace homeomorphic to an uncountable regular cardinal, in ZFC only.

Of course we have to sacrifice hereditary separability, since Todorčević (and a little later Baumgartner) showed that it is consistent to assume that hereditarily separable regular spaces are Lindelöf (in short: 'There are no S-spaces'). For details see [10].

In Section 2 we associate with each topological space X a group $B(X)$, which has a topology such that all translations are continuous, and show that in some special cases $B(X)$ is a topological group.

In Section 3 we give a space X whose $B(X)$ will be the desired example. Our $B(X)$ construction was somewhat inspired by the work done concerning free topological groups. For details and references, see Smith-Thomas [11].

1. Definitions and preliminaries

For set theory consult [8] and for topology consult [1].

1.0. The free Boolean group of a set.

A group G is called *Boolean* if every element of it has order at most 2. In this case G is Abelian.

Now let X be a set. The free Boolean group $B(X)$ of X is a Boolean group containing X such that every function from X to a Boolean group extends to a unique homomorphism from $B(X)$ to that group. We give two visualizations of $B(X)$.

(a) The set of finite subsets of X, with symmetric difference as group operation.

(b) The subgroup $\{x \in {}^{X}2 : |x^{\leftarrow}(1)| < \omega\}$ of ${}^{X}2$.

We shall write the elements of $B(X)$ as formal Boolean sums of elements of X. For every $n \in \mathbb{N}$ we define a function $\varphi_n : X^n \to B(X)$ by $\varphi_n(x) = x_1 + x_2 + \cdots + x_n$. We let $X_n = \varphi_n(X^n)$. Notice that

$$X_1 \subseteq X_3 \subseteq X_5 \subseteq \cdots \quad \text{and} \quad X_2 \subseteq X_4 \subseteq X_6 \subseteq \cdots.$$

Let $E = \bigcup_{n\in\mathbb{N}} X_{2n}$ and $O = \bigcup_{n\in\mathbb{N}} X_{2n-1}$. Observe that $E \cap O = \emptyset$ and that the identity element of $B(X)$ is in E.

1.1. Permutations and X^n.

As usual S_n is the group of permutations of $\{1, 2, \ldots, n\}$. Let X be a set. Every $\sigma \in S_n$ induces a function $\sigma : X^n \to X^n$ via $\sigma(x) = \langle x_{\sigma(1)}, \ldots, x_{\sigma(n)} \rangle$. The set of orbits under this action of S_n on X^n will be denoted by $\hat{X}_n$ and $\pi_n : X^n \to \hat{X}_n$ will be the natural projection. Furthermore for each $\sigma \in S_n$, $\varphi_n \circ \sigma = \varphi_n$ so that φ_n induces a function $\psi_n : \hat{X}_n \to X_n$ such that $\psi_n \circ \pi_n = \varphi_n$.

Now assume that X is a topological space. Then each $\sigma \in S_n$ is an autohomeomorphism of X^n. So, since for each $A \subseteq X^n$ we have $\pi_n^{\leftarrow}\pi_n(A) = \bigcup_{\sigma\in S_n} \sigma(A)$, if we give $\hat{X}_n$ the quotient topology determined by X^n and π_n, then π_n will be a closed and open map since S_n is finite.

1.2. Adjunction spaces [3].

Let X and Y be topological spaces, $A \subseteq X$ closed and $f : A \to Y$ continuous. Consider the sum $X \oplus Y$ and let $\sim$ be the smallest equivalence relation on $X \oplus Y$ satisfying $f(x) \sim x$ for all $x \in A$. The quotient of $X \oplus Y$ determined by $\sim$ will be denoted by $X \cup_f Y$ and is called the *adjunction space of X and Y by f*. Actually, Dugundji uses the term 'X is attached to Y by f' but 'adjunction space' is more common these days. We mention the following facts:

(a) $X \setminus A$ is an open subspace of $X \cup_f Y$.

(b) Y is a closed subspace of $X \cup_f Y$.

(c) The adjunction space preserves the following properties: T_1, normal and compact Hausdorff.

2. The general construction

In this section we associate with each topological space a homogeneous space. In the second part we shall show that in some special cases we actually have a topological group.

To avoid trivial nuisance we shall assume that all spaces under consideration are infinite and Hausdorff.

2.0. Construction.

Let X be a topological space. We shall topologize the group $B(X)$ as follows. First we give each set X_n the quotient topology determined by X^n and φ_n. We then define

$$\tau=\{U\subseteq B(X)\colon\ U\cap X_n \text{ is open in } X_n \text{ for all } n\},$$

i.e., τ is the topology on $B(X)$ determined by the sets X_n, $n\in\mathbb{N}$. This is the topology on $B(X)$ we are interested in and from now on we shall assume that $B(X)$ carries this topology.

The referee pointed out that our construction looks like the one of Ordman [9]. The precise relationship seems unclear.

Straight from the definition we can prove the following proposition.

2.1. Proposition. (a) *$B(X)$ is homogeneous.*

(b) *Both E and O are clopen in $B(X)$.*

Proof. (a) We show that translations are continuous, and of course it suffices to consider words of length 1, i.e. elements of X_1, only. Let $x\in X$ and let $U\subseteq B(X)$ be open. To show that $U+x$ is open we must show that for each $n\in\mathbb{N}$ the set

$$\overleftarrow{\varphi_n}((U+x)\cap X_n)$$

is open in X^n. Let $n\in\mathbb{N}$ and define $\boldsymbol{i}_x\colon X^n\to X^{n+1}$ by $\boldsymbol{i}_x(\boldsymbol{x})=\langle\boldsymbol{x},x\rangle$. Then $\overleftarrow{\varphi_n}((U+x)\cap X_n)=\{\boldsymbol{x}\in X^n\colon x+\varphi_n(\boldsymbol{x})\in U\}=\{\boldsymbol{x}\in X^n\colon\langle\boldsymbol{x},x\rangle\in\overleftarrow{\varphi}_{n+1}(U\cap X_{n+1})\}=\overleftarrow{\boldsymbol{i}_x}\overleftarrow{\varphi}_{n+1}(U\cap X_{n+1})$ is open in X^n, because $U\cap X_{n+1}$ is open in X_{n+1} and $\boldsymbol{i}_x$ is continuous.

(b) Note that $E\cap X_n=X_n$ if n is even and $E\cap X_n=\emptyset$ if n is odd. So E is clopen and hence so is $O=B(X)\setminus E$. $\square$

The following proposition shows that each X_n is a closed subspace of $B(X)$.

2.2. Proposition. *X_n is a closed subspace of X_{n+2} for each n.*

Proof. (a) Since $\overleftarrow{\varphi}_{n+2}(X_n)=\bigcup_{1\leqslant i<j\leqslant n+2}\{\boldsymbol{x}\in X^{n+2}\colon x_i=x_j\}=\bigcup_{\sigma\in S_{n+2}}\sigma(X^n\times\Delta X)$ (here ΔX denotes the diagonal of X in X^2), which is closed in X^{n+2} (X is Hausdorff), X_n is closed in X_{n+2}.

(b) Let $F\subseteq X_n$ be closed in X_n. Then $\overleftarrow{\varphi}_{n+2}(F)=\bigcup_{\sigma\in S_{n+2}}\sigma(\overleftarrow{\varphi_n}(F)\times\Delta X)$ is closed in X^{n+2}. Hence F is closed in X_{n+2}.

(c) Let $F\subseteq X_n$ be closed in X_{n+2}. Fix $x\in X$ and define $\boldsymbol{j}_x\colon X^n\to X^{n+2}$ by $\boldsymbol{j}_x(\boldsymbol{x})=\langle\boldsymbol{x},x,x\rangle$. Then $\overleftarrow{\varphi_n}(F)=\overleftarrow{\boldsymbol{j}_x}\overleftarrow{\varphi}_{n+2}(F)$ is closed in X_n, by continuity of $\boldsymbol{j}_x$, so that F is closed in X_n. $\square$

It is easy to estimate the density of $B(X)$.

2.3. Proposition. $d(B(X)) \leq d(X)$.

Proof. If D is dense in X then $\bigcup_{n\in\mathbb{N}} \varphi_n(D^n)$ is dense in $B(X)$. As $d(X) \geq \omega$, this shows that $d(B(X)) \leq d(X)$. □

We now give an alternative description of the topologies on the sets X_n. From this description it is easy to derive certain properties of $B(X)$.

2.4. Lemma. *X_n also has the quotient topology determined by $\hat{X}_n$ and ψ_n.*

Proof. (a) If $\psi_n^{\leftarrow}(F)$ is closed then $\varphi_n^{\leftarrow}(F) = \pi_n^{\leftarrow}\psi_n^{\leftarrow}(F)$ is closed.

(b) If $\varphi_n^{\leftarrow}(F)$ is closed then $\psi_n^{\leftarrow}(F) = \pi_n(\varphi_n^{\leftarrow}(F))$ is closed because the map π_n is closed. □

We let $\hat{\psi}_{n+2}$ denote the restriction $\psi_{n+2} \restriction \psi_{n+2}^{\leftarrow}(X_n)$.

2.5. Proposition. $X_{n+2} = \hat{X}_{n+2} \bigcup_{\hat{\psi}_{n+2}} X_n$.

Proof. Outside $\psi_{n+2}^{\leftarrow}(X_n)$ the map ψ_{n+2} is one-to-one, so we can assume that the underlying set of $\hat{X}_{n+2} \bigcup_{\hat{\psi}_{n+2}} X_n$ is X_{n+2}. If we let $\pi: \hat{X}_{n+2} \oplus X_n \to X_{n+2}$ be the adjunction projection then $\pi \restriction \hat{X}_{n+2} = \psi_{n+2}$ and $\pi \restriction X_n = \mathrm{id}$.

Let $F \subseteq X_{n+2}$. Then F is closed in $X_{n+2} \leftrightarrow F$ is closed in X_{n+2} and $F \cap X_n$ is closed $\leftrightarrow \psi_{n+2}^{\leftarrow}(F)$ is closed in $\hat{X}_{n+2}$ and $F \cap X_n$ is closed, $\leftrightarrow F$ is closed in $\hat{X}_{n+2} \bigcup_{\hat{\psi}_{n+2}} X_n$. □

We are now in a position to give a sufficient condition for normality of $B(X)$.

2.6. Proposition. *If X^n is normal then X_n is normal.*

Proof. We use induction.

$n = 1$. X_1 is homeomorphic to X hence normal.

$n = 2$. $\hat{X}_2$ is normal since π_2 is closed. Furthermore, X_2 is obtained from $\hat{X}_2$ by collapsing $\pi_2(\Delta X)$ to a point, hence $\hat{X}_2$ is normal.

$n \geq 3$. Since X^n is normal, X^{n-2} is also normal. Hence X_{n-2} is normal by induction hypothesis. Also $\hat{X}_n$ is normal since π_n is closed. We conclude that $X_n = \hat{X}_n \bigcup_{\hat{\psi}_n} X_{n-2}$ is normal (Proposition 2.5). □

This gives us:

2.7. Theorem. *If X^n is normal for all $n \in \mathbb{N}$ then $B(X)$ is normal.*

Proof. We show that E is normal. This is sufficient since $O = E + x$ for some (all) $x \in X$ and both E and O are clopen (Proposition 2.1). Let $A \subseteq E$ be closed and

$f: A \to I$ continuous. Using the fact that $\{X_{2n}\}_{n\in\mathbb{N}}$ is an increasing closed cover of E by normal spaces, it is easy to build an extension $F: E \to I$ of f such that $F \restriction X_{2n}$ is continuous for all $n \in \mathbb{N}$. By the definition of the topology on $B(X)$ it follows that F is continuous. $\square$

We now show that for a restrictive class of spaces $B(X)$ is a topological group. First we show that $B(X)$ is a topological group if X is compact.

2.8. Theorem. *If X is compact then $B(X)$ is a topological group.*

Proof. As E is clopen in $B(X)$ it suffices to show that $+: E \times E \to E$ is continuous. First of all by Proposition 2.5 each X_n is compact and Hausdorff. We need something like Proposition 2.5 for this since arbitrary quotients of Hausdorff spaces need not be Hausdorff. Furthermore to start the inductive proof we must show that X_1 and X_2 are compact Hausdorff: X_1 is homeomorphic to X and π_2 and φ_2 are closed quotient mappings so that X_2 is compact Hausdorff.

Second, using compactness of the spaces X_{2n} $(n \in \mathbb{N})$ it is easy to show that the sequence $\{X_{2n} \times X_{2n}\}_{n\in\mathbb{N}}$ determines the topology of $E \times E$ (i.e. $U \subseteq E \times E$ is open iff $U \cap (X_{2n} \times X_{2n})$ is open in $X_{2n} \times X_{2n}$ for all $n \in \mathbb{N}$). It remains to show that $+: X_{2n} \times X_{2n} \to X_{4n}$ is continuous for each n. Consider the diagram

$$\begin{tikzcd} X^{2n} \times X^{2n} \arrow[r, "h"] \arrow[d, "\varphi_{2n} \times \varphi_{2n}"'] & X^{4n} \arrow[d, "\varphi_{4n}"] \\ X_{2n} \times X_{2n} \arrow[r, "+"] & X_{4n} \end{tikzcd}$$

where $h(x, y) = \langle x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n} \rangle$ is the obvious homeomorphism. The diagram commutes and by compactness the map $\varphi_{2n} \times \varphi_{2n}$ is closed. Hence for a closed set $F \subseteq X_{4n}$ the set $+^{\leftarrow}(F) = \varphi_{2n} \times \varphi_{2n}(h^{\leftarrow}\varphi_{4n}^{\leftarrow}(F))$ is closed. $\square$

We shall now show that for certain spaces $B(X)$ is a subspace of $B(\beta X)$, as $B(X)$ is already a subgroup of $B(\beta X)$, for these spaces we then know that $B(X)$ is a topological group.

2.9. Proposition. *Suppose that X^n is normal and that $(\beta X)^n = \beta(X^n)$. Then X_n is a subspace of $(\beta X)_n$ and in fact $(\beta X)_n = \beta(X_n)$.*

Proof. Consider the following diagram:

$$\begin{tikzcd} X^n \arrow[r, hook, "\beta"] \arrow[d, "\varphi_n"'] & \beta X^n \arrow[d, "\varphi_n^\beta"] \\ X_n \arrow[r, hook, "i"] & (\beta X)_n \end{tikzcd}$$

where β and i are the natural inclusion maps.

(a) We have that $i \circ \varphi_n = \varphi_n^\beta \circ \beta$ and that $\varphi_n^\beta \circ \beta$ is continuous. Hence i is continuous since φ_n is a quotient map.

(b) Let $f: X_n \to I$ be continuous. We shall find a continuous $g: (\beta X)_n \to I$ such that $g \circ i = f$. Let $\bar{f} = f \circ \varphi_n$, and let $\bar{g}: \beta X^n \to I$ be the Stone extension of $\bar{f}$. We show that $\bar{g}$ induces a map $g: (\beta X)_n \to I$. Suppose that $\varphi_n^\beta(x) = \varphi_n^\beta(y)$ for $x, y \in \beta X^n$.

Since for each $\sigma \in S_n$ we have that $\varphi_n \circ \sigma = \varphi_n$ it follows that $\bar{f} \circ \sigma = \bar{f}$ for each $\sigma \in S_n$. Consequently, $\bar{g} \circ \sigma = \bar{g}$ for each $\sigma \in S_n$.

So we may assume that $x = \langle x_1, \dots, x_p, x_{p+1}, \dots, x_n \rangle$ and $y = \langle x_1, \dots, x_p, y_{p+1}, \dots, y_n \rangle$ with $x_{p+1} = x_{p+2}, \dots, x_{n-1} = x_n$ and $y_{p+1} = y_{p+2}, \dots, y_{n-1} = y_n$. Now choose nets in X converging to each coordinate in question, say $z_i^\alpha \to x_i$, $1 \leqslant i \leqslant p$, $u_{p+1}^\alpha \to x_{p+1}, \dots, u_{n-1}^\alpha \to x_{n-1}$, $v_{p+1}^\alpha \to y_{p+1}, \dots, v_{n-1}^\alpha \to y_{n-1}$. Then

$$
\begin{array}{ll}
 & \bar{f}(\langle z_1^\alpha, \dots, z_p^\alpha, u_{p+1}^\alpha, u_{p+1}^\alpha, \dots, u_{n-1}^\alpha, u_{n-1}^\alpha \rangle) \to \bar{g}(x) \\
f(z_1^\alpha + \dots + z_p^\alpha) \begin{array}{l} \nearrow\!\!\!\!\!= \\ \searrow\!\!\!\!\!= \end{array} & \\
 & \bar{f}(\langle z_1^\alpha, \dots, z_p^\alpha, v_{p+1}^\alpha, v_{p+1}^\alpha, \dots, v_{n-1}^\alpha, v_{n-1}^\alpha \rangle) \to \bar{g}(y)
\end{array}
$$

so that $\bar{g}(x) = \bar{g}(y)$. This proves the existence of g. Now g satisfies $\bar{g} = g \circ \varphi_n^\beta$. Since $\bar{g}$ is continuous and φ_n^β is quotient, this implies that g is continuous. In addition, for $x \in X^n$,

$$g \circ i \circ \varphi_n(x) = g \circ \varphi_n^\beta \circ \beta(x) = \bar{g} \circ \beta(x) = \bar{f}(x) = f \circ \varphi_n(x).$$

Hence $g \circ i = f$.

(c) Now by (a) and (b) X_n is a subspace of $(\beta X)_n$ and $(\beta X)_n = \beta X_n$, *provided* X_n is completely regular. However X_n is normal since X^n is normal (Proposition 2.6). $\square$

2.10. Remark. Something like normality of X^n is needed for Proposition 2.9. To see this consider X_2. It is easy to see that regularity of X_2 implies that ΔX has a closed neighborhood base in X^2. In [6] an example is given of a space with very strong separation properties which does not possess this last property and for which X_2 and hence $B(X)$ is therefore not even regular.

2.11. Theorem. *If for all $n \in \mathbb{N}$ the space X^n is normal and $(\beta X)^n = \beta(X^n)$, then $B(X)$ is a subspace of $B(\beta X)$ and consequently a topological group.*

Proof. (a) Let $A \subseteq B(\beta X)$ be closed. Then for each $n \in \mathbb{N}$ (Proposition 2.9)

$$A \cap B(X) \cap X_n = A \cap X_n = A \cap \beta X_n \cap X_n$$

is closed in X_n. Hence $A \cap B(X)$ is closed in $B(X)$.

(b) Let $A \subseteq B(X)$ be closed. For each $n \in \mathbb{N}$ let $A_n = \overline{A \cap X_n}$, where the closure is taken in $B(\beta X)$ or equivalently in βX_n (Proposition 2.9). Let $A' = \bigcup_{n \in \mathbb{N}} A_n$. Let

$n, m \in \mathbb{N}$ with $n \equiv m \pmod 2$. If $n \leqslant m$ then $A_n \subseteq A_m$. If $n > m$ then

$$\begin{aligned} A_n \cap \beta X_n &= \overline{A \cap X_n} \cap \overline{X_m} \\ &= \overline{A \cap X_n \cap X_m} \quad \text{(in } \beta X_m\text{, using the fact that } X_m \text{ is normal)} \\ &= \overline{A \cap X_m} = A_m. \end{aligned}$$

Hence for each n we have that $A' \cap \beta X_n = A_n$ is closed in βX_n. So A' is closed in $B(\beta X)$, due to the definition of the topology of $B(\beta X)$. Since

$$A' \cap X_n = A' \cap X_n \cap \beta X_n = A_n \cap X_n = A \cap X_n,$$

we find that $A' \cap B(X) = A$.

Now (a) and (b) together show that $B(X)$ is a subspace of $B(\beta X)$. $\square$

The above result is quite restrictive of course. It is however exactly what is needed for the example in Section 3, which is how we got to it in the first place.

3. The example

We shall now describe a particular space X for which $B(X)$ will be a separable normal topological group without any nice covering properties.

To begin with we introduce some notions and notations (for these and other notions see [2]).

3.0. For (infinite) subsets $A, B \subseteq \omega$, $A \subset_* B$ means $|A \setminus B| < \omega$ and $|B \setminus A| = \omega$. If γ is an ordinal number then a *tower of length* γ is a sequence $\{A_\alpha : \alpha \in \gamma\}$ of subsets of ω, such that $(\alpha \in \beta \in \gamma) \rightarrow (A_\alpha \subset_* A_\beta)$.

A tower $\{A_\alpha : \alpha \in \gamma\}$ is *maximal* if whenever $X \subseteq \omega$ satisfies $(\forall \alpha \in \gamma)(A_\alpha \subset_* X)$ then X is cofinite in ω. By $\mathfrak{t}$ we denote the minimum length of a maximal tower. Note that $\mathfrak{t}$ is a regular uncountable cardinal.

3.1. Now let $\{A_\alpha : \omega \leqslant \alpha < \mathfrak{t}\}$ be a maximal tower. We topologize $\mathfrak{t}$ as follows:

(i) Points of ω will be isolated.

(ii) The nth neighborhood $U(\omega, n)$ of ω will be $\{\omega\} \cup A_\omega \setminus n$.

(iii) If $\omega \leqslant \beta \in \alpha \in \mathfrak{t}$ and $n \in \omega$ then $U(\alpha, \beta, n) = (\beta, \alpha] \cup (A_\alpha \setminus (A_\beta \cup n))$ is a basic neighborhood of α.

This will be our space X.

3.2. The space X is due to Franklin and Rajagopalan [4], we refer to [2, Example 7.1] for the proof that X is:

(1) Separable: ω is dense in X.

(2) Sequentially compact: this uses maximality of the tower; and

(3) Normal: the basic open sets are clopen and of two disjoint closed sets at least one must be compact.

At the end of the section we shall show that X^n is normal for each $n \in \mathbb{N}$.

3.3. Example. $B(X)$ is a separable normal topological group containing $\mathfrak{t}$ as a closed subspace.

Proof. By Proposition 2.3 $B(X)$ is separable and by Proposition 2.7 $B(X)$ is normal. As $\mathfrak{t}$ is closed in X and X is a closed subspace of $B(X)$, $\mathfrak{t}$ is also a closed subspace of $B(X)$. Finally X^n is sequentially compact for each n, hence pseudocompact and it follows from Glicksberg's Theorem [5] that $(\beta X)^n = \beta X^n$ for all n. So now by Theorem 2.11 $B(X)$ is a topological group. $\square$

3.4. We finish by showing that X^n is normal for every $n \in \mathbb{N}$. We use induction on n.

By 3.2, X is normal. Let $n \geqslant 2$ and assume that X^{n-1} is normal. To begin with, it is an easy exercise on closed unbounded subsets of $\mathfrak{t}$ to establish the known fact that disjoint closed subsets of $\mathfrak{t}^n$ have disjoint closures in $(\mathfrak{t}+1)^n$. Now let $F, G \subseteq X^n$ be closed and disjoint. Applying the above-mentioned fact we can find a finite sequence $\omega = \alpha_0 < \alpha_1 < \cdots < \alpha_m < \mathfrak{t}$ of ordinals, such that if we put

$$\mathcal{O} = \{[\omega, \alpha_1], \ldots, (\alpha_{m-1}, \alpha_n], (\alpha_m, \mathfrak{t})\},$$

and

$$\mathcal{O}^{(n)} = \left\{ \prod_{i=1}^{n} O_i : \langle O_i \rangle_{i=1}^{n} \in \mathcal{O}^n \right\},$$

then $\mathcal{O}^{(n)}$ is a finite disjoint clopen (in $[\omega, \mathfrak{t})^n$) cover of $[\omega, \mathfrak{t})^n$ satisfying

$$\forall O \in \mathcal{O}^{(n)}, \quad (O \cap F = \emptyset) \vee (O \cap G = \emptyset).$$

Put $\mathscr{C}_F = \{O \in \mathcal{O}^{(n)} : O \cap F \neq \emptyset\}$, $C_F = \bigcup \mathscr{C}_F$, $\mathscr{C}_G = \{O \in \mathcal{O}^{(n)} : O \cap F = \emptyset\}$, and $C_G = \bigcup \mathscr{C}_G$, respectively. Using the facts that the sequence $\langle \alpha_1, \ldots, \alpha_m \rangle$ is finite and that $\mathfrak{t}$ is regular, one can find a $p_0 \in \omega$ and an unbounded $S \subseteq \mathfrak{t}$ such that

(i) $1 \leqslant i < n \to A_{\alpha_i} \setminus A_{\alpha_{i+1}} \subseteq p_0$,

(ii) $\alpha \in S \to \alpha > \alpha_m \wedge A_{\alpha_m} \setminus A_\alpha \subseteq p_0$,

(iii) $\bigcup_{\alpha \in S} A_\alpha \setminus (A_{\alpha_n} \cup p_0) = \omega \setminus (A_{\alpha_n} \cup p_0)$ ($\bigcup_{\alpha \in S} A_\alpha$ satisfies $(\forall \beta \in \mathfrak{t})$ $(A_\beta \subset_* \bigcup_{\alpha \in S} A_\alpha)$ and hence is cofinite).

For $O \in \mathcal{O}$ and $q \geqslant p_0$ let

$$O(q) = \begin{cases} [\omega, \alpha_1] \cup (A_{\alpha_1} \setminus q) & \text{if } O = [\omega, \alpha_1], \\ U(\alpha_{i+1}, \alpha_i, q) & \text{if } O = (\alpha_i, \alpha_{i+1}] \ (1 \leqslant i < m), \\ \bigcup_{\alpha \in S} U(\alpha, \alpha_m, q) & \text{if } O = (\alpha_m, \mathfrak{t}). \end{cases}$$

For $O = \prod_{i=1}^n O_i \in \mathcal{O}^{(n)}$ put $O(q) = \prod_{i=1}^n O_i(q)$. Finally, let
$\mathcal{O}(q) = \{O(q) : O \in \mathcal{O}\}$ and define $\mathcal{O}(q)^{(n)}$ similarly to $\mathcal{O}^{(n)}$,
$\mathscr{C}_{F,q} = \{O(q) : O \in \mathscr{C}_F\}$, and $O_{F,q} = \bigcup \mathscr{C}_{F,q}$, and
$\mathscr{C}_{G,q} = \{O(q) : O \in \mathscr{C}_G\}$, and $O_{G,q} = \bigcup \mathscr{C}_{G,q}$.

We now have that: $\mathcal{O}(q)$ is a disjoint clopen collection by choice of p_0 and $\bigcup \mathcal{O}(q) = [\omega, t) \cup (\omega \backslash q)$; $\mathcal{C}_{F,q} \cup \mathcal{C}_{G,q} = \mathcal{O}(q)^{(n)}$, so that $O_{F,q} \cup O_{G,q} = ([\omega, t) \cup (\omega \backslash q))^n$;

$\bigcap_{q \geq p_0} O_{F,q} = C_F$ and $\bigcap_{q \geq p_0} O_{G,q} = C_G$; $C_F \cap G = \emptyset$ so by sequential compactness $O_{F,q_1} \cap G = \emptyset$ for a $q_1 \in \omega$; and similarly $O_{G,q_2} \cap F = \emptyset$ for a $q_2 \in \omega$.

Let $q = \max(q_1, q_2)$. Then $O_{F,q}$ and $O_{G,q}$ are disjoint clopen subsets of X^n, $O_{F,q}$ separates $F \cap [q, t)^n$ from G and $O_{G,q}$ separates $G \cap [q, t)^n$ from F. So we only have to take care of $F \backslash O_{F,q}$ and $G \backslash O_{G,q}$. But

$$X^n \backslash (O_{F,q} \cup O_{G,q}) = \bigcup_{j \in q} \bigcup_{\sigma \in S_n} \sigma(\{j\} \times X^{n-1})$$

is a clopen normal subspace of X^n by our inductive assumption. We conclude that F and G can be separated in X^n.

3.5. Remark and question. It can be shown that in our example X_2 and hence $B(X)$ are not hereditarily normal. This leaves open the following question: Is there in ZFC a separable hereditarily normal topological group without nice covering properties?

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