

CHAPTER 11

An Introduction to $\beta\omega$

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Contents

Introduction	505
0. Preliminaries	506
1. The spaces $\beta\omega$ and $\beta\omega \setminus \omega$ under CH	508
2. The spaces $\beta\omega$ and $\beta\omega \setminus \omega$ under \neg CH	527
3. Partial orderings on $\beta\omega$	539
4. Weak P -points and other points in ω^*	548
5. Remarks	562
Open problems	563
References	564

Introduction

The aim of this paper is to give an introduction to the space $\beta\omega$, i.e. the Stone space of the Boolean algebra $\mathcal{P}(\omega)$ of subsets of ω . There are several arguments in favour of writing such a paper. Firstly, in the last five years several important questions concerning the structure of $\beta\omega$ were solved. We have a good picture of $\beta\omega$ now. Secondly, results about $\beta\omega$ usually have wide applications in various parts of mathematics. The space $\beta\omega$ is an exciting place where topologists, set theorists, infinite combinatorists, Boolean algebraists, and sometimes even number theorists and analysts, meet.

Since this is a chapter in the Handbook of Set Theoretic Topology, I have written this paper from the perspective of a topologist. Our language is topological but at several places it was more natural to use Boolean algebras instead of their Stone spaces, so we freely did this. We mention our perspective at this early stage of the introduction since this gives the reader an idea about what types of results are to be expected in this paper. In addition, we do not aim to be complete. Several important results will not be proven in detail, or will not even be mentioned. For this reason we have called this paper "An introduction to $\beta\omega$ ". Also, we will not give lengthy historical comments giving proper credit to everybody, but we will usually only refer to the paper giving the final solution of the problem we are discussing.

It is probably true that the following facts are the most important results obtained in $\beta\omega$ in recent years:

(1) it is consistent that P -points do not exist in $\beta\omega \setminus \omega$ (Shelah; see MILLS [1978] or WIMMERS [1978]).

(2) some but not all points in $\beta\omega \setminus \omega$ are weak P -points (KUNEN [1978]),

(3) every point in $\beta\omega \setminus \omega$ is a c -point (BALCAR & VOJTÁŠ [1980]).

(1) will interest set theorists most, (2) fascinates topologists and (3) is connected with and important in Boolean algebras as well as topology. Due to our perspective, we will discuss (2), but we leave (1) and (3) untouched.

The space $\beta\omega$ is a monster having three heads. If one works in a model in which the Continuum Hypothesis (abbreviated CH) holds, then one will see only the first head. This head is smiling, friendly, and makes you feel comfortable working with $\beta\omega$. I do not know many open problems on $\beta\omega$ the answers of which are unknown under CH. In fact, one usually does not work with $\beta\omega$ or with $\beta\omega \setminus \omega$, but with a Boolean algebra satisfying a certain completeness property which characterizes the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ under CH. This is the theme in Section 1. Here we discuss the spaces $\beta\omega$ and $\beta\omega \setminus \omega$ under CH. We begin by identifying the completeness property which characterizes $\mathcal{P}(\omega)/\text{fin}$ and then work in Boolean algebras satisfying this completeness property. Because of the presence of the CH, transfinite inductions have length ω_1 and because of the special properties of the Boolean algebras under consideration, we can always continue the transfinite inductions until stage ω_1 . The reader should observe that nowhere in Section 1 do

we use the special structure of $\beta\omega$, with the exception, of course, of the completeness property of $\mathcal{P}(\omega)/\text{fin}$. If one works in a model in which CH does not hold, then one will see the second head of $\beta\omega$. This head constantly tries to confuse you and you will never be able to decide whether it speaks the truth. This head of $\beta\omega$ will be discussed in Section 2. It turns out that all but three of the CH results derived in Section 1 are consistently false. After reading the first two sections, the reader might feel that $\beta\omega$ is a horrible creature since it seems that all statements about it depend on special set theoretic assumptions. What can there 'really' (=in ZFC) be said about $\beta\omega$? The answer to this question is: quite a bit. The third head of $\beta\omega$ is its head in ZFC. Because of the first two heads, this head is rather vague, but some parts of it are very clear. If one wants to see the clear part, one will have to work like a slave, inventing ingenious combinatorial arguments. One will have to use special properties of $\beta\omega$ and not only global properties. Some ZFC results on $\beta\omega$ are discussed in Sections 3 and 4.

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0. Preliminaries

In order to be able to understand the arguments in this paper, one should know some elementary facts about Boolean algebras and Čech–Stone compactifications. All one needs to know can be found in COMFORT & NEGREPONTIS [1974, §2]. A Boolean algebra is usually denoted by \mathcal{B} , its universal bounds by 0 and 1, concepts such as homomorphism, embedding, isomorphism, etc., should be familiar. Cardinals are initial (von Neumann) ordinals, and get the discrete topology. If α is an ordinal, then $W(\alpha)$ denotes the topological space with underlying set α equipped with the order topology. What should one know about Čech–Stone compactifications? Well, one should know that βX is the unique compactification of the (completely regular Hausdorff) space X with the property that disjoint zero-sets in X have disjoint closures. This easily implies that given a map $f: X \rightarrow K$, where K is compact, there exists a unique map $\beta f: \beta X \rightarrow K$ extending f . This map is called the *Stone extension* of f . I often hear the remark that $\beta\omega$ is clear, since it is the Stone space of $\mathcal{P}(\omega)$, but βX , for arbitrary X , is not clear, partly because it is not the Stone space of a Boolean algebra. For this reason in this paper we almost exclusively work with *strongly zero-dimensional* spaces, i.e. those spaces X for which βX is zero-dimensional, or equivalently, those spaces X for which βX is equivalent to the Stone space of the Boolean algebra $\mathcal{B}(X)$ consisting of all clopen (=both closed and open) subsets of X . Observe that in this case the existence of the Stone extension βf discussed above is clear, since the existence of f implies that $\mathcal{B}(K)$ can be embedded in $\mathcal{B}(X)$. Henceforth, all topological spaces under discussion are assumed to be completely regular and Hausdorff. The Stone space of a Boolean algebra \mathcal{B} is denoted by $\text{st}(\mathcal{B})$. Recall that a subset U of a space X is called

regular open provided that $U = \text{int}_X \text{cl}_X U$. Let $\text{RO}(X) = \{U \subseteq X : U \text{ is regular open}\}$. Then $\text{RO}(X)$ becomes a complete Boolean algebra under the following operations:

$$\begin{aligned} U \wedge V &= U \cap V, \\ U \vee V &= \text{int}_X \text{cl}_X (U \cup V), \\ U' &= \text{int}_X (X \setminus U). \end{aligned}$$

If X is compact, then the Stone space of $\text{RO}(X)$ will be denoted by EX . Since $\text{RO}(X)$ is complete, EX is *extremally disconnected* (=closure of an open set is open). It is easily seen that topologically, EX is characterized as follows: EX is the unique extremally disconnected space which admits an *irreducible* (a continuous surjection $f: S \rightarrow T$ is called irreducible provided that $f(A) \neq T$ for all closed $A \subseteq S$ with $A \neq S$) perfect map $\pi: EX \rightarrow X$. The space EX is called the *projective cover* of X . For a recent survey on projective covers, see WOODS [1979]. A space X is called *basically disconnected* if the closure of each open F_σ is again open. Observe that, trivially, each extremally disconnected space is basically disconnected, but not conversely. As usual, fin denotes the ideal of finite subsets of ω , and $\mathcal{P}(\omega)/\text{fin}$ is the Boolean algebra we obtain from $\mathcal{P}(\omega)$ by calling $A, B \in \mathcal{P}(\omega)$ equivalent iff $A \Delta B \in \text{fin}$ ($A \Delta B = (A \setminus B) \cup (B \setminus A)$). As remarked above, $\beta\omega$ denotes $\text{st}(\mathcal{P}(\omega))$. If $n > \omega$, then we identify n with the point

$$\{A \in \mathcal{P}(\omega) : n \in A\}$$

from $\text{st}(\mathcal{P}(\omega))$. Points from $\beta\omega \setminus \omega$ are called *free ultrafilters*. Obviously, $\beta\omega \setminus \omega \approx \text{st}(\mathcal{P}(\omega)/\text{fin})$. If $A \subseteq \omega$, we put

$$A^* = \{x \in \beta\omega \setminus \omega : A \in x\}.$$

It is clear that the collection $\{A^* : A \in \mathcal{P}(\omega)\}$ is a base for $\beta\omega \setminus \omega$. Also observe that $\beta\omega \setminus \omega = \omega^*$.

0.1. LEMMA. (a) *If $V \subseteq \omega$ is infinite, then \bar{V} is homeomorphic to $\beta\omega$.*

(b) *If $V, W \subseteq \omega$ are infinite, then $V^* \cap W^* = \emptyset$ iff $|V \cap W| < \omega$.*

PROOF. (a) Is clear since $\mathcal{P}(V)$ is isomorphic to $\mathcal{P}(\omega)$. We leave the proof of (b) as an exercise to the reader. \square

A point x of a space X is called a *P-point* if the intersection of countably many neighborhoods of x is again a neighborhood of x .

Whenever X is a set and κ is a cardinal we define (as usual)

$$[X]^\kappa = \{A \subseteq X : |A| = \kappa\},$$

$$[X]^{\leq \kappa} = \{A \subseteq X : |A| \leq \kappa\},$$

$$[X]^{< \kappa} = \{A \subseteq X : |A| < \kappa\},$$

respectively. We also let \subset denote *proper inclusion*.

If X is a space, then X^* denotes $\beta X \setminus X$ and if $U \subseteq X$ is open, then

$$\text{Ex}(U) = \beta X \setminus \text{cl}_{\beta X}(X \setminus U).$$

Observe that $\text{Ex}(U)$ is open and that $\text{Ex}(U) \cap X = U$. The reader can easily verify that the collection

$$\{\text{Ex}(U) : U \subseteq X \text{ is open}\}$$

is a base for the topology of βX . If $U \subseteq X$ is open, let

$$U' = \text{Ex}(U) \cap X^*.$$

If X is normal and Y is closed in X , then $\text{cl}_{\beta X} Y = \beta Y$. We identify $\text{cl}_{\beta X} Y \setminus Y$ and Y^* in this case. For definitions such as character, π -weight, cellularity etc., see JUHÁSZ [1980], or HODEL [1983]. By “ X is ccc” we mean that X satisfies the countable chain condition. We say that a family of sets \mathcal{F} has the n -intersection property ($n < \omega$) provided that $\bigcap \mathcal{G} \neq \emptyset$ for all $\mathcal{G} \in [\mathcal{F}]^{\leq n}$. To indicate that two spaces X and Y are homeomorphic, we write $X \approx Y$.

A zero-set of a space X is any set of the form $f^{-1}(\{0\})$, where $f: X \rightarrow I$ is continuous. A cozero-set is the complement of a zero-set.

Let α and κ be cardinals. We define, as usual,

$$\alpha^\kappa = \sum \{\alpha^\lambda : (\lambda \text{ is a cardinal and}) \lambda < \kappa\}.$$

1. The spaces $\beta\omega$ and $\beta\omega \setminus \omega$ under CH

In this section we will see how $\beta\omega$ and ω^* behave under CH.

1.1. A characterization of $\mathcal{P}(\omega)/\text{fin}$

Let \mathcal{B} be a Boolean algebra and let $F, G \subseteq \mathcal{B}$. We say that $F < G$ provided that for all $F' \in [F]^{<\omega}$, $G' \in [G]^{<\omega}$ we have that $\bigvee F' < \bigwedge G'$.

1.1.1. DEFINITION. Let \mathcal{B} be a Boolean algebra. We say that \mathcal{B} satisfies condition H_ω provided that for all $F \in [\mathcal{B} \setminus \{1\}]^{<\omega}$ and $G \in [\mathcal{B} \setminus \{0\}]^{<\omega}$ such that $F < G$ there is an element $x \in \mathcal{B}$ such that $F < \{x\} < G$.

1.1.2. LEMMA. $\mathcal{P}(\omega)/\text{fin}$ satisfies condition H_ω .

PROOF. We will begin by proving the following assertion: if $A \in [\mathcal{P}(\omega)/\text{fin}]^{<\omega}$ and $\{0\} < A$, then there is a $y \in \mathcal{P}(\omega)/\text{fin}$ such that $\{0\} < \{y\} < A$. Indeed, enumerate A

as $\{a_n : n < \omega\}$, and for all $n < \omega$, let $C_n \in [\omega]^\omega$ be a representative of a_n . By induction, pick points y_n for all $n < \omega$, such that

$$y_n \in \bigcap_{0 \leq i \leq n} C_i \setminus \{y_0, y_1, \dots, y_{n-1}\}$$

Let $Y = \{y_n : n < \omega\}$ and let y be the element of $\mathcal{P}(\omega)/\text{fin}$ corresponding to y . It is a good exercise to show that $\{0\} < \{y\} < A$.

Now let us return to the proof of the lemma. Suppose that $F, G \in [\mathcal{P}(\omega)/\text{fin}]^{<\omega}$, $1 \notin F, 0 \notin G$ and $F < G$. If $\vee F$ or $\wedge G$ exists, then using the above assertion, it is easy to find the required x . So assume that this is not the case. Enumerate F as $\{f_n : n < \omega\}$ and G as $\{g_n : n < \omega\}$. It is clear that without loss of generality we may assume that $f_0 < f_1 < \dots$ and $g_0 > g_1 > \dots$. For each $n < \omega$ take representatives $A_n, B_n \in [\omega]^\omega$ of f_n , resp. g_n . By induction on $k < \omega$, pick a point $d_k < \omega$ such that

$$(1) \ d_k \in \bigcap_{0 \leq i \leq k} B_i \setminus \left(\bigcup_{0 \leq i \leq k} A_i \cup \{d_0, \dots, d_{k-1}\} \right)$$

and put $D = \{d_k : k < \omega\}$. In addition, define

$$A' = \bigcup_{k < \omega} \left(A_k \cap \bigcap_{0 \leq i \leq k} B_i \right).$$

Put $C = A' \cup D$. Then $C \in [\omega]^\omega$ while moreover

(2) if $n < \omega$, then $|A_n \setminus C| < \omega$, and

(3) if $m < \omega$, then $|C \setminus B_m| < \omega$.

Let x be the element of $\mathcal{P}(\omega)/\text{fin}$ corresponding to C . It is easy to see that (2) and (3) imply that $F < \{x\} < G$. \square

1.1.3. DEFINITION. Let \mathcal{B} be a Boolean algebra. We say that \mathcal{B} satisfies condition R_ω provided that for all nonempty $F \in [\mathcal{B} \setminus \{1\}]^{<\omega}$, $G \in [\mathcal{B} \setminus \{0\}]^{<\omega}$ and $H \in [\mathcal{B}]^{<\omega}$ such that

(1) $F < G$, and

(2) $\forall \bar{F} \in [F]^{<\omega} \forall \bar{G} \in [G]^{<\omega} \forall h \in H : h \not\leq \vee \bar{F}$ and $\wedge \bar{G} \not\leq h$,

there is an element $x \in \mathcal{B}$ such that

(3) $F < \{x\} < G$, and

(4) $\forall h \in H : h \not\leq x$ and $x \not\leq h$.

The main reason that ω^* is relatively easy to deal with under CH is, as we will see later, because of the following lemma.

1.1.4. LEMMA. *If a Boolean algebra \mathcal{B} satisfies condition H_ω , then it satisfies condition R_ω .*

PROOF. Let $F \in [\mathcal{B} \setminus \{1\}]^{\leq \omega}$, $G \in [\mathcal{B} \setminus \{0\}]^{\leq \omega}$ and $H \in [\mathcal{B}]^{\leq \omega}$ be as in 1.1.3 (1) and (2). Enumerate F as $\{f_n : n < \omega\}$, G as $\{g_n : n < \omega\}$ and H as $\{h_n : n < \omega\}$. For each $h \in H$ and finite $\tilde{F} \in [F]^{\leq \omega}$ we have that $(\vee \tilde{F}) \wedge h \neq 0$, consequently there exists, by applying condition H_ω for all $n < \omega$, an element $d_n \in \mathcal{B} \setminus \{0\}$ such that

$$(1) \quad d_n < h_n \quad \text{and} \quad \forall f \in F, f \wedge d_n = 0.$$

Similarly, we can find $e_n \in \mathcal{B} \setminus \{0\}$ such that

$$(2) \quad \{e_n\} < G \quad \text{and} \quad e_n \wedge h_n = 0.$$

If the d_n 's and e_n 's are chosen with a little extra care, we can assure that $e_n \wedge d_m = 0$ for all $n, m < \omega$. Now define for all $n < \omega$,

$$\tilde{f}_n = f_n \vee e_n, \quad \text{and} \quad \tilde{g}_n = g_n \wedge d'_n.$$

Notice if $n, m < \omega$ then $\bigvee_{0 \leq i \leq n} \tilde{f}_i \leq \bigwedge_{0 \leq j \leq m} \tilde{g}_j$. By H_ω , we can therefore find an element $x \in \mathcal{B}$ such that for all $n, m < \omega$,

$$\bigvee_{0 \leq i \leq n} \tilde{f}_i \leq x \leq \bigwedge_{0 \leq j \leq m} \tilde{g}_j.$$

An easy check shows that x is as required. \square

1.1.5. COROLLARY. $\mathcal{P}(\omega)/\text{fin}$ satisfies condition R_ω .

We now come to the main result of this section. The proof we give is slightly incomplete. The reader is encouraged to fill in all missing details (in case of problems, see COMFORT & NEGREPONTIS [1974, Lemma 6.10]). If \mathcal{B} is a Boolean algebra (abbreviated: BA) and if $A \subseteq \mathcal{B}$, then $\langle\langle A \rangle\rangle \subseteq \mathcal{B}$ denotes the subalgebra of \mathcal{B} generated by A .

1.1.6. THEOREM (CH). If \mathcal{B} is a Boolean algebra of cardinality at most c satisfying condition H_ω , then \mathcal{B} is isomorphic to $\mathcal{P}(\omega)/\text{fin}$.

PROOF. Let \mathcal{B} and \mathcal{E} be BA's satisfying condition H_ω such that $|\mathcal{B}|, |\mathcal{E}| \leq c$. By CH list \mathcal{B} as $\{b_\alpha : \alpha < \omega_1\}$ and \mathcal{E} as $\{e_\alpha : \alpha < \omega_1\}$.

Without loss of generality we may assume that $e_0 = 0$ and $b_0 = 0$. By transfinite induction, for $\alpha < \omega_1$ we will construct countable subalgebras $\mathcal{B}_\alpha \subseteq \mathcal{B}$ and $\mathcal{E}_\alpha \subseteq \mathcal{E}$ and an isomorphism $\sigma_\alpha : \mathcal{B}_\alpha \rightarrow \mathcal{E}_\alpha$ such that

$$(1) \quad b_\alpha \in \mathcal{B}_\alpha \quad \text{and} \quad e_\alpha \in \mathcal{E}_\alpha,$$

$$(2) \quad \text{if } \beta < \alpha, \text{ then } \mathcal{B}_\beta \subseteq \mathcal{B}_\alpha, \mathcal{E}_\beta \subseteq \mathcal{E}_\alpha \text{ and } \sigma_\alpha \upharpoonright \mathcal{B}_\beta = \sigma_\beta.$$

Let $\mathcal{B}_0 = \{0, 1\}$ and $\mathcal{E}_0 = \{0, 1\}$ and let $\sigma_0 : \mathcal{B}_0 \rightarrow \mathcal{E}_0$ be defined in the obvious way. Suppose that $\mathcal{B}_\beta, \mathcal{E}_\beta$ and σ_β are defined for all $\beta < \alpha < \omega_1$ satisfying (1) and (2). If $b_\alpha \in \bigcup_{\beta < \alpha} \mathcal{B}_\beta$ and $e_\alpha \in \bigcup_{\beta < \alpha} \mathcal{E}_\beta$, then define $\mathcal{B}_\alpha = \bigcup_{\beta < \alpha} \mathcal{B}_\beta$, $\mathcal{E}_\alpha = \bigcup_{\beta < \alpha} \mathcal{E}_\beta$ and

$\sigma_\alpha = \bigcup_{\beta < \alpha} \sigma_\beta$. Suppose next that e.g. $b_\alpha \notin \bigcup_{\beta < \alpha} \mathcal{B}_\beta = \mathcal{F}$. Let $\sigma = \bigcup_{\beta < \alpha} \sigma_\beta$. Put

$$\mathcal{F}_0 = \{f \in \mathcal{F} : f < b_\alpha\}, \quad \mathcal{F}_1 = \{f \in \mathcal{F} : b_\alpha < f\}, \quad \text{and} \quad \mathcal{F}_2 = \mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_1).$$

By Lemma 1.1.4 there is an element $e \in \mathcal{E}$ such that $\sigma(\mathcal{F}_0) < \{e\}$, $\{e\} < \sigma(\mathcal{F}_1)$, and for all $\tilde{e} \in \sigma(\mathcal{F}_2)$, $\tilde{e} \not\leq e$ and $e \not\leq \tilde{e}$. If we put $\sigma(b_\alpha) = e$ and $\sigma(b'_\alpha) = e'$, then σ can be extended to an isomorphism $\tilde{\sigma} : \langle\langle \mathcal{F} \cup \{b_\alpha\} \rangle\rangle \rightarrow \langle\langle \sigma(\mathcal{F}) \cup \{e\} \rangle\rangle$. If $e_\alpha \notin \langle\langle \sigma(\mathcal{F}) \cup \{e\} \rangle\rangle$, then we do the same thing as above with σ replaced by σ^{-1} . This shows how to construct \mathcal{B}_α and \mathcal{E}_α .

We conclude that \mathcal{B} and \mathcal{E} are isomorphic and by Corollary 1.1.5, this also shows that both \mathcal{B} and \mathcal{E} are isomorphic to $\mathcal{P}(\omega)/\text{fin}$. \square

1.1.7. REMARK. Observe that each BA satisfying condition H_ω has cardinality at least c .

1.2. A topological translation

In this section we will translate the results of Section 1.1 in topological language.

Let X be a space. A subset $A \subseteq X$ is called *C^* -embedded* in X provided that each map $f : A \rightarrow [0, 1]$ can be extended to a map $f : X \rightarrow [0, 1]$.

1.2.1. DEFINITION. A space X is called an *F -space* if each cozero-set in X is C^* -embedded in X .

The following lemma summarizes some relevant information on F -spaces.

1.2.2. LEMMA. (a) X is an F -space iff βX is F -space.

(b) A normal space X is an F -space iff any two disjoint open F_σ subsets of X have disjoint closures in X .

(c) Each basically disconnected space is an F -space.

(d) Any closed subspace of a normal F -space is again an F -space.

(e) If an F -space X satisfies the countable chain condition, then it is extremally disconnected.

PROOF. For (a), use that X is C^* -embedded in βX . The proof of (b) is routine and (c) is trivial. The proof of (d) is easy if one uses the characterization of normal F -spaces stated in (b). For (e), first observe that it suffices to show that disjoint open subsets of X have disjoint closures. Let $U, V \subseteq X$ be open and disjoint. Use the fact that X is ccc to find dense cozero-sets $U' \subseteq U$ and $V' \subseteq V$. The function $f : U' \cup V' \rightarrow [0, 1]$ defined by $f(x) = 0$ if $x \in U'$ and $f(x) = 1$ if $x \in V'$ can be extended to a map $\bar{f} : X \rightarrow [0, 1]$. Since $U \subseteq \bar{f}^{-1}(\{0\})$ and $V \subseteq \bar{f}^{-1}(\{1\})$, we conclude that $\bar{U} \cap \bar{V} = \emptyset$. \square

The following result gives a topological translation of condition H_ω .

1.2.3. LEMMA. *Let X be a compact zero-dimensional space. The following statements are equivalent:*

- (1) $\mathcal{B}(X)$ satisfies condition H_ω ,
- (2) X is an F -space and each nonempty G_δ in X has infinite interior.

PROOF. (1) \Rightarrow (2) follows directly from Lemma 1.2.2(b) and the fact that in a compact zero-dimensional space every open F_σ is a countable union of clopen sets. That (2) implies (1) is routine. \square

1.2.4. COROLLARY (CH). *Let X be a space. The following statements are equivalent:*

- (1) $X \approx \omega^*$,
- (2) X is a compact zero-dimensional F -space of weight c in which each nonempty G_δ has infinite interior.

PROOF. Follows directly from Theorem 1.1.6 and Lemma 1.2.3. \square

A compact zero-dimensional F -space of weight c in which each non-empty G_δ has infinite interior, will be called a *Parovičenko space* from now on. Corollary 1.2.4 says that, under CH, ω^* is, up to homeomorphism, the only Parovičenko space.

Corollary 1.2.4 is a very useful result since it turns out that the class of Parovičenko spaces is quite large. The following result, which is of independent interest, is the key in finding more Parovičenko spaces.

1.2.5. THEOREM. *Let X be a locally compact, σ -compact and noncompact space. Then X^* is an F -space and each nonempty G_δ in X^* has infinite interior.*

PROOF. Let $F \subseteq X^*$ be any F_σ and let $f: F \rightarrow [0, 1]$ be continuous. Since $Y = X \cup F$ is σ -compact, it is normal and therefore, since F is closed in Y , the Tietze Extension Theorem implies that f can be extended to a map $\bar{f}: Y \rightarrow [0, 1]$. Put $g = \bar{f} \upharpoonright X$. Then g can be extended to a map $\bar{g}: \beta X \rightarrow [0, 1]$. Since clearly $\bar{g} \upharpoonright F = f$, we see that $\bar{f} = \bar{g} \upharpoonright X^*$ is the required extension of f .

Let $S \subseteq X^*$ be a nonempty G_δ . Since the set $\{U': U \text{ open in } X\}$ is a basis for X^* , it is clear that we can find open sets $U_n \subseteq X$ for all $n < \omega$, such that

$$\bar{U}_{n+1} \subseteq U_n \quad \text{and} \quad \emptyset \neq \bigcap_{n < \omega} U'_n \subseteq S$$

(since $U_n \subseteq X$ for all n , the bar means closure in X of course). Since X is locally compact and σ -compact, we can write X as $\bigcup_{n < \omega} K_n$, where each K_n is compact and moreover each compact $K \subseteq X$ is contained in some K_n . For each $n < \omega$ choose a nonempty open set $V_n \subseteq U_n$ such that

$$\bar{V}_n \text{ is compact and misses } K_n.$$

Put $V = \bigcup_{n < \omega} V_n$. If $n < \omega$, then $V \setminus U_n$ has compact closure in X , whence

$$V' \subseteq \bigcap_{n < \omega} U'_n \subseteq S.$$

In addition, $V' \neq \emptyset$ since V does not have compact closure in X . The easy proof that V' is infinite is left to the reader. \square

We now present an interesting topological consequence of Theorem 1.1.6.

1.2.6. THEOREM (CH). *Let X be a zero-dimensional, locally compact, σ -compact, noncompact space of weight at most c . Then X^* and ω^* are homeomorphic.*

PROOF. Since X is a zero-dimensional Lindelöf space, X is strongly zero-dimensional and has at most $c^\omega = c$ clopen sets. It follows that X^* is a zero-dimensional compact space of weight at most c . By Theorem 1.2.5 and Lemma 1.2.3 $\mathcal{B}(X^*)$ satisfies condition H_ω . This implies that $\mathcal{B}(X^*)$ and $\mathcal{P}(\omega)/\text{fin}$ are isomorphic (Theorem 1.1.6), and consequently, by Stone duality, that X^* and ω^* are homeomorphic. \square

*1.3. Continuous images of ω^**

In this section we characterize the continuous images of ω^* .

1.3.1. THEOREM. *Let \mathcal{B} be a Boolean algebra of cardinality at most ω_1 . Then \mathcal{B} can be embedded in $\mathcal{P}(\omega)/\text{fin}$.*

PROOF. Use the same technique as in the proof of Theorem 1.1.6. \square

By Stone duality, Theorem 1.3.1 is equivalent to the statement that each compact and zero-dimensional space of weight at most ω_1 is a continuous image of ω^* . This result suggests the question whether the same result holds without the assumption on zero-dimensionality. This is indeed the case, see Theorem 1.3.3 below.

1.3.2. LEMMA. *Let X be a compact space of weight κ . Then there is a compact zero-dimensional space Y of weight κ which can be mapped onto X .*

PROOF. Let $\mathcal{B} \in [\text{RO}(X)]^\kappa$ be such that \mathcal{B} is a basis and put $\mathcal{E} = \langle\langle \mathcal{B} \rangle\rangle \subseteq \text{RO}(X)$. Observe that $|\mathcal{E}| = \kappa$. Let Y be the Stone space of \mathcal{E} . \square

1.3.3. THEOREM. *Each compact space of weight at most ω_1 is a continuous image of ω^* .*

PROOF. Let X be a compact space of weight ω_1 and let Y be as in Lemma 1.3.2. By Theorem 1.3.1, $\mathcal{B}(Y)$ embeds in $\mathcal{P}(\omega)/\text{fin}$. Consequently, by Stone duality, ω^* can be mapped onto Y . \square

1.3.4. COROLLARY (CH). *Each compact space of weight at most c is a continuous image of ω^* .*

1.4. Closed subspaces of $\beta\omega$

In this section we characterize topologically the closed subspaces of $\beta\omega$. If X is a closed subspace of $\beta\omega$, then X must clearly be of weight at most c and X must be a zero-dimensional compact F -space by Lemma 1.2.2(d). It turns out that, under CH, these conditions are not only necessary but also sufficient.

Let X be a space. A subset of $B \subseteq X$ is called a P -set provided that the intersection of countably many neighborhoods of B is again a neighborhood of B .

Let X and Y be compact spaces. Let $A \subseteq X$ be closed and let $f: A \rightarrow Y$ be a continuous surjection. It is easily seen that the collection

$$\mathcal{B} = \{f^{-1}(y) : y \in Y\} \cup \{\{x\} : x \in X \setminus A\}$$

is an upper-semicontinuous decomposition of X ; the decomposition space X/\mathcal{B} will be denoted by $X \cup_f Y$. If $\pi: X \rightarrow X \cup_f Y$ is the decomposition map, then we identify Y and $\pi(A)$.

1.4.1. LEMMA. *Let X and Y be compact F -spaces, let $A \subseteq X$ be a closed P -set, and let $f: A \rightarrow Y$ be a continuous surjection. Then $X \cup_f Y$ is an F -space.*

PROOF. Let U and V be disjoint open F_σ subsets of $X \cup_f Y$. Since Y is an F -space, $(U \cap Y)^- \cap (V \cap Y)^- = \emptyset$. Let E and F be closed G_δ neighborhoods of $(U \cap Y)^-$ and $(V \cap Y)^-$ such that $E \cap F = \emptyset$. Then $U \setminus E$ and $V \setminus F$ are disjoint open F_σ subsets of X which both do not meet A . Since X is an F -space and A is a P -set,

$$(1) (U \setminus E)^- \cap (V \setminus F)^- = \emptyset, \text{ and}$$

$$(2) ((U \setminus E)^- \cup (V \setminus F)^-) \cap A = \emptyset,$$

This easily implies that $\bar{U} \cap \bar{V} = \emptyset$.

By Lemma 1.2.2(b), we may now conclude that $X \cup_f Y$ is an F -space. \square

1.4.2. LEMMA. *Let X be a compact space with the property that each nonempty G_δ has infinite interior. If $A \subseteq X$ is closed and nowhere dense and if $f: A \rightarrow Y$ is a continuous surjection, then the space $X \cup_f Y$ has also the property that each nonempty G_δ has infinite interior.*

PROOF. Obvious. \square

1.4.3. LEMMA (CH). ω^* contains a nowhere dense closed P -set A which is homeomorphic to ω^* .

PROOF. By Theorem 1.2.6, we can represent ω^* by

$$Z = (\omega \times W(\omega_1 + 1))^* .$$

Let $A = (\omega \times \{\omega_1\})^*$. Trivially, $A \approx \omega^*$ and that A is a P -set follows easily from the fact that ω_1 is a P -point in $W(\omega_1 + 1)$. That A is nowhere dense is clear. \square

We now come to the main result of this section.

1.4.4. THEOREM (CH). Let X be a space. The following statements are equivalent:

- (1) X is a compact zero-dimensional F -space of weight at most c ,
- (2) X can be embedded in $\beta\omega$ as a closed subspace,
- (3) X can be embedded as a nowhere dense closed P -set in ω^* .

PROOF. The implications (2) \Rightarrow (1) and (3) \Rightarrow (2) are trivial, so it suffices to prove that (1) \Rightarrow (3). To this end, let X be a compact zero-dimensional F -space of weight at most c . By Lemma 1.4.3 we can find a closed nowhere dense P -set A of ω^* such that $A \approx \omega^*$. In addition, by Corollary 1.3.4, there is a continuous surjection $f: A \rightarrow X$. Put $Z = \omega^* \cup_f X$. It is routine to verify that

- (a) Z is zero-dimensional,
- (b) Z is of weight c ,
- (c) X is a nowhere dense closed P -set of Z .

Lemma 1.4.1 followed by Lemma 1.4.2 imply that Z is a compact F -space in which each nonempty G_δ has infinite interior. Consequently, by Corollary 1.2.4, $Z \approx \omega^*$. \square

Let \mathcal{B} be a Boolean algebra. We say that \mathcal{B} is *weakly countably complete* (abbreviated: WCC) iff the Stone space of \mathcal{B} is an F -space. In Boolean algebraic language, \mathcal{B} is a WCC BA iff \mathcal{B} is a BA and

$$\forall B, C \in [\mathcal{B}]^{\leq \omega} \text{ such that } \forall b \in B \forall c \in C: b \wedge c = 0, \\ \text{there is an } a \in \mathcal{B} \text{ with } b \leq a \leq c' \text{ for all } b \in B, c \in C.$$

The following result is a purely Boolean algebraic consequence of Theorem 1.4.4.

1.4.5. THEOREM (CH). Let \mathcal{B} be a Boolean algebra. The following statements are equivalent:

- (1) \mathcal{B} is WCC and $|\mathcal{B}| \leq c$,
- (2) \mathcal{B} is a homomorphic image of $\mathcal{P}(\omega)$.

1.4.6. COROLLARY (CH). *Each WCC Boolean algebra of cardinality at most c is a homomorphic image of a complete Boolean algebra.*

We will now prove an interesting result without the aid of the CH.

1.4.7. THEOREM. *Let X be a compact extremally disconnected space of weight at most c . Then X can be embedded in $\beta\omega$.*

PROOF. We may assume that $X \subseteq I^c$, where as usual, $I = [0, 1]$. Since I^c is separable, ENGELKING [1977, 2.3.16], there is a continuous surjection $f: \beta\omega \rightarrow I^c$. Let $g = f \upharpoonright f^{-1}(X)$ and take a closed $Z \subseteq f^{-1}(X)$ such that $g \upharpoonright Z: Z \rightarrow X$ is irreducible. The existence of Z easily follows from Zorn's Lemma (order all closed sets of $f^{-1}(X)$ that map onto X by reverse inclusion). We claim that $h = g \upharpoonright Z$ is a homeomorphism. For this it suffices to show that h is one to one. To this end, take distinct points $x, y \in Z$. Find disjoint clopen neighborhoods U and V of, respectively, x and y (in Z). Since h is irreducible,

$$h(x) \in \overline{\text{int } h(U)}, \quad h(y) \in \overline{\text{int } h(V)}, \quad \text{and} \quad \text{int } h(U) \cap \text{int } h(V) = \emptyset.$$

Consequently, by the extremal disconnectivity of X ,

$$\overline{\text{int } h(U)} \cap \overline{\text{int } h(V)} = \emptyset,$$

and we conclude that $h(x) \neq h(y)$. \square

1.5. C^* -embedded subspaces of $\beta\omega$

In this section we will characterize those subspaces of $\beta\omega$ that are C^* -embedded in $\beta\omega$. It is interesting that being C^* -embedded in $\beta\omega$ turns out to be a topological property and does not depend on how a given set is placed in $\beta\omega$.

1.5.1. DEFINITION. A space X is called *weakly Lindelöf* provided that for any open cover \mathcal{U} of X there is a countable subfamily $\mathcal{E} \subseteq \mathcal{U}$ such that $(\cup \mathcal{E})^- = X$.

Observe that each space satisfying the countable chain condition is weakly Lindelöf.

The following important result shows that $\beta\omega$ has 'many' C^* -embedded subspaces.

1.5.2. THEOREM. *Let $X \subseteq \beta\omega$ be weakly Lindelöf. Then X is C^* -embedded in $\beta\omega$.*

PROOF. It clearly suffices to show that disjoint zero-sets in X have disjoint closures in $\beta\omega$, COMFORT & NEGREPONTIS [1974, Theorem 2.6]. To prove this, let Z_0 ,

$Z_1 \subseteq X$ be a disjoint zero-sets. There are disjoint open neighborhoods U and V of Z_0 and Z_1 such that $\text{cl}_X U \cap \text{cl}_X V = \emptyset$. For each $x \in X$ let $C_x \subseteq \beta\omega$ be a clopen neighborhood of x in $\beta\omega$ such that

- (1) if $x \in \text{cl}_X U$, then $C_x \cap \text{cl}_X V = \emptyset$,
- (2) if $x \in \text{cl}_X V$, then $C_x \cap \text{cl}_X U = \emptyset$,
- (3) if $x \notin (\text{cl}_X U \cup \text{cl}_X V)$, then $C_x \cap (\text{cl}_X U \cup \text{cl}_X V) = \emptyset$.

Since X is weakly Lindelöf, there is a sequence x_n ($n < \omega$) in X such that $\bigcup_{n < \omega} (C_{x_n} \cap X)$ is dense in X . For each $n < \omega$, put

$$E_n = C_{x_n} \setminus \bigcup_{i < n} C_{x_i}.$$

Then the family $\{E_n : n < \omega\}$ is a pairwise disjoint collection clopen subsets of $\beta\omega$ such that

- (1) $\bigcup_{n < \omega} (E_n \cap X)$ is dense in X , and
- (2) each E_n meets at most one of the $\text{cl}_X U$ and $\text{cl}_X V$.

Put $E = \bigcup \{E_n : n < \omega \text{ \& } E_n \cap U \neq \emptyset\}$ and $F = \bigcup \{E_n : n < \omega \text{ \& } E_n \cap V \neq \emptyset\}$. Then $E \cap F = \emptyset$ and therefore, since $\beta\omega$ is an F -space, $\bar{E} \cap \bar{F} = \emptyset$. Since obviously $U \subseteq \bar{E}$ and $V \subseteq \bar{F}$, we conclude that Z_0 and Z_1 have disjoint closures in $\beta\omega$. \square

We will now show that, under CH, the converse of Theorem 1.5.2 is true, thus giving a *topological* characterization of those subspaces of $\beta\omega$ that are C^* -embedded in $\beta\omega$.

1.5.3. THEOREM (CH). *Let $X \subseteq \beta\omega$. The following statements are equivalent:*

- (1) X is weakly Lindelöf,
- (2) X is C^* -embedded in $\beta\omega$,
- (3) $|C^*(X)| = c$.

PROOF. (1) \Rightarrow (2) is shown in Theorem 1.5.2 and (2) \Rightarrow (3) is clear since $|C^*(\beta\omega)| = c$. It therefore suffices to prove that the negation of (1) implies the negation of (3).

If X is not weakly Lindelöf then, by CH, there is a family $\{C_\alpha : \alpha < \omega_1\}$ of clopen subsets of $\beta\omega$ such that

- (1) $X \subseteq \bigcup_{\alpha < \omega_1} C_\alpha$,
- (2) for each $\alpha < \omega_1$, $X \setminus (\bigcup_{\beta < \alpha} C_\beta \cap X)^- \neq \emptyset$.

We can therefore find a strictly increasing sequence of ordinals $\kappa_\alpha < \omega_1$ ($\alpha < \omega_1$) and for each $\alpha < \omega_1$ a clopen set $D_\alpha \subseteq C_{\kappa_\alpha}$ such that

- (3) $D_\alpha \cap X \neq \emptyset$,
- (4) $D_\alpha \cap (\bigcup_{\beta < \alpha} (C_\beta \cap X) \cup \bigcup_{\beta < \alpha} (D_{\kappa_\beta} \cap X))^- = \emptyset$.

For each $\alpha < \omega_1$, put $\tilde{C}_\alpha = C_\alpha \cap \bar{X}$ and let $D = \bigcup_{\alpha < \omega_1} D_\alpha \cap \bar{X}$.

Claim. D is C^* -embedded in $\bigcup_{\alpha < \omega_1} \tilde{C}_\alpha$.

Let $f : D \rightarrow [0, 1]$ be given. For each $\alpha < \omega_1$ put

$$f_\alpha = f \upharpoonright D \cap \bigcup_{\beta \leq \alpha} \tilde{C}_\beta.$$

Observe that (4) implies that $\text{dom}(f_\alpha)$ is an open F_σ -subset of \bar{X} for all $\alpha < \omega_1$. We will construct, for each $\alpha < \omega_1$, an extension $g_\alpha : \bigcup_{\beta \leq \alpha} \tilde{C}_\beta \rightarrow I$ of f_α such that for all $\beta < \alpha$,

$$g_\beta \subseteq g_\alpha.$$

Suppose this is done for all $\beta < \alpha$. The function $\bigcup_{\beta < \alpha} g_\beta \cup f_\alpha$ is continuous on $\bigcup_{\beta < \alpha} \tilde{C}_\beta \cup (D \cap \tilde{C}_\alpha)$, and this set is an open F_σ -subset of \bar{X} . Therefore, since \bar{X} is an F -space (Lemma 1.2.2(d)), this function can be extended to get the required g_α .

Finally put $g = \bigcup_{\alpha < \omega_1} g_\alpha$. It is clear that g is as required.

Since D is a union of ω_1 pairwise disjoint nonempty clopen sets, $|C^*(D)| \leq 2^{\omega_1}$ and consequently, by the Claim, $|C^*(\bigcup_{\alpha < \omega_1} \tilde{C}_\alpha)| \geq 2^{\omega_1}$. Since X is dense in $\bigcup_{\alpha < \omega_1} \tilde{C}_\alpha$ this implies that

$$|C^*(X)| \geq 2^{\omega_1} > c,$$

which is a contradiction. \square

1.5.4. COROLLARY (CH). *If $x \in \omega^*$, then $\omega^* \setminus \{x\}$ is not C^* -embedded in ω^* .*

PROOF. If $\omega^* \setminus \{x\}$ is C^* -embedded in ω^* , then $\omega^* \setminus \{x\}$ is C^* -embedded in $\beta\omega$ since ω^* is C^* -embedded in $\beta\omega$. By Theorem 1.5.3 it therefore suffices to prove the following easy

Fact. If $x \in \omega^$, then $\omega^* \setminus \{x\}$ is not weakly Lindelöf.*

Assume, to the contrary, that $\omega^* \setminus \{x\}$ is weakly Lindelöf. Put $\mathcal{U} = \{C \subseteq \omega^* : C \text{ is clopen and } x \notin C\}$. Since by assumption $\omega^* \setminus \{x\}$ is weakly Lindelöf, there are $C_n \in \mathcal{U}$ ($n < \omega$) such that $\bigcup_{n < \omega} C_n$ is dense in ω^* . By Lemma 1.1.2 or by Theorem 1.2.5, there is a nonempty clopen $E \subseteq \omega^*$ such that $E \cap (\bigcup_{n < \omega} C_n) = \emptyset$. It is clear that without loss of generality we may assume that $x \notin E$. Then E must meet $\bigcup_{n < \omega} C_n$, which is not the case and therefore we have obtained the desired contradiction. \square

1.6. Autohomeomorphisms of ω^*

In this section we will concentrate on autohomeomorphisms of ω^* . Our main results are Theorems 1.6.4 and 1.6.5.

If $\pi : \omega \rightarrow \omega$ is a permutation, then $\beta\pi \upharpoonright \omega^*$ is an autohomeomorphism of ω^* .

Let π_0 and π_1 be two permutations of ω . We claim that if $\beta\pi_0 \upharpoonright \omega^* = \beta\pi_1 \upharpoonright \omega^*$, then $|\{n < \omega : \pi_0(n) \neq \pi_1(n)\}| < \omega$. If not, then we can find an infinite set $E \subseteq \omega$ such that $\pi_0(E) \cap \pi_1(E) = \emptyset$. Take $x \in \omega^*$ such that $E \in x$. Since $\pi_i(E) \in \beta\pi_i(x)$ for $i < 2$, we conclude that $\beta\pi_0(x) \neq \beta\pi_1(x)$, which contradicts our assumptions. Since it is clear that we can find a family $\{\pi_\xi : \xi < c\}$ of permutations of ω such that for all $\eta < \xi < c$ we have that $\{n : \pi_\eta(n) \neq \pi_\xi(n)\}$ is infinite, this shows that ω^* has at least c autohomeomorphisms which are induced from permutations on ω . Are there others? Under CH, there are.

1.6.1. LEMMA (CH). ω^* has precisely 2^c autohomeomorphisms.

PROOF. Since by Theorem 1.2.6, $\omega^* \approx (\omega \times 2^c)^*$ (here 2^c denotes the Cantor cube of weight c) and since 2^c has 2^c autohomeomorphisms, being a topological group of cardinality 2^c , it easily follows that ω^* has at least 2^c autohomeomorphisms. Since ω^* has weight c , it cannot have more than 2^c autohomeomorphisms. \square

We will now prove two results which are steps in the proof of Theorem 1.6.4.

1.6.2. LEMMA. Let U and V be noncompact open F_σ -subsets of ω^* . Then there is an autohomeomorphism $h : \omega^* \rightarrow \omega^*$ with $h(U) = V$.

PROOF. Find partitions $\{A_n : n < \omega\}$ and $\{B_n : n < \omega\}$ of ω in infinite sets such that

$$U = \bigcup_{n < \omega} A_n^* \quad \text{and} \quad V = \bigcup_{n < \omega} B_n^*.$$

Let $\pi : \omega \rightarrow \omega$ be a permutation such that $\pi(A_n) = B_n$ for all $n < \omega$. Then $h = \beta\pi \upharpoonright \omega^*$ is clearly as required. \square

1.6.3. COROLLARY (CH). Let S and T be nowhere dense P -sets in ω^* such that $S \approx T \approx \omega^*$. Then there is an autohomeomorphism

$$h : \omega^* \rightarrow \omega^* \quad \text{with} \quad h(S) = T.$$

PROOF. Let X be a homeomorph of $\omega \times \omega^*$ disjoint from ω^* . Since, by Theorem 1.2.6, $X^* \approx \omega^* \approx S$, we can identify X^* and S . In other words, we assume that $\beta X \cap \omega^* = S$. We topologize $Z_0 = \omega^* \cup X$ by pasting βX and ω^* together. Formally,

$$U \subseteq Z_0 \text{ is open} \quad \text{iff} \quad \begin{aligned} &U \cap \beta X \text{ is open in } \beta X \\ &\text{and } U \cap \omega^* \text{ is open in } \omega^*. \end{aligned}$$

By using similar arguments as in the proofs of Lemmas 1.4.1 and 1.4.2, the reader

can easily verify that Z_0 is a Parovičenko space, consequently $Z_0 \approx \omega^*$. Similarly, take a homeomorph Y of $\omega \times \omega^*$ disjoint from ω^* such that $\beta Y \cap \omega^* = T$ and form the Parovičenko space $Z_1 = \omega^* \cup Y$. By Lemma 1.6.2 there is a homeomorphism $\tilde{h}: Z_0 \rightarrow Z_1$ with $\tilde{h}(X) = Y$. Then $h = \tilde{h} \upharpoonright \omega^*$ is clearly as required. \square

1.6.4. THEOREM (CH). *Let $S, T \subseteq \omega^*$ be nowhere dense P -sets such that $S \approx T \approx \omega^*$ and let $h: T \rightarrow S$ be a homeomorphism. Then h can be extended to a homeomorphism $\tilde{h}: \omega^* \rightarrow \omega^*$.*

PROOF. Let $f: \omega^* \rightarrow \omega^*$ be a homeomorphism such that $f(S) \cap T = \emptyset$. It is clear that such homeomorphism exists since all clopen subsets of ω^* are homeomorphic to ω^* and $S \cup T$ is nowhere dense. Put $Z = f(S) \cup T$ and define $\varphi: Z \rightarrow Z$ by

$$\begin{cases} \varphi(t) = f(h(t)) & \text{if } t \in T, \\ \varphi(t) = h^{-1}(f^{-1}(t)) & \text{if } t \in f(S). \end{cases}$$

Now if we can extend $\varphi: Z \rightarrow Z$ to a homeomorphism $\bar{\varphi}: \omega^* \rightarrow \omega^*$ then $\bar{h} = f^{-1} \circ \bar{\varphi}$ is a homeomorphism of ω^* extending h . Since $Z \approx \omega^*$, in view of Corollary 1.6.3, it therefore suffices to prove the following

Fact. There is a nowhere dense P -set $A \subseteq \omega^$ such that $A \approx \omega^*$ and each autohomeomorphism of A extends to an autohomeomorphism of ω^* .*

Put $X = \omega \times W(\omega_1 + 1) \times \omega^*$ and $Y = \omega \times \{\omega_1\} \times \omega^*$. It is easy to see that $Y^* \subseteq X^*$ is a nowhere dense P -set. The projection $\pi: Y \rightarrow \omega^*$ extends to a map $\beta\pi: \beta Y \rightarrow \omega^*$. Let $f = \beta\pi \upharpoonright Y^*$ and define $B = X^* \cup_f Y^*$. By Lemmas 1.4.1 and 1.4.2, B is a Parovičenko space. Obviously, $A = \omega^*$ is a nowhere dense P -set in B . Let $h: A \rightarrow A$ be any homeomorphism. The homeomorphism $\tilde{h} = \text{id} \times \text{id} \times h$ of X extends to a homeomorphism $\beta\tilde{h}: \beta X \rightarrow \beta X$. Define $\bar{h}: B \rightarrow B$ by

$$\begin{cases} \bar{h}(x) = h(x) & \text{if } x \in A, \\ \bar{h}(x) = \beta\tilde{h}(x) & \text{if } x \notin A. \end{cases}$$

An easy check shows that \bar{h} is an autohomeomorphism of B extending h . Since $B \approx \omega^*$ (Corollary 1.2.4) this is as required. \square

The following result is, in a sense, also a result on extending homeomorphisms.

1.6.5. THEOREM (CH). *Let $p, q \in \omega^*$ be P -points. Then there is an autohomeomorphism $h: \omega^* \rightarrow \omega^*$ with $h(p) = q$.*

PROOF. Adapt the proof of Theorem 1.1.6. \square

Observe that Theorem 1.4.4 implies that P -points in ω^* exist.

1.6.6. REMARK. Theorem 1.6.4 and 1.6.5 have a common generalization. In VAN DOUWEN & VAN MILL [1981b] it will be shown that any homeomorphism between (arbitrary) nowhere dense closed P -sets extends to an autohomeomorphism of ω^* . The proof which van Douwen and I have of this result is conceptually simple, but technically complicated. Since we believe that the proof is not in its final form yet, in this section we have only worked out some special cases which have simpler proofs.

1.7. P -points and nonhomogeneity of ω^*

Since ω is homogeneous, ω^* looks homogeneous and the question naturally arises whether ω^* is homogeneous. We will prove that, under CH, this is not the case. We will show later that ω^* is not homogeneous in ZFC. Observe that Theorem 1.4.4 implies that, under CH, ω^* contains a P -point. If all points of ω^* are P -points, then the compactness of ω^* implies that ω^* is finite, which is clearly not the case. Therefore, ω^* contains both P -points and non P -points and we conclude that ω^* is not homogeneous under CH. The proof, just given here that there are P -points in ω^* , is not very economical. We will therefore give an easier proof of this fact.

1.7.1. LEMMA. ω^* cannot be covered by ω_1 nowhere dense sets.

PROOF. Let $\{D_\alpha : \alpha < \omega_1\}$ be a family of ω_1 nowhere dense subsets of ω^* . By using Lemma 1.1.2 or Theorem 1.2.5, find a family $\{C_\alpha : \alpha < \omega_1\}$ of nonempty clopen subsets of ω^* such that for all $\alpha < \omega_1$,

$$(1) C_\alpha \cap D_\alpha = \emptyset,$$

$$(2) \text{ if } \beta < \alpha, \text{ then } C_\alpha \subseteq C_\beta.$$

Consequently, any point of $\bigcap_{\alpha < \omega_1} C_\alpha$ misses $\bigcup_{\alpha < \omega_1} D_\alpha$. \square

1.7.2. COROLLARY (CH). ω^* contains P -points.

PROOF. Let $\mathcal{A} = \{\bar{U} \setminus U : U \subseteq \omega^* \text{ is an open } F_\sigma\}$. By CH, $|\mathcal{A}| \leq \omega_1$. By Lemma 1.7.1, $\omega^* \setminus \bigcup \mathcal{A} \neq \emptyset$ and each point of this set is a P -point. \square

Since by Theorem 1.6.5, for any two P -points $x, y \in \omega^*$, under CH there is an autohomeomorphism $h : \omega^* \rightarrow \omega^*$ with $h(x) = y$, all P -points are topologically the same. In view of the above results, one therefore naturally wonders whether P -points and non P -points are the only types of points in ω^* . This is not true, as the next result shows.

A point x of a space X is called a *weak P -point* provided that $x \notin \bar{F}$ for all countable $F \subseteq X \setminus \{x\}$.

- 1.7.3. THEOREM (CH).** (1) *There is a weak P -point in ω^* which is not a P -point,*
 (2) *There is a point $x \in \omega^*$ such that*
 (a) *for some countable $F \subseteq \omega^* \setminus \{x\}$ we have that $x \in \bar{F}$,*
 (b) *for all countable discrete $D \subseteq \omega^* \setminus \{x\}$ we have that $x \notin \bar{D}$.*

PROOF. Let \mathcal{M} be the BA of Lebesgue measurable subsets of $[0, 1]$ and let \mathcal{N} be the ideal of null-sets. We put $\mathcal{B} = \mathcal{M}/\mathcal{N}$. It is well-known, and easy to prove, that $|\mathcal{B}| = \mathfrak{c}$ and that \mathcal{B} is complete. Consequently, $X = \text{st}(\mathcal{B})$ is an extremally disconnected compactum of weight \mathfrak{c} . If $M \in \mathcal{M}$, then the \mathcal{N} -equivalence class of M is denoted by $[M]$. λ denotes Lebesgue measure.

Fact 1. If $D \subseteq X$ is countable, then D is nowhere dense.

Take $M \in \mathcal{M}$ and list D as $\{d_n : n < \omega\}$. Since d_n is an ultrafilter in the BA \mathcal{B} , there exists an element $M_n \in \mathcal{B}$ such that

- (1) $[M_n] \in d_n$, and
- (2) $\lambda(M_n) < 2^{-2^{-n}} \cdot \lambda(M)$.

Then $\{x \in X : [M \setminus \bigcup_{n < \omega} M_n] \in x\}$ is a nonempty open subset of $[M]$ which misses $\{d_n : n < \omega\}$.

Fact 2. X is ccc.

Let $\mathcal{A} \subseteq \mathcal{M}$ be uncountable such that $\lambda(A) > 0$ for all $A \in \mathcal{A}$ while moreover the family

$$\{\{x \in X : [A] \in x\} : A \in \mathcal{A}\}$$

is pairwise disjoint. Let \mathcal{U} be a countable open basis for $[0, 1]$ which is closed under finite unions and for all $U \in \mathcal{U}$, put

$$\mathcal{A}(U) = \{A \in \mathcal{A} : \lambda(A \cap U) > \frac{1}{2}\lambda(U)\}.$$

If $A \in \mathcal{A}$, then there is a compact $K \subseteq A$ with $\lambda(K) > 0$. For this K there is an element $U \in \mathcal{U}$ with $K \subseteq U$ and $\lambda(K) > \frac{1}{2}\lambda(U)$. We conclude that $A \in \mathcal{A}(U)$ and since A is arbitrarily chosen, this implies that

$$\bigcup_{U \in \mathcal{U}} \mathcal{A}(U) = \mathcal{A}.$$

Hence there must be an element $U \in \mathcal{U}$ such that $\mathcal{A}(U)$ is uncountable. But this contradicts the definition of $\mathcal{A}(U)$.

Fact 3. There is a family \mathcal{D} of \mathfrak{c} nowhere dense subsets of X such that each nowhere dense subset of X is contained in an element of \mathcal{D} .

Since X has weight \mathfrak{c} , by Fact 2 we can take \mathcal{D} to be the collection of all nowhere dense closed G_δ 's.

Fact 4 (CH). There is a point $x \in Y^$, where $Y = \omega \times X$, such that*

- (1) *x is a P -point of Y^* ,*
- (2) *if $D \subseteq \omega \times X$ is any nowhere dense set, then $x \notin \bar{D}$.*

By Fact 3 and by CH there is a family \mathcal{E} of ω_1 nowhere dense subsets of Y such that each nowhere dense subset of Y is contained in an element of \mathcal{E} . By Theorem 1.2.5, $Y^* \approx \omega^*$, and therefore by Lemma 1.7.1, there is a point

$$x \in Y^* \setminus (\cup \{E^* : E \in \mathcal{E}\} \cup \cup \{\bar{U} \setminus U : U \subseteq Y^* \text{ is an open } F_\sigma\})$$

(it is easily seen that if $E \in \mathcal{E}$, then $E^* \subseteq Y^*$ is nowhere dense). It is clear that x is as required.

Now, since βY is an extremally disconnected compactum of weight c , by Theorem 1.4.4 (3), βY can be embedded in ω^* as a closed P -set. If we take $x \in \beta Y$ such as in Fact 4, then Facts 1 and 2 imply (if we identify βY with a P -set in ω^*) that x is a weak P -point which is not a P -point (in ω^* as well as in βY). This proves (1). To prove (2), substitute X by the projective cover of the Cantor set and proceed similarly. The details of checking this out are left to the reader. \square

1.7.4. REMARK. As we will see later, Theorem 1.7.3 is true in ZFC by a more complicated argument. We have included the above proof since this way of constructing points will be used frequently in the remaining part of this paper. If the reader understands the proof of Theorem 1.7.3, she or he will have less trouble understanding the more complicated forthcoming constructions.

*1.8. Retracts of $\beta\omega$ and ω^**

In this section we study subspaces of $\beta\omega$ or ω^* on which $\beta\omega$ or ω^* can be retracted.

1.8.1. THEOREM (CH). *Let X be a closed P -set of ω^* . Then X is a retract of ω^* .*

PROOF. By CH, we can enumerate $\mathcal{B}(X)$ by $\{C_\alpha : \alpha < \omega_1\}$. It is easy, using the fact that X is a P -set, to construct for each $\alpha < \omega_1$ a countable subalgebra $\mathcal{B}_\alpha \subseteq \mathcal{B}(X)$ and an embedding $\rho_\alpha : \mathcal{B}_\alpha \rightarrow \mathcal{B}(\omega^*)$ such that

- (1) if $\beta < \alpha$, then $\mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$,
- (2) if $\beta < \alpha$, then $\rho_\alpha \upharpoonright \mathcal{B}_\beta = \rho_\beta$,
- (3) $C_\alpha \in \mathcal{B}_\alpha$,
- (4) if $C \in \mathcal{B}_\alpha$, then $\rho_\alpha(C) \cap X = C$.

Define $\rho : \mathcal{B}(X) \rightarrow \mathcal{B}(\omega^*)$ by $\rho(C) = \rho_\alpha(C)$ if $C \in \mathcal{B}_\alpha$.

Now define $r : \omega^* \rightarrow X$ by

$$\{r(x)\} = \bigcap \{C \in \mathcal{B}(X) : x \in \rho(C)\}.$$

An easy check shows that r is a retraction. \square

We will now prove a result on retracts of $\beta\omega$ which does not use CH.

1.8.2. THEOREM. *Let $X \subseteq \beta\omega$ be a closed subspace of countable π -weight. Then X is a retract of $\beta\omega$.*

PROOF. Let \mathcal{C} be a countable subalgebra of $\mathcal{B}(X)$ which forms a π -basis for X (for the definition of π -basis, see JUHÁSZ [1980] or HODEL [1983]). It is trivial to find a function $\rho: \mathcal{C} \rightarrow \mathcal{B}(\beta\omega)$ such that

- (1) ρ is an embedding,
- (2) if $C \in \mathcal{C}$, then $\rho(C) \cap X = C$.

Define a function $\kappa: \mathcal{B}(X) \rightarrow \mathcal{B}(\beta\omega)$ by

$$\kappa(A) = (\bigcup \{\rho(C) : C \in \mathcal{C}, C \subseteq A\})^-.$$

Since $\mathcal{P}(\omega)$ is complete or, in topological language, since $\beta\omega$ is extremally disconnected, κ is well-defined.

Fact 1. *If $A \in \mathcal{B}(X)$, then $\kappa(A) \cap X = A$.*

Since $\bigcup \{\rho(C) : C \in \mathcal{C}, C \subseteq A\} \cap \bigcup \{\rho(C) : C \in \mathcal{C}, C \subseteq X \setminus A\} = \emptyset$, this is immediate.

Fact 2. *κ is an embedding.*

This follows easily from the fact that ρ is an embedding.

Now, as in the proof of the previous theorem, define $r: \beta\omega \rightarrow X$ by

$$\{r(x)\} = \bigcap \{A \in \mathcal{B}(X) : x \in \kappa(A)\}.$$

By Stone duality, r is continuous, and by Fact 1, $r \upharpoonright X = \text{identity}$. \square

Observe that the above theorem is interesting since it shows that a certain class of subspaces of $\beta\omega$ is always a retract of $\beta\omega$ no matter how these sets are placed in $\beta\omega$. However, topologically, there are not many closed subspaces of $\beta\omega$ which have countable π -weight, so in this sense the theorem is quite restrictive. M. TALAGRAND [1981] has given recently a quite complicated example of a separable closed subspace of $\beta\omega$ which is not a retract of $\beta\omega$ (his construction is under CH; it is desirable to find such an example in ZFC only). Therefore, the above theorem cannot be generalized. The following result shows precisely how far one can go, and it also illustrates the complexity of Talagrand's Example.

1.8.3. THEOREM. *Any separable, extremally disconnected compact space can be embedded in ω^* in such a way that it is a retract of $\beta\omega$.*

PROOF. Let X be a separable, extremally disconnected compact space. Since $\beta\omega$ maps onto each separable compact space, there is a continuous surjection $f: \beta\omega \rightarrow X$. Let $Z \subseteq \beta\omega$ be such that $f \upharpoonright Z$ is an irreducible surjection from Z onto X . Since X is extremally disconnected, as in the proof of Theorem 1.4.7, $f \upharpoonright Z$ is a homeomorphism.

Since ω^* is infinite, it contains a countable relatively discrete subspace D . By Theorem 1.5.2, $\bar{D} \approx \beta\omega$, and by Theorem 1.8.2, \bar{D} is a retract of $\beta\omega$. Since, as was shown above, X embeds in \bar{D} as a retract, the desired result follows. \square

1.9. Nowhere dense P -sets in ω^*

Nowhere dense P -sets have played an important role in this section. The question therefore naturally arises whether each point $x \in \omega^*$ is contained in a nowhere dense P -set. Under CH, we will show that this is not the case.

1.9.1. DEFINITION. Let $\kappa \geq \omega$. A subset A of a space X is called a P_κ -set provided that the intersection of fewer than κ neighborhoods of A is again a neighborhood of A .

P_{ω_1} -sets are precisely the P -sets of course, and any subset of any space is a P_ω -set.

1.9.2. LEMMA. Let X be a space of π -weight $\leq \kappa$, where κ is regular and $\kappa \geq \omega$. For each $1 \leq n < \omega$ there is a family \mathcal{F}_n of closed subsets of X such that

- (1) \mathcal{F}_n has the n -intersection property,
- (2) if $K \subseteq X$ is any nowhere dense P_κ -set, then for some $F \in \mathcal{F}_n$ we have that $F \cap K = \emptyset$.

PROOF. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be a π -basis for X and let \mathcal{D} be the family of all nowhere dense subsets of X . For each $D \in \mathcal{D}$, put

$$H(D) = \{\alpha < \kappa : \bar{U}_\alpha \cap D = \emptyset\}.$$

Define ordinals $\mu(D, m) < \kappa$ ($1 \leq m < \omega$) as follows

$$\begin{aligned} \mu(D, 1) &= \min H(D), \\ \mu(D, m) &= \min\{\alpha < \kappa : \forall \beta \leq \mu(D, m-1) \exists \xi \leq \alpha \bar{U}_\xi \subseteq U_\beta \setminus \bar{D}\} \end{aligned}$$

(observe that $\mu(D, 2)$ need not be defined if κ is singular).

Define $F(D, n) = \bigcup\{\bar{U}_\alpha : \alpha \in H(D) \text{ and } \alpha \leq \mu(D, n)\}$.

Fact. $\{F(D, n) : D \in \mathcal{D}\}$ has the n -intersection property. In fact, if $\mathcal{E} \in [\mathcal{D}]^n$ then there is an $\alpha \leq \max\{\mu(D, n) : D \in \mathcal{E}\}$ such that $U_\alpha \subseteq \bigcap\{F(D, n) : D \in \mathcal{E}\}$.

Induction on n . The case $n = 1$ is trivial. Suppose the fact to be true for all $i \leq n$ and take $D_1, D_2, \dots, D_{n+1} \in \mathcal{D}$. We may assume that for all $i \leq n + 1$ we have that $\mu(D_i, n) \leq \mu(D_{n+1}, n)$.

By induction hypothesis, there is an $\alpha \leq \max\{\mu(D_i, n) : 1 \leq i \leq n\}$ such that $U_\alpha \subseteq \bigcap\{F(D_i, n) : 1 \leq i \leq n\} \subseteq \bigcap\{F(D_i, n+1) : 1 \leq i \leq n\}$. Since $\alpha \leq \mu(D_{n+1}, n)$, there is a $\beta \leq \mu(D_{n+1}, n+1)$ with $\bar{U}_\beta \subseteq U_\alpha \setminus \bar{D}_{n+1}$.

Now let $\mathcal{F}_n = \overline{\{F(K, n) : K \subseteq X \text{ is a nowhere dense } P_\kappa\text{-set}\}}$. Since $F(K, n)$ is a union of less than κ closed sets each of which do not intersect K , clearly $\overline{F(K, n)} \cap K = \emptyset$. Consequently, \mathcal{F}_n is as required. \square

We now come to the main result of this section.

1.9.3. THEOREM. *Let X be a compact space of π -weight $\leq \kappa$ ($\kappa > \omega$). Then there is an $x \in X$ such that $x \notin K$ for all nowhere dense P_κ -sets $K \subseteq X$.*

PROOF. Suppose first that κ is regular. Let $\{V_n : n < \omega\}$ be a sequence of countably many nonempty pairwise disjoint open subsets of X . By Lemma 1.9.2, there is a family \mathcal{F}_n of closed subsets of \overline{V}_n such that

(1) \mathcal{F}_n has the n -intersection property,

(2) if $K \subseteq X$ is a nowhere dense P_κ -set, then there is an $F \in \mathcal{F}_n$ with $F \cap K = \emptyset$. (Observe that if $K \subseteq X$ is a nowhere dense P_κ -set, then $K \cap \overline{V}_n$ is a nowhere dense P_κ -set of \overline{V}_n). Take any point x in the intersection

$$\bigcap \left\{ \overline{\bigcup_{n < \omega} g(n)} : g \in \prod_{n < \omega} \mathcal{F}_n \right\}.$$

Since $\kappa > \omega$, x is as required.

Now observe that if κ is singular, then any P_κ -set of X is a P_{κ^+} -set. Therefore, the theorem for singular κ follows from the theorem for regular κ . \square

1.9.4. COROLLARY (CH). *There is a point $x \in \omega^*$ such that $x \notin K$ for all nowhere dense P -sets $K \subseteq \omega^*$.*

Notes for Section 1

Theorem 1.1.6 is due to PAROVIČENKO [1963]. Lemma 1.2.2 is well known. For other results of this type see COMFORT & NEGREPONTIS [1974]. Corollary 1.2.4 is due to PAROVIČENKO [1963]. Theorem 1.2.5 can be found in GILLMAN & JERISON [1960]. The argument given here is due to NEGREPONTIS [1967]. That Theorem 1.2.6 is a consequence of Parovičenko's characterization of ω^* , was observed by many people. Theorem 1.3.1 is due to PAROVIČENKO [1963]. For another proof of this result see BLASZCZYK & SZYMAŃSKI [1980b]. Lemmas 1.4.1 and 1.4.2 are implicit in BALCAR, FRANKIEWICZ & MILLS [1980]. Theorem 1.4.4 (1) \Leftrightarrow (2) is due to LOUVEAU [1973]; the equivalence (1) \Leftrightarrow (3) can be found in BALCAR, FRANKIEWICZ & MILLS [1980]. Theorem 1.4.7 is due to EFIMOV [1970]. A result stronger than Theorem 1.5.2 is due to COMFORT, HINDMAN & NEGREPONTIS [1969]. A result stronger than Theorem 1.5.3 is due to WOODS [1976a]. For a related result, see WOODS [1976b]. An important step in the proof of Theorem 1.5.3 is due to FINE &

GILLMAN [1960]. Corollary 1.5.4 is due to GILLMAN [1966]. Lemma 1.6.1 is due to W. RUDIN [1956] but the proof we give is due to van DOUWEN & VAN MILL [1981d]. Lemma 1.6.2 is well-known and Theorem 1.6.4 is a special case of a result in VAN DOUWEN & VAN MILL [1981b]. Theorem 1.6.5 is due to W. RUDIN [1956]. Corollary 1.7.2 is also due to W. RUDIN [1956]. Theorem 1.7.3 is due to KUNEN [1976]. Theorem 1.8.1 can be found in VAN DOUWEN & VAN MILL [1981b]. The easy Theorem 1.8.2 seems to be new. The proof is in the spirit of VAN MILL [1979b]. For a related result see VAN DOUWEN & VAN MILL [1980]. Theorem 1.8.3 is well-known. I don't know who proved this for the first time. Corollary 1.9.4 is due to Kunen and Theorem 1.9.3 is due to KUNEN, VAN MILL & MILLS [1980]. The proof presented here, which was suggested to me by Alan Dow, is different from the proof given in KUNEN, VAN MILL & MILLS [1980]. It is in the spirit of CHAE & SMITH [1980] and VAN DOUWEN [1981].

2. The spaces $\beta\omega$ and $\beta\omega \setminus \omega$ under \neg CH

In this section we will see how $\beta\omega$ and ω^* behave in various models in which CH is not true. All the CH results derived in Section 1 are consistently false, except for Theorem 1.7.3, which is true in ZFC (see Theorems 4.3.3 and 4.4.1), and Lemma 1.4.3 and Theorem 1.7.3 of which we do not know whether they can be false.

2.1. A characterization of $\mathcal{P}(\omega)/\text{fin}$, II

The main result in Section 1.1, namely Theorem 1.1.6, is false under \neg CH. In fact, Theorem 1.1.6 is equivalent to CH.

2.1.1. THEOREM. *CH is equivalent to the statement that all Boolean algebras of cardinality c which satisfy condition H_ω are isomorphic.*

PROOF. Our proof is in topological language, since this is more convenient here. The Boolean algebraic reader can easily translate this proof in Boolean algebraic language.

We will construct two Parovičenko spaces that cannot be homeomorphic under \neg CH.

Example 1. A Parovičenko space S having a point p such that $\chi(p, X) = \omega_1$.

By Lemma 1.2.3 there is an ω_1 -sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ of clopen subsets in ω^* with $C_\alpha \subset C_\beta$ if $\beta < \alpha < \omega_1$. Let $P = \bigcap_{\alpha < \omega_1} C_\alpha$, and let $S = X/P$, the quotient space obtained from X by collapsing P to one point. By Lemma 1.4.1, S is an F -space. The other properties of S required in the definition of a Parovičenko space are easily checked. If we put $p = \{P\}$, then, obviously, $\chi(p, S) = \omega_1$.

Example 2. A Parovičenko space T with $\pi(x, T) = c$ for all $x \in T$.

Put $T = (\omega \times 2^c)^*$. Since $\omega \times 2^c$ is a zero-dimensional Lindelöf space of weight c , T is zero-dimensional and has weight c . Consequently, Theorem 1.2.5 implies that T is a Parovičenko space.

For $\alpha < c$ denote the α -th projection $2^c \rightarrow 2$ by π_α . For $\alpha < c$ and $i < 2$ define

$$K(\alpha, i) = T \cap (\omega \times \pi_\alpha^{-1}(\{i\}))^-.$$

Note that $K(\alpha, i)$ is a nonempty clopen subset of T and that $K(\alpha, i) = K(\alpha', i')$ iff $\alpha = \alpha'$ and $i = i'$. Define

$$\mathcal{K} = \{K(\alpha, i) : \alpha < c, i < 2\}.$$

Claim. Any intersection of ω_1 distinct members of \mathcal{K} has empty interior.

For symmetry reasons it suffices to prove that $I = \bigcap_{\alpha < \omega_1} K(\alpha, 0)$ has empty interior. Suppose that this is not true. Then there is a clopen $U \subseteq \beta(\omega \times 2^c)$ such that $\emptyset \neq U \cap T \subseteq I$. For every $\alpha < \omega_1$ the set $U \setminus (\omega \times \pi_\alpha^{-1}(\{0\}))$ is a compact subset of $\omega \times 2^c$, and since $U \cap (\omega \times 2^c)$ is not compact, there is an integer n_α such that $\emptyset \neq U \cap (\{n_\alpha\} \times 2^c) \subseteq \{n_\alpha\} \times \pi_\alpha^{-1}(\{0\})$. There is an integer n such that $A = \{\alpha < \omega_1 : n_\alpha = n\}$ is infinite. But then $\{n\} \times \bigcap_{\alpha \in A} \pi_\alpha^{-1}(\{0\})$ is a subset of $\{n\} \times 2^c$ with nonempty interior, which is absurd.

Let $x \in T$ be arbitrary, and let \mathcal{U} be a π -base for x . The family $\mathcal{F} = \{K \in \mathcal{K} : x \in K\}$ has cardinality c . For each $K \in \mathcal{F}$ there is a $U(K) \in \mathcal{U}$ with $U(K) \subseteq K$, hence $|\mathcal{U}| \geq |\mathcal{F}| = c$ since the Claim implies that $\{K \in \mathcal{K} : U(K) = U\} \leq \omega$ for all $U \in \mathcal{U}$. It follows that $\pi(x, T) = c$ since we know already that T has weight at most c . \square

2.1.2. REMARK. The above result suggests the following interesting question: is CH equivalent to the statement that (*) all Boolean algebras of cardinality c which satisfy condition H_ω have isomorphic completions? This question was first considered by Broverman & Weiss [1981], who showed that (*) is not a theorem of ZFC. Subsequently, VAN MILL & WILLIAMS [1983] proved that (*) implies that $c < 2^{\omega_1}$. Whether (*) iff CH is still unknown. (See the remarks on p. 564.)

2.2. A topological translation, II

Section 2.1 of course implies that Corollary 1.2.4 is equivalent to CH. Whether Theorem 1.2.6 is equivalent to CH is unknown, although it is easy to show it is not a Theorem of ZFC. In the proof of Theorem 2.1.1 we showed that, among others, $(\omega \times 2^c)^*$ contains a nonempty intersection of ω_1 clopen sets with empty interior. However, $MA + \neg CH$ implies that each nonempty intersection of ω_1 clopen subsets of ω^* has nonempty interior, see 2.3. Consequently, $MA + \neg CH$ implies that $(\omega \times 2^c)^*$ is not homeomorphic to ω^* . This argument does not apply to prove that a space such as $(\omega \times 2^\omega)^*$ is not homeomorphic to ω^* , since $MA + \neg CH$

easily implies that if X is a locally compact, σ -compact, noncompact space of countable π -weight, then each nonempty intersection of fewer than c open subsets of X^* has nonempty interior. One might therefore hope that if X is topologically very 'close' to ω , then X^* and ω^* are homeomorphic. Even this is not true.

2.2.1. THEOREM. *It is consistent that ω^* and $(\omega \times W(\omega + 1))^*$ are not homeomorphic.*

PROOF. SHELAH [1978] has recently shown that it is consistent that all homeomorphisms of ω^* are induced, i.e. that for each autohomeomorphism $h : \omega^* \rightarrow \omega^*$ there is a permutation $\pi : \omega \rightarrow \omega$ such that $h = \beta\pi \upharpoonright \omega^*$. We will show that for any permutation π of ω , the set of fixed points of $\beta\pi$ is a clopen subset of $\beta\omega$, consequently, the set of fixed points of $\beta\pi \upharpoonright \omega^*$ is a clopen subset of ω^* . Let $\pi : \omega \rightarrow \omega$ be a permutation, and let $p \in \omega^*$ be a fixed point of $\beta\pi$. Let $E = \{n < \omega : \pi(n) = n\}$. If $E \in p$, then p has clearly a clopen neighborhood consisting of fixed points of $\beta\pi$, namely, the closure of E in $\beta\omega$. So assume that $E \notin p$. Define $F = \omega \setminus E$. Since for all $n \in F$ we have that $\pi(n) \neq n$, it is easy to split F in two sets F_0 and F_1 such that $\pi(F_0) \cap F_0 = \emptyset$ and $\pi(F_1) \cap F_1 = \emptyset$. Without loss of generality, $F_0 \in p$. Then $\pi(F_0) \in \beta\pi(p)$, whence $p \neq \beta\pi(p)$, a contradiction. We conclude that the set of fixed points of $\beta\pi$ is open, whence clopen. To prove that in Shelah's model, ω^* and $(\omega \times W(\omega + 1))^*$ are not homeomorphic, it therefore suffices to produce an autohomeomorphism h of $(\omega \times W(\omega + 1))^*$ such that the set $\text{Fix}(h)$ of fixed points of h is not clopen. To this end, let $E, F \subseteq \omega$ be two complementary infinite sets and let $\pi : \omega \rightarrow \omega$ be a permutation such that $\pi(E) = F$ (which implies that $\pi(F) = E$). Define $f : \omega \times W(\omega + 1) \rightarrow \omega \times W(\omega + 1)$ by

$$\begin{cases} f(\langle n, m \rangle) = \langle n, \pi(m) \rangle & (m \in \omega), \\ f(\langle n, \omega \rangle) = \langle n, \omega \rangle. \end{cases}$$

Put $h = \beta f \upharpoonright (\omega \times W(\omega + 1))^*$. It is easily seen that

$$\text{Fix}(h) = (\omega \times \{\omega\})^*,$$

which implies that $\text{Fix}(h)$ is not clopen. \square

2.2.2. REMARK. Observe that in the proof of Theorem 2.2.1 we found an easily described topological property that distinguishes between ω^* and $(\omega \times W(\omega + 1))^*$.

2.3. Continuous images of ω^* , II

It is well known that Corollary 1.3.4 is not a result of ZFC. KUNEN [1968, 12.7 and 12.3] proved that in a model formed by adding ω_2 Cohen reals to a model of CH, there is no ω_2 sequence of subsets of ω which is strictly decreasing (mod fin).

Therefore, in this model ω^* cannot be mapped onto $W(c+1)$. VAN DOUWEN & PRZYMUSIŃSKI [1980] have used results of ROTHBERGER [1952] and PRZYMUSIŃSKI [1978] to show that Corollary 1.3.4 is not true under the following hypothesis:

$$(*) \quad \omega_2 \leq c < 2^{\omega_1} = \omega_{\omega_2}.$$

This is interesting since (*) only involves cardinals.

For a discussion of Martin's Axiom (abbreviated MA), see M.E. RUDIN [1977]. The following statements are consequences of MA:

P(c): If \mathcal{A} is a family of less than c subsets of ω such that for all $\mathcal{B} \in [\mathcal{A}]^{<\omega}$ we have that $|\bigcap \mathcal{B}| = \omega$, then there is an infinite $B \subseteq \omega$ such that $|B \setminus A| < \omega$ for all $A \in \mathcal{A}$,

S(c): Suppose that \mathcal{A} and \mathcal{B} are families of less than c subsets of ω such that for all $A \in \mathcal{A}$ and $\mathcal{F} \in [\mathcal{B}]^{<\omega}$ we have that $|A \cap \bigcap \mathcal{F}| = \omega$. Then there is an infinite $C \subseteq \omega$ such that $|C \cap A| = \omega$ for each $A \in \mathcal{A}$, and $|C \setminus B| < \omega$ for all $B \in \mathcal{B}$.

We are now in a position to generalize Theorem 1.3.3.

2.3.1. THEOREM (MA). *Each compact space of weight less than c is a continuous image of ω^* .*

PROOF. Let Y be a compact space of weight κ , where $\kappa < c$. We may assume that Y is a nowhere dense subspace of $[0, 1]^\kappa$. Since $\kappa \leq c$, $[0, 1]^\kappa$ is separable, and we can therefore find a countable dense set D of $[0, 1]^\kappa$ which misses Y . Let \mathcal{E} be an open base for $[0, 1]^\kappa$ which is closed under finite unions and such that $|\mathcal{E}| = \kappa$. Put

$$\mathcal{A} = \{E \cap D : E \in \mathcal{E} \text{ \& } E \cap Y \neq \emptyset\},$$

and

$$\mathcal{B} = \{E \cap D : E \in \mathcal{E} \text{ \& } Y \subseteq E\},$$

respectively. It is easily seen that \mathcal{A} and \mathcal{B} satisfy the hypotheses of *S(c)*. Consequently, we can find a subset $J \subseteq D$ such that

- (1) $|A \cap J| = \omega$ for all $A \in \mathcal{A}$,
- (2) $|J \setminus B| < \omega$ for all $B \in \mathcal{B}$.

Let $Z = Y \cup J$. We claim that Z is compact and that J is a dense set of isolated points of Z . If this is true, then Z is a compactification of ω , which implies that Y is a continuous image of ω^* .

If $E \in \mathcal{E}$ and $E \cap Y \neq \emptyset$, then $E \cap D \in \mathcal{A}$ which implies that $E \cap J$ is infinite. Hence J is dense in Z . Let x be a limit point of J which does not belong to Y . Since \mathcal{E} is closed under finite unions and since Y is compact, there are disjoint $E_0, E_1 \in \mathcal{E}$ with $Y \subseteq E_0$ and $x \in E_1$. Then $E_0 \cap D \in \mathcal{B}$ which implies, by (2), that $J \setminus E_0$

is finite. But E_1 contains infinitely many points of J , contradiction. We conclude that Z is compact and that J is relatively discrete. \square

Of course, the above theorem suggests the question, due to VAN DOUWEN & PRZYMUSIŃSKI [1980, 2.8], whether MA implies that each compact space of weight c is a continuous image of ω^* . In the remaining part of this section we will show that this is not the case.

Let κ and λ be infinite cardinals and consider the following statement:

$G(\kappa, \lambda)$: there are a κ -sequence $\langle U_\xi : \xi < \kappa \rangle$ of clopen sets in ω^* and a λ -sequence $\langle V_\xi : \xi < \lambda \rangle$ of clopen sets in ω^* such that

- (1) $U_\xi \subset U_\eta$ if $\xi < \eta < \kappa$,
- (2) $V_\xi \subset V_\eta$ if $\xi < \eta < \lambda$,
- (3) $(\bigcup_{\xi < \kappa} U_\xi) \cap (\bigcup_{\xi < \lambda} V_\xi) = \emptyset$, but
- (4) $(\bigcup_{\xi < \kappa} U_\xi)^- \cap (\bigcup_{\xi < \lambda} V_\xi)^- \neq \emptyset$.

This has a straightforward translation in terms of the existence of certain families of subsets of ω which we leave to the reader.

By Lemma 1.1.2, $G(\omega, \omega)$ is false, but interestingly, $G(\omega_1, \omega_1)$ is true, HAUSDORFF [1936].

2.3.2. THEOREM. *There is a compact space X and a continuous surjection $f : X \rightarrow \omega^*$ such that, under $\text{MA} + \text{---CH} + \text{---}G(c, c)$, X has weight c , f is irreducible, and ω^* cannot be mapped onto X .*

PROOF. Let $Y = \omega^*$ with the $G_{<c}$ -topology, i.e the underlying set of Y is ω^* and the intersections of fewer than c clopen subsets of ω^* form an open basis for Y . Let \mathcal{E} be a basis for Y of cardinality $w(Y)$ consisting of clopen sets. By transfinite induction, for each $\alpha < c$ we will construct subalgebras $\mathcal{E}_\alpha \subseteq \mathcal{B}(Y)$ such that

- (1) $\mathcal{E}_0 = \langle\langle \mathcal{E} \rangle\rangle$,
- (2) $|\mathcal{E}_\alpha| \leq |\bigcup_{\beta < \alpha} \mathcal{E}_\beta|^\xi$,
- (3) if $\beta < \alpha < c$ and if $\mathcal{F} \in [\mathcal{E}_\beta]^{<c}$, then $\bigcup \mathcal{F} \in \mathcal{E}_\alpha$.

It is straightforward to construct these algebras since the union of fewer than c clopen subsets of Y is clopen.

Put $\mathcal{B} = \bigcup_{\alpha < c} \mathcal{E}_\alpha$ and observe that if c is regular, then \mathcal{B} is a $<c$ -closed subalgebra of $\mathcal{B}(Y)$. Let $X = \text{st}(\mathcal{B})$. We claim that X is as required. It is clear that the function $f : X \rightarrow \omega^*$ defined by

$$\{f(x)\} = \bigcap \{\bar{B} : B \in x\}$$

is a continuous surjection.

From now on, assume $\text{MA} + \text{---CH} + \text{---}G(c, c)$. We also identify Y and the subspace of X consisting of the fixed ultrafilters on \mathcal{B} .

First observe that (2) implies that X has weight c , since MA implies that $2^c = c$ for all $\omega \leq \kappa < c$, see M.E. RUDIN [1977]. We first claim that f is irreducible. To this end, let $A \subseteq X$ be a proper closed subset. Since Y is dense in X , we can find a point $y \in Y \setminus A$. Choose $E \in \mathcal{E}$ such that $\bar{E} \cap A = \emptyset$ and $y \in \bar{E}$ (the closure is taken in X). Since E is an intersection of fewer than c clopen subsets of ω^* , by $P(c)$ we can find a nonempty clopen $C \subseteq \omega^*$ such that $C \subseteq E$. It is clear that $C \cap f(A) = \emptyset$.

We will now show that ω^* cannot be mapped onto X . Fix $y \in Y$. We will construct a family $\{B(y, \alpha) : \alpha < c\}$ and a family $\{E(y, \alpha) : \alpha < c\}$ of clopen subsets of X such that

- (4) $\alpha < \beta < c \rightarrow B(y, \alpha) \subset B(y, \beta) \subset X \setminus \{y\}$,
- (5) $\alpha < \beta < c \rightarrow E(y, \alpha) \subset E(y, \beta) \subset X \setminus \{y\}$,
- (6) $\alpha < \beta < c \rightarrow B(y, \alpha) \cap E(y, \beta) = \emptyset$,
- (7) $(\bigcup \{B(y, \alpha) : \alpha < c\})^- \cap (\bigcup \{E(y, \alpha) : \alpha < c\})^- = \{y\}$,
- (8) $\bigcup \{B(y, \alpha) : \alpha < c\} \cup \bigcup \{E(y, \alpha) : \alpha < c\} = X \setminus \{y\}$.

(This construction is a triviality of course). Let $\{Z_\alpha : \alpha < c\}$ enumerate the family of all clopen subsets of X containing y . To achieve (7) and (8), we will make the required families of clopen sets such that

- (9) $Z_\alpha \cap B(y, \alpha) \neq \emptyset$, $Z_\alpha \cap E(y, \alpha) \neq \emptyset$ and $X \setminus Z_\alpha \subseteq B(y, \alpha) \cup E(y, \alpha)$.

So our induction hypotheses are (4), (5), (6) and (9). Suppose that we have completed the construction for all $\alpha < \beta < c$. Put

$$B = \bigcup_{\alpha < \beta} B(y, \alpha) \quad \text{and} \quad E = \bigcup_{\alpha < \beta} E(y, \alpha).$$

Then \bar{B} and \bar{E} are both open, since \mathcal{B} is $<c$ -closed, which implies, by (6), that $\bar{B} \cap \bar{E} = \emptyset$. Since y is a P_c -point of X , i.e. the intersection of fewer than c neighborhoods of y is again a neighborhood of y , $y \notin \bar{B} \cup \bar{E}$. Let $F \subseteq Z_\alpha$ be a clopen neighborhood of y which misses $\bar{B} \cup \bar{E}$. Take two disjoint clopen nonempty subsets $G, H \subseteq F$ which do not contain y . Define

$$B(y, \alpha) = \bar{B} \cup G \cup (X \setminus (\bar{E} \cup Z_\alpha)) \quad \text{and} \quad E(y, \alpha) = \bar{E} \cup H.$$

It is clear that our inductive hypotheses are satisfied.

Now suppose that there is a continuous surjection $g : \omega^* \rightarrow X$. Put

$$B_y = (\bigcup \{g^{-1}(B(y, \alpha)) : \alpha < c\})^- \quad \text{and} \quad E_y = (\bigcup \{g^{-1}(E(y, \alpha)) : \alpha < c\})^-.$$

By $\neg G(c, c)$, $B_y \cap E_y = \emptyset$. Observe that y is the unique point of Y with the property that $g^{-1}(y)$ meets both B_y and E_y .

Let $Y_0 = \{y \in Y : B_y \cup E_y = \omega^*\}$. Then B_y and E_y are both clopen and since if $y_0, y_1 \in Y$ are distinct, then $B_{y_0} \neq B_{y_1}$, we have that $|Y_0| \leq c$.

Let $Y_1 = \{y \in Y : B_y \cup E_y \neq \omega^*\}$. If $y \in Y_1$, then $g^{-1}(y)$ has nonempty interior in ω^* , which implies that $|Y_1| \leq c$.

Since by 3.1.2 (c), $|Y| = 2^c$ and $Y = Y_0 \cup Y_1$, we have the desired contradiction. \square

2.3.3. REMARK. An inspection of the proof of Theorem 2.3.2 will show that we ‘only’ need the following hypotheses:

- (1) $2^\kappa = c$ if $\omega \leq \kappa < c$, and
- (2) $\neg G(c, c)$.

2.3.4. REMARK. If \mathcal{B} is the BA of clopen subsets of the space of Theorem 2.3.2, then $\mathcal{P}(\omega)/\text{fin}$ can be embedded in \mathcal{B} , $\mathcal{P}(\omega)/\text{fin}$ and \mathcal{B} have isomorphic completions, but $|\mathcal{B}| = c$ and \mathcal{B} cannot be embedded in $\mathcal{P}(\omega)/\text{fin}$.

2.3.5. REMARK. Let X be the space of Theorem 2.3.2. Observe that $Z = (\omega \times X)^*$ is a Parovičenko space which is not a continuous image of ω^* , since Z can be mapped onto X .

Until now it is not clear yet that Theorem 2.3.2 has some use, for it is not obvious at all that $\text{MA} + \neg\text{CH} + \neg G(c, c)$ can be true. Fortunately, KUNEN [1981] has shown the following.

2.3.6. THEOREM. (A) *It is consistent with $\text{MA} + \neg\text{CH}$ that $G(\omega_1, c)$ and $G(c, c)$ both are false,*
 (B) *it is consistent with $\text{MA} + \neg\text{CH}$ that $G(\omega_1, c)$ and $G(c, c)$ both are true.*

Let X be the space constructed in the proof of Theorem 1.7.3, i.e. X is the Stone space of the reduced measure algebra of $[0, 1]$. It is unknown whether it is consistent that X is not a continuous image of ω^* . It will not be possible to deduce this from rather global properties of X , since BELL [1980] has constructed in ZFC examples of spaces which are very similar to X and which are continuous images of ω^* .

Let us finally notice that PRZYMUSIŃSKI [1982] has shown that each perfectly normal compact space is a continuous image of ω^* . The big open question in this area is whether every first countable compactum is a continuous image of ω^* .

2.4. Closed subspaces of $\beta\omega$, II

In Section 1.4 we showed that every compact zero-dimensional F -space of weight c embeds, under CH, in $\beta\omega$. This suggests to consider the following statement:

FE: *Every compact zero-dimensional F -space can be embedded in an Extremally disconnected space.*

Observe that in Boolean algebraic language FE is the statement that each WCC BA is a homomorphic image of some complete BA.

It is convenient to factor FE as FB + BE, where

FB: *Every compact zero-dimensional F-space can be embedded in a Basically disconnected space,*

and

BE: *Every Basically disconnected compact space can be embedded in an Extremely disconnected space.*

Of course, both FB and BE have straightforward Boolean algebraic translations. In Section 1.4 we showed, in particular, that the restriction of FE to spaces of weight \mathfrak{c} holds under CH. VAN DOUWEN & VAN MILL [1980] construct, under $MA + \mathfrak{c} = \omega_2$, an example of a compact zero-dimensional F-space V of weight \mathfrak{c} that cannot be embedded in any basically disconnected space. As a consequence, neither FE nor FB are theorems of ZFC. Very little is known about BE, it is known however that the Čech–Stone compactification of any P -space embeds in an extremely disconnected space, see Section 4.4. We conclude that Theorems 1.4.4, 1.4.5 and Corollary 1.4.6 are false under $MA + \mathfrak{c} = \omega_2$.

2.5. C^* -embedded subspaces of $\beta\omega$, II

It is easy to see that Theorem 1.5.3 need not be true. If $2^{\omega_1} = \mathfrak{c}$, then by Theorem 1.4.7, $\beta\omega_1$ embeds in ω^* , say by the embedding h . It is clear that $h(\omega_1)$ is C^* -embedded in $\beta\omega$, but $h(\omega_1)$ is not weakly Lindelöf. It is not so clear that Corollary 1.5.4 need not be true.

2.5.1. LEMMA [$\forall \kappa < \mathfrak{c}$, $\neg G(\kappa, \omega) + \neg G(\mathfrak{c}, \mathfrak{c})$]. *If $A \subseteq \omega^*$ is a closed $P_{\mathfrak{c}}$ -set, then $\omega^* \setminus A$ is C^* -embedded in ω^* .*

PROOF. Striving for a contradiction, assume there are disjoint, nonempty, closed G_{δ} -subsets $Z_0, Z_1 \subseteq \omega^* \setminus A$ such that $\bar{Z}_0 \cap \bar{Z}_1 \neq \emptyset$. Pick a point $a \in \bar{Z}_0 \cap \bar{Z}_1$ and let $\{C_{\alpha} : \alpha < \mathfrak{c}\}$ enumerate the family of all clopen subsets of ω^* containing a . By transfinite induction on $\alpha < \mathfrak{c}$, we will construct clopen subsets G_{α}^i ($i < 2$) of ω^* such that

- (1) $G_{\alpha}^i \subseteq Z_i$ and $G_{\alpha}^i \cap C_{\alpha} \neq \emptyset$,
- (2) if $\beta < \alpha$, then $G_{\beta}^i \subseteq G_{\alpha}^i$.

If we can complete the induction, then we contradict $\neg G(\mathfrak{c}, \mathfrak{c})$. Suppose that the sets G_{β}^i are defined for all $\beta < \alpha$, $i < 2$. Since A is a $P_{\mathfrak{c}}$ -set, there is a clopen $C \subseteq \omega^* \setminus A$ such that

$$\bigcup_{\beta < \alpha} G_{\beta}^0 \cup \bigcup_{\beta < \alpha} G_{\beta}^1 \subseteq C.$$

By $\neg G(\alpha, \omega)$, we can find clopen sets $C_i \subseteq C \cap Z_i$ such that

$$\bigcup_{\beta < \alpha} G_{\beta}^i \subseteq C_i.$$

Put $C'_\alpha = C_\alpha \cap (\omega^* \setminus C)$. Take $x_i \in Z_i \cap C'_\alpha$. Let $E_i \subseteq \omega^*$ be clopen neighborhoods of x_i not meeting A . By Theorem 1.2.5, $E_i \cap Z_i$ contains a non-empty clopen set, say F_i . Define $G^i_\alpha = C_i \cup F_i$. This completes the induction, which gives us the required contradiction. \square

2.5.2. COROLLARY [$\forall \kappa < c, \neg G(\kappa, \omega) + \neg G(c, c)$]. *Let $A = \{x \in \omega^* : \exists$ closed nowhere dense P_c -set $B \subseteq \omega^*$ containing $x\}$. If $x \in A$, then $\omega^* \setminus \{x\}$ is C^* -embedded in ω^* .*

PROOF. By Lemma 2.5.1, if $x \in A$ and if B is a nowhere dense closed P_c -set containing x , then $\beta(\omega^* \setminus B) = \omega^*$. But this easily implies that $\beta(\omega^* \setminus \{x\}) = \omega^*$. \square

The question arises of course whether Corollary 2.5.2 is of any use, i.e. is it possible that the set A is nonempty, while moreover the combinatorial hypotheses required for the proof of Corollary 2.5.2 hold. The answer is yes of course. By Theorem 2.3.6, there is a model in which $MA + \neg CH + \neg G(c, c)$ is true. It is easy to show that MA implies there are P_c -points and that MA implies $\neg G(\omega, \kappa)$ for all $\kappa < c$.

Consequently, we obtain

2.5.3. COROLLARY TO COROLLARY. *It is consistent that for some $x \in \omega^*$ we have that $\beta(\omega^* \setminus \{x\}) = \omega^*$.*

It is unpleasant that the point x of Corollary 2.5.3 does not ‘really’ exist, since it is a P_c -point. We will show that there are many points which ‘really’ exist for which it is consistent that their complements in ω^* are C^* -embedded in ω^* .

2.5.4. THEOREM [$\forall \kappa < c, \neg G(\kappa, \omega) + \neg G(c, c)$]. *If $x \in \omega^*$ is not a P -point, then $\omega^* \setminus \{x\}$ is C^* -embedded in ω^* .*

PROOF. Let $x \in \omega^*$ be not a P -point and let $U \subseteq \omega^*$ be an open F_σ such that $x \in \bar{U} \setminus U$. Let $f: \omega^* \setminus \{x\} \rightarrow [0, 1]$ be continuous. Let $f_0 = f \upharpoonright \bar{U} \setminus \{x\}$ and $f_1 = f \upharpoonright \omega^* \setminus \bar{U}$. By $\forall \kappa < c, \neg G(\kappa, \omega)$, we have that \bar{U} is a P_c -set in ω^* . Consequently, by Lemma 2.5.1, we can extend f_1 to a continuous map $g_1: \omega^* \setminus \bar{U} \rightarrow [0, 1]$. By Theorem 1.2.5, $\omega^* \setminus \bar{U} = (\omega^* \setminus \bar{U}) \cup (\bar{U} \setminus U)$. This implies that $g_1(t) = f(t)$ for all $t \in \bar{U} \setminus (U \cup \{x\})$. By Theorem 1.5.2, U is C^* -embedded in \bar{U} , consequently, $\bar{U} \setminus \{x\}$ is C^* -embedded in \bar{U} . We therefore conclude that we can extend f_0 to a continuous map $g_0: \bar{U} \rightarrow [0, 1]$. Since $g_0 \upharpoonright \bar{U} \setminus (U \cup \{x\}) = g_1 \upharpoonright \bar{U} \setminus (U \cup \{x\})$, we conclude that $g_0 \upharpoonright \bar{U} \setminus U = g_1 \upharpoonright \bar{U} \setminus U$ since x is not isolated (Theorem 1.2.5). Define $g: \omega^* \rightarrow [0, 1]$ by

$$\begin{cases} g(t) = g_0(t) & \text{if } t \in \bar{U}, \\ g(t) = g_1(t) & \text{if } t \notin \bar{U}. \end{cases}$$

It is obvious that g is continuous and that g extends f . \square

Since P -points need not exist in ω^* , one might hope that the following statements can be simultaneously true:

- (1) $\forall \kappa < c, \neg G(\kappa, \omega)$,
- (2) $\neg G(c, c)$,
- (3) no P -points,

for then, there would be a model in which $\beta(\omega^* \setminus \{x\}) = \omega^*$ for all $x \in \omega^*$. Unfortunately, as was pointed out to me by Ken Kunen, (1) implies $\neg(3)$. Define an order $<^*$ on ω^ω by

$$f <^* g \text{ iff } |\{n < \omega : f(n) \geq g(n)\}| < \omega.$$

A subset $A \subseteq \omega^\omega$ is called *dominating* if for each $f \in \omega^\omega$ there is a $g \in A$ with $f <^* g$.

2.5.5. LEMMA. *Suppose that no subset of ω^ω of cardinality less than c dominates. Then ω^* contains a P -point.*

PROOF. Let $\{f_\alpha : \alpha < c\}$ enumerate ω^ω . By transfinite induction on $\alpha < c$ we will construct a filter $\mathcal{F}_\alpha \subseteq \mathcal{P}(\omega)$ such that

- (1) finite intersections of elements of \mathcal{F}_α have infinite intersections,
- (2) there is an element $F \in \mathcal{F}_\alpha$ such that either $|f_\alpha^{-1}(n) \cap F| < \omega$ for all $n < \omega$, or $F \subseteq f_\alpha^{-1}(\{0, 1, \dots, n\})$ for certain $n < \omega$,
- (3) if $\kappa < \alpha$ then $\mathcal{F}_\kappa \subseteq \mathcal{F}_\alpha$ and $|\mathcal{F}_\alpha| \leq |\alpha| \cdot \omega$.

Suppose we have constructed everything for all $\kappa < \alpha$ and define $\mathcal{F} = \bigcup_{\kappa < \alpha} \mathcal{F}_\kappa$. Observe that $|\mathcal{F}| \leq |\alpha| \cdot \omega < c$. For each $F \in \mathcal{F}$ define a function $g(F) : \omega \rightarrow \omega$ by

$$g(F)(n) = \begin{cases} \min(F \cap f_\alpha^{-1}(n)) & \text{if } F \cap f_\alpha^{-1}(n) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|\mathcal{F}| < c$, we can find a function $f \in \omega^\omega$ such that $f \not<^* g(F)$ for all $F \in \mathcal{F}$. Define

$$X = \bigcup_{n < \omega} f_\alpha^{-1}(n) \cap \{j < \omega : j \leq f(n)\}$$

and define \mathcal{F}_α to be the filter generated by $\mathcal{F} \cup \{X\}$ if $|F \cap X| = \omega$ for all $F \in \mathcal{F}$. Otherwise, define $\mathcal{F}_\alpha = \mathcal{F}$.

It is clear that any ultrafilter extending $\bigcup_{\alpha < c} \mathcal{F}_\alpha$ is a P -point. \square

In view of the above Lemma it therefore suffices to prove the following.

2.5.6. LEMMA. $(\forall \kappa < c, \neg G(\kappa, \omega)) \rightarrow (\text{no subset of } \omega^\omega \text{ of cardinality less than } c \text{ dominates}).$

PROOF. Let $\kappa = \min\{\lambda : \exists F \subseteq \omega^\omega \text{ such that } F \text{ dominates and } |F| = \lambda\}$. Choose $F \subseteq \omega^\omega$ of cardinality κ such that F dominates. We may assume that $F = \{f_\alpha : \alpha < \kappa\}$, where $\alpha < \beta$ implies that $f_\alpha <^* f_\beta$. For each $\alpha < \kappa$, let

$$S_\alpha = \{\langle m, n \rangle : n < f(m)\}.$$

If $T_n = (\omega \setminus \{0, 1, \dots, n\}) \times \omega$ for all $n < \omega$, then the families $\{S_\alpha : \alpha < \kappa\}$ and $\{T_n : n < \omega\}$ form a (κ, ω) gap (defined on $\omega \times \omega$). \square

2.6. *Autohomeomorphisms of ω^* . II*

As remarked in Section 2.2, SHELAH [1978] has shown it to be consistent that all autohomeomorphisms of ω^* are induced by a permutation of ω . Consequently, in this model ω^* has precisely \mathfrak{c} autohomeomorphisms and we conclude that Lemma 1.6.1 can be false. I do not know whether Theorem 1.6.4 is a result of ZFC. This is caused by the fact that I do not know whether in ZFC there is a nowhere dense P -set in ω^* which is homeomorphic to ω^* . Theorem 1.6.5 is false under $\text{MA} + \neg\text{CH}$, since this axiom easily implies that there are P_c -points in ω^* and P -points which are not P_c -points.

2.7. *P-points and nonhomogeneity of ω^* . II*

It was an open problem for many years whether P -points in ω^* could be constructed without using additional set theoretic hypotheses. Finally, Shelah, see MILLS [1980] or WIMMERS [1980], proved it to be consistent that P -points in ω^* do not exist. Therefore, Corollary 1.7.2 cannot be established in ZFC. In Sections 3 and 4 we will give several proofs that ω^* is not homogeneous.

Theorem 1.7.3 is true in ZFC, see 4.3.3 and 4.4.1.

2.8. *Retracts of $\beta\omega$ and ω^* . II*

Theorem 1.8.1 has to be consistently false of course. This can be seen in various ways, one of which we give below. As usual,

$$U(\omega_1) = \{p \in \beta\omega_1 : \forall P \in \mathcal{P}, |P| = \omega_1\}.$$

2.8.1. THEOREM ($\text{MA} + \neg\text{CH}$). *There is a nowhere dense closed P -set X in ω^* which is not a retract of ω^* .*

PROOF. By KUNEN [1976, 1.2], the space $\beta\omega_1$ embeds in ω^* as a P_c -set under $\text{MA} + \neg\text{CH}$. Let $h : \beta\omega_1 \rightarrow \omega^*$ be an embedding such that $h(\beta\omega_1)$ is a P -set and put $X = h(U(\omega_1))$. Since $U(\omega_1)$ is a P -set in $\beta\omega_1$, X is a P -set in ω^* . By COMFORT & NEGREPONTIS [1974, 12.2], it follows that there is a family of \mathcal{A} of ω_2 uncountable subsets of ω_1 such that for distinct $A, B \in \mathcal{A}$ the set $A \cap B$ is

countable. This immediately implies that the cellularity of $U(\omega_1)$ is at least ω_2 . Since ω_1 is dense in $\beta\omega_1$, there cannot be a retraction from $\beta\omega_1$ onto $U(\omega_1)$, which immediately implies that there cannot be a retraction from ω^* onto X . \square

2.9. Nowhere dense P -sets in ω^* , II

BALCAR, FRANKIEWICZ & MILLS [1980] prove it to be consistent that ω^* can be covered by nowhere dense closed P -sets. Consequently, Corollary 1.9.4 is not a result of ZFC.

DOW & VAN MILL [1981] show that no compact space can be covered by nowhere dense ccc P -sets i.e. P -sets satisfying the countable chain condition. It is not known whether there is a compact space that can be covered by nowhere dense P -sets of cellularity at most ω_1 , however it is known that ω^* is not a consistent example.

2.9.1. PROPOSITION. *There is a point $x \in \omega^*$ such that $x \notin K$ for any nowhere dense P -set $K \subseteq \omega^*$ of cellularity at most ω_1 .*

PROOF. Under CH, this is a consequence of Theorem 1.9.3. So assume \neg CH. It is left to the reader to prove that the R -points constructed in the proof of Lemma 3.3.4 have the required property. \square

It is unknown whether in ZFC there is a nowhere dense P -set in ω^* of cellularity at most ω_1 .

The cover of ω^* constructed by BALCAR, FRANKIEWICZ and MILLS consists of P -sets of different ‘cofinalities’. Interestingly, NYIKOS [1982] has recently shown that it is consistent that ω^* can be covered by nowhere dense closed P -sets which are all an intersection of a chain of ω_1 clopen subsets of ω^* .

Notes for Section 2

Theorem 2.1.1 is due to VAN DOUWEN & VAN MILL [1978]. Theorem 2.2.1 is due to VAN DOUWEN & VAN MILL [1981d]. That Theorem 2.3.1 holds was established in VAN DOUWEN & PRZYMUSIŃSKI [1980]. It was known from the work of Baumgartner that MA does not imply that each compact space of weight \mathfrak{c} is a continuous image of ω^* . Theorem 2.3.2, which is due to the author, gives another proof of this result. As noticed in Section 2.3, the interesting Theorem 2.3.6 is due to KUNEN [1980].

Section 2.5, with the exception of 2.5.5 and 2.5.6, was taken from VAN DOUWEN & VAN MILL [1981c]. Lemma 2.5.5 is due to KETONEN [1976]. That $\forall \kappa < \mathfrak{c}$, $\neg G(\kappa, \omega)$ implies that P -points in ω^* exist, was pointed out to me by Ken Kunen. Lemma 2.5.6 is due to Kunen and HECHLER [1975]. Theorem 2.8.1 is well-known. Proposition 2.9.1 is due to DOW & VAN MILL [1980].

3. Partial orderings on $\beta\omega$

In this section we will concentrate on various partial orderings on $\beta\omega$, which can be used to prove that certain spaces are not homogeneous.

3.1. The Rudin–Keisler order on $\beta\omega$

Let $f: \omega \rightarrow \omega$ be a function and let $\beta f: \beta\omega \rightarrow \beta\omega$ be its Stone extension. It is easily verified that

$$(*) \quad \beta f(p) = q \text{ iff } \forall P \in p: f(P) \in q \text{ iff } \forall Q \in q: f^{-1}(Q) \in p.$$

Define an equivalence relation \sim on $\beta\omega$ by

$$p \sim q \text{ iff } \exists \text{ permutation } \pi: \omega \rightarrow \omega \text{ with } \beta\pi(p) = q.$$

It is clear that \sim is indeed an equivalence relation.

Let $p, q \in \beta\omega$ and write

$$p \leq q \text{ iff } \exists f \in \omega^\omega \text{ with } \beta f(q) = p.$$

The following theorem, which we will not prove in detail, summarizes relevant information about \sim and \leq .

3.1.1. THEOREM. *Let $p, q, r \in \beta\omega$. Then*

- (a) $p \leq p$,
- (b) if $p \leq q$ and $q \leq r$, then $p \leq r$,
- (c) if $p \leq q$ and $q \leq p$, then $p \sim q$.

Observe that only 3.1.1(c) requires proof. For information concerning the proof of Theorem 3.1.1(c) and many related things, see COMFORT & NEGREPONTIS [1974, section 9].

Observe that Theorem 3.1.1 shows that the quotient relation defined by \leq on $\beta\omega/\sim$ is a partial ordering.

The relation \leq on $\beta\omega$ is called the *Rudin–Keisler order* on $\beta\omega$. If $p \in \beta\omega$, then the set $\{q \in \beta\omega : q \leq p\}$ is equal to

$$\{\beta f(p) : f \in \omega^\omega\}$$

and therefore has cardinality at most c , since $|\omega^\omega| = c$. Is there for all $p \in \beta\omega$ a point $q \in \beta\omega$ such that $q \not\leq p$? It seems strange, but at this moment we do not have the tools yet to answer this question, since we have almost not deduced any results about $\beta\omega$ in ZFC alone. In fact, we did not even find the cardinality of $\beta\omega$. Let us quickly compute $|\beta\omega|$, in order to answer the above question.

3.1.2. LEMMA. (a) *There is a family $\{A_\alpha : \alpha < c\}$ of infinite subsets of ω such that if $\alpha < \beta$, then $A_\alpha \cap A_\beta$ is finite.*

(b) *There is a family $\{(A_\alpha^0, A_\alpha^1) : \alpha < c\}$ of pairs of disjoint subsets of ω such that for all finite $F \subseteq c$ and for each $f : F \rightarrow 2$ we have that $\bigcap_{\alpha \in F} A_\alpha^{f(\alpha)}$ is infinite.*

(c) $|\beta\omega| = 2^c$; in fact, if $A \subseteq \beta\omega$ is countably infinite, then $|\bar{A}| = 2^c$.

PROOF. For each irrational number $r \in \mathbb{R}$ choose a sequence $S(r)$ of rational numbers converging to r . The family $\{S(r) : r \text{ irrational}\}$ is obviously as required in (a), except that it does not consist of subsets of ω , but of the countable set \mathbb{Q} . But this causes no problems of course.

Let $\{A_\alpha : \alpha < c\}$ be a family of subsets of ω as in (a). For each $\alpha < c$, define

$$B_\alpha^0 = \{F \in [\omega]^{<\omega} : F \cap A_\alpha \neq \emptyset\}, \quad \text{and} \quad B_\alpha^1 = [\omega]^{<\omega} \setminus B_\alpha^0.$$

An easy check shows that the family $\{(B_\alpha^0, B_\alpha^1) : \alpha < c\}$ has the properties of the required family in (b), except that it is not defined on ω , but on the countable set $[\omega]^{<\omega}$. But this again causes no problems of course.

Let $\{(A_\alpha^0, A_\alpha^1) : \alpha < c\}$ be a family as in (b).

If $f \in 2^c$, take a point $p_f \in \beta\omega$ such that $\{A_\alpha^{f(\alpha)} : \alpha < c\} \subseteq p_f$. It is clear that $|\beta\omega| \geq |\{p_f : f \in 2^c\}| = 2^c$. Since $|\mathcal{P}(\omega)| = c$, $|\beta\omega| \leq 2^c$ which proves that $|\beta\omega| = 2^c$. Statement (c) now follows from Theorem 1.5.2 and from the fact that each countably infinite space contains a countably infinite relatively discrete subspace. \square

The proof of the above lemma tells us two important facts, namely that combinatorial arguments are important if one wishes to study $\beta\omega$ without extra hypotheses, and that for obtaining certain families of subsets of ω , one should not try to define them directly on ω but rather on a suitable countable set which is, in the given situation, easier to handle than ω .

Let us now return to our question: given $p \in \beta\omega$, is there a point $q \in \beta\omega$ such that $q \not\leq p$? The answer is now easy of course, since $|\{q \in \beta\omega : q \leq p\}| \leq c$ and $|\beta\omega| = 2^c$, by Lemma 3.1.2(c). Let us specify the question a little bit: given $p \in \omega^*$, is there a point $q \in \beta\omega$ such that $p \not\leq q$ and $q \not\leq p$? It may come as a shock, but the answer to this question is not known. Under CH it is easy to show that the answer is yes, but in ZFC the answer is unknown. It is known, however, that at least there are points $p, q \in \beta\omega$ with $p \not\leq q$ and $q \not\leq p$ and in the remaining part of this section we will construct such points.

3.1.3. DEFINITION. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter no element of which is finite. An indexed family $\{(A_i^0, A_i^1) : i \in I\}$ of pairs of disjoint subsets of ω is called an *independent family with respect to \mathcal{F}* provided that for all $\sigma \in [I]^{<\omega}$, $f \in 2^\sigma$ and $F \in \mathcal{F}$ the set $F \cap \bigcap_{i \in \sigma} A_i^{f(i)}$ is infinite.

Let $\mathcal{C}\mathcal{F}$ denote the filter of cofinite subset of ω .

3.1.4. LEMMA. *There is an independent family $\{\{A_\alpha^0, A_\alpha^1\} : \alpha < c\} \subseteq \mathcal{P}(\omega)$ w.r.t. \mathcal{CF} .*

PROOF. Lemma 3.1.2 (b). \square

If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we denote by $\langle \mathcal{A} \rangle$ the (possibly improper) filter on ω generated by \mathcal{A} .

3.1.5. LEMMA. *Let $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(\omega)$ be filters and assume that $\{\{A_i^0, A_i^1\} : i \in I\}$ is independent w.r.t. \mathcal{F} as well as \mathcal{G} . For each $f \in \omega^\omega$ there is a finite $J \subseteq I$ and a subset $A \subseteq \omega$ such that $\{\{A_i^0, A_i^1\} : i \in I \setminus J\}$ is independent w.r.t. $\langle \mathcal{F} \cup \{A\} \rangle$ as well as $\langle \mathcal{G} \cup \{\omega \setminus f^{-1}(A)\} \rangle$.*

PROOF. Fix $a \in I$ arbitrarily.

Case 1: $\{\{A_i^0, A_i^1\} : i \in I \setminus \{a\}\}$ is independent w.r.t. $\langle \mathcal{G} \cup \{\omega \setminus f^{-1}(A_a^0)\} \rangle$.

We then put $A = A_a^0$ and $J = \{a\}$. An easy check shows that A and J are as required.

Case 2: Not Case 1.

Then there are a finite $K \subseteq I \setminus \{a\}$ and a function $\tilde{f} \in 2^K$ and an element $G \in \mathcal{G}$ such that

$$(*) \quad \left| \bigcap_{i \in K} A_i^{\tilde{f}(i)} \cap G \cap (\omega \setminus f^{-1}(A_a^0)) \right| < \omega.$$

Now put $A = \omega \setminus A_a^0$ and $J = K \cup \{a\}$. It is clear that $\{\{A_i^0, A_i^1\} : i \in I \setminus J\}$ is independent w.r.t. $\langle \mathcal{F} \cup \{A\} \rangle$, so it remains to be shown that $\{\{A_i^0, A_i^1\} : i \in I \setminus J\}$ is independent w.r.t. $\langle \mathcal{G} \cup \{\omega \setminus f^{-1}(A)\} \rangle$. To this end, let $L \subseteq I \setminus J$ be finite and take $g \in 2^L$. Choose $G_0 \in \mathcal{G}$ arbitrarily. Then

$$\begin{aligned} \bigcap_{i \in L} A_i^{g(i)} \cap G_0 \cap (\omega \setminus f^{-1}(A)) &= \bigcap_{i \in L} A_i^{g(i)} \cap G_0 \cap f^{-1}(A_a^0) \\ &\supseteq \bigcap_{i \in L} A_i^{g(i)} \cap \bigcap_{i \in K} A_i^{\tilde{f}(i)} \cap (G_0 \cap G) \cap f^{-1}(A_a^0), \end{aligned}$$

which is infinite by (*) and by our assumption that $\{\{A_i^0, A_i^1\} : i \in I\}$ is independent w.r.t. \mathcal{G} . \square

We now come to the main result of this section.

3.1.6. THEOREM. *There are points $p, q \in \beta\omega$ such that $p \not\leq q$ and $q \not\leq p$.*

PROOF. By Lemma 3.1.4 there is an independent family $\{\{A_\alpha^0, A_\alpha^1\} : \alpha < c\} \subseteq \mathcal{P}(\omega)$ w.r.t. \mathcal{CF} . Let $\{f_\alpha : 1 \leq \alpha < c\}$ enumerate ω^ω . By transfinite induction on α we will construct $\mathcal{F}_\alpha, \mathcal{G}_\alpha$ and $K_\alpha \subseteq 2^\omega$ so that

(1) \mathcal{F}_α and \mathcal{G}_α are filters on ω and $\{\langle A_\xi^0, A_\xi^1 \rangle : \xi \in K_\alpha\}$ is independent w.r.t. \mathcal{F}_α as well as \mathcal{G}_α ,

(2) $K_0 = 2^\omega$ and $\mathcal{F}_0 = \mathcal{G}_0 = \mathcal{CF}$,

(3) $\kappa < \alpha$ implies $\mathcal{F}_\kappa \subseteq \mathcal{F}_\alpha$, $\mathcal{G}_\kappa \subseteq \mathcal{G}_\alpha$ and $K_\alpha \subseteq K_\kappa$,

(4) for each α , $|2^\alpha \setminus K_\alpha| \leq |\alpha| \cdot \omega$,

(5) for each $\alpha \geq 1$ there are sets $A, B \subseteq \omega$ with $\{A, \omega \setminus f_\alpha^{-1}(B)\} \subseteq \mathcal{F}_\alpha$ and $\{B, \omega \setminus f_\alpha^{-1}(A)\} \subseteq \mathcal{G}_\alpha$.

Suppose that we have completed the construction for all $\kappa < \alpha$, $\alpha < c$. Put $K = \bigcap_{\kappa < \alpha} K_\kappa$, $\mathcal{G} = \bigcup_{\kappa < \alpha} \mathcal{G}_\kappa$ and $\mathcal{F} = \bigcup_{\kappa < \alpha} \mathcal{F}_\kappa$. Observe that, by (4), $|K| = c$ and that, by (1), $\{\langle A_\xi^0, A_\xi^1 \rangle : \xi \in K\}$ is independent w.r.t. \mathcal{F} as well as \mathcal{G} . Using Lemma 3.1.5 twice, it is easy to construct \mathcal{F}_α , \mathcal{G}_α and K_α satisfying (1) through (5).

Now let $p \in \beta\omega$ extend $\bigcup_{\alpha < c} \mathcal{F}_\alpha$ and let $q \in \beta\omega$ extend $\bigcup_{\alpha < c} \mathcal{G}_\alpha$. By (5) it easily follows that $p \not\leq q$ and $q \not\leq p$. \square

3.1.7. REMARK. Points p and q as in the above Theorem 3.1.6 are called \leq -incomparable. Observe that if $p, q \in \beta\omega$ are \leq -incomparable, then both p and q belong to ω^* .

3.1.8. REMARK. The technique of proof used in Theorem 3.1.6 is quite important. At each stage of the construction we give up a negligible number of the initial independent family in order to obtain in return a required property of the filter(s) we wish to construct. Under CH (or MA) such a delicate process is not necessary, since then at each stage of the construction we are at a countable level and one can then construct by hand enough sets to continue the induction. Under \neg CH, in a transfinite induction of length c one has to pass level ω_1 , and if one then for example in the previous steps constructed a family of sets which constitute a Hausdorff gap (i.e. a family of sets which witnesses the fact that $G(\omega_1, \omega_1)$ holds), then there is usually no way to continue the induction. In the proof of Theorem 3.1.6 this cannot happen, since before starting the induction enough sets were identified which ensure that one can always pick new sets to continue the induction.

3.1.9. REMARK. If $c^+ = 2^c$, then ω^* has a \leq -cofinal well-ordered subset (of cardinality 2^c); and the condition $c^+ < 2^c$ is equivalent to the statement that any subset of ω^* of cardinality 2^c has a pairwise \leq -incomparable subset of cardinality 2^c . For details, see COMFORT & NEGREPONTIS [1974, Corollaries 10.11 and 10.15].

3.2. The Rudin–Frolík order on ω^*

The Rudin–Frolík (pre-)order \sqsubseteq on ω^* is defined as follows:

$$p \sqsubseteq q \text{ iff there is an embedding } h : \beta\omega \rightarrow \omega^* \text{ with } h(p) = q.$$

This order and the Rudin–Keisler order \leq of Section 3.1 are related by the following lemma.

3.2.1. LEMMA. *If $p, q \in \omega^*$ and if $p \sqsubseteq q$, then $p \leq q$.*

PROOF. Let $h : \beta\omega \rightarrow \omega^*$ be an embedding with $h(p) = q$. Since $h(\omega)$ is a relatively discrete, there is a sequence C_n of subsets of ω such that for all $n < \omega$,

- (1) $h(n) \in \bar{C}_n$, and
- (2) if $n < m$, then $C_n \cap C_m = \emptyset$.

By adding $\omega \setminus \bigcup_{n < \omega} C_n$ to C_0 we may assume that the sequence $\{C_n : n < \omega\}$ is a partition of ω . Define $g : \omega \rightarrow \omega$ by

$$g(k) = n \quad \text{if } k \in C_n.$$

and let $\beta g : \beta\omega \rightarrow \beta\omega$ be its Stone extension. We claim that $\beta g(q) = p$. Take $Q \in q$ arbitrarily. The set $\{n : h(n) \in \bar{Q}\}$ must belong to p , since $h(p) = q$, but

$$\{n : h(n) \in \bar{Q}\} \subseteq g(Q),$$

and we therefore may conclude that $g(Q) \in p$. Consequently, $\beta g(q) = p$, which is as required. \square

3.2.2. COROLLARY. *If $p, q \in \omega^*$ and if $p \sqsubseteq q$ and $q \sqsubseteq p$, then $p \sim q$.*

PROOF. Apply Lemma 3.2.1 and Theorem 3.1.1(c). \square

3.2.3. LEMMA. *If $p, q, r \in \omega^*$ and if $p \sqsubseteq q$ and $q \sqsubseteq r$, then $p \sqsubseteq r$.*

PROOF. Let $f : \beta\omega \rightarrow \omega^*$ be an embedding with $f(p) = q$ and, similarly, let $g : \beta\omega \rightarrow \omega^*$ be an embedding with $g(q) = r$. Let $h = (g \upharpoonright f(\beta\omega)) \circ f$. Then h is an embedding with $h(p) = r$. \square

Observe that Corollary 3.2.2, Lemma 3.2.3 and Theorem 3.1.1(c) show that the quotient relation defined by \sqsubseteq on $\beta\omega/\sim$ is a partial ordering.

We will now show that the orders \leq and \sqsubseteq are powerful tools if one wishes to study $\beta\omega$. First a preliminary lemma.

3.2.4. LEMMA. *Let $f : \omega^* \rightarrow \omega^*$ be a homeomorphism and let $q \in \omega^*$. Then*

$$\{p \in \omega^* : p \sqsubseteq q\} = \{p \in \omega^* : p \sqsubseteq f(q)\}.$$

PROOF. Obvious. \square

This enables us to give our first ‘real’ proof that ω^* is not homogeneous.

3.2.5. THEOREM *ω^* is not homogeneous.*

PROOF. Let $D = \{d_n : n < \omega\}$ be a relatively discrete subset of ω^* and take a point $x \in \bar{D} \setminus D$. Observe that D is C^* -embedded in ω^* (Theorem 1.5.2), hence $\bar{D} = \beta D$. Put $A = \{y \in \bar{D} \setminus D : \exists \text{ homeomorphism } f : \omega^* \rightarrow \omega^* \text{ with } f(x) = y\}$. Let $h : \omega \rightarrow D$ be defined by $h(n) = d_n$ and let βh be its Stone extension. If $y \in A$, then clearly $(\beta h)^{-1}(y) \sqsubseteq y$ and consequently, by Lemma 3.2.3, $(\beta h)^{-1}(y) \sqsubseteq x$. Since βh is one to one, and since by Lemma 3.2.1 $|\{q \in \beta\omega : q \sqsubseteq x\}| \leq c$, we conclude that $|A| \leq c$. In Lemma 3.1.2 (c) we proved that $|\beta\omega| = 2^c$. Since $\beta D \approx \beta\omega$, we therefore can find 2^c points in $\bar{D} \setminus A$. \square

3.3. Another order on $\beta\omega$

Define an order \leq on $\beta\omega$ by

$$p \leq q \text{ if there is a finite to one } f \in \omega^\omega \text{ with } \beta f(q) = p.$$

This order is obviously quite similar to the Rudin–Keisler order. One might hope that at least on ω^* , the orders \leq and \leq are the same. The aim of this section is to show that this is not true. We will construct points $p, q \in \omega^*$ such that $p \leq q$, but p and q are \leq -incomparable. We first need a generalization of the concept of an independent family of subsets of ω . We will directly translate the new concept in terms of clopen subsets of ω^* .

3.3.1. DEFINITION. An indexed family $\{A_j^i : I \in I, j \in J\}$ of clopen subsets of ω^* is called a J by I independent matrix if

- (1) for all distinct $j_0, j_1 \in J$ and $i \in I$ we have that $A_{j_0}^i \cap A_{j_1}^i = \emptyset$,
- (2) if $F \in [I]^{<\omega}$ and $f \in J^F$, then

$$\bigcap \{A_{f(\alpha)}^\alpha : \alpha \in F\} \neq \emptyset.$$

We will first show that large families of this type exist.

3.3.2. LEMMA. *There is a c by c independent matrix of clopen subsets of ω^* .*

PROOF. Let $S = \{(k, f) : k < \omega \ \& \ f \in \mathcal{P}(k)^{\mathcal{P}(k)}\}$. For each $X, Y \in \mathcal{P}(\omega)$, put

$$A_X^Y = \{(k, f) \in S : f(Y \cap k) = X \cap k\}.$$

An easy check shows that the family $\{A_X^Y : X, Y \in \mathcal{P}(\omega)\}$, defined on the countable set S , gives us a c by c independent matrix of clopen subsets of ω^* . \square

3.3.3. DEFINITION. A closed subset $A \subseteq \omega^*$ is called an R -set if there is an open F_σ $U \subseteq \omega^*$ such that $A \subseteq \bar{U} \setminus U$ and $A \cap \bar{F} = \emptyset$ for all $F \cap [U]^{<c}$. An R -set consisting of precisely one point is called an R -point.

3.3.4. LEMMA. *There exists an R -point in ω^* .*

PROOF. Let $\{C_n : n < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^* . Put $C = \bigcup_{n < \omega} C_n$. For each $n < \omega$, let $\{A_\alpha^i(n) : i < \omega, \alpha < c\}$ be a c by ω independent matrix of clopen subsets of C_n (Lemma 3.3.2). Put

$$\mathcal{F} = \{F \subseteq C : \forall n < \omega \forall i \leq n \exists \alpha < c \text{ such that } A_\alpha^i(n) \subseteq F\}.$$

Notice that if $\mathcal{G} \in [\mathcal{F}]^n$, then $\bigcap \mathcal{G} \cap C_i \neq \emptyset$ for all $i \leq n - 1$. Let $D \in [C]^{<c}$. For each $n < \omega$ and $i \leq n$ choose $\alpha(n, i) < c$ such that $A_{\alpha(n,i)}^i \cap D = \emptyset$ and put

$$F = \bigcup_{n < \omega} \bigcup_{i \leq n} A_{\alpha(n,i)}^i(n).$$

Then $F \in \mathcal{F}$ and $F \cap D = \emptyset$. Since F is clopen (in C) and since disjoint clopen subsets of C have disjoint closures in ω^* (recall that ω^* is an F -space), we conclude that $\bar{F} \cap \bar{D} = \emptyset$. (Observe that $D \subseteq C \setminus F$). Consequently, each point of $\bigcap_{F \in \mathcal{F}} \bar{F}$ is an R -point of ω^* . \square

The following result is the key in deriving our main result of this section.

3.3.5. THEOREM. *Let \mathcal{A} be a family of c R -sets in ω^* . If $\{C_n : n < \omega\}$ is a family of countably many nonempty clopen subsets of ω^* , then for each $n < \omega$ there is a point $x_n \in C_n$ such that*

$$\bigcup \mathcal{A} \cap \{x_n : n < \omega\}^- = \emptyset.$$

PROOF. List \mathcal{A} as $\{A_\alpha : \alpha < c\}$. By induction, for each $\alpha < c$ we will construct for each $n < \omega$ a nonempty closed subset $F_\alpha^n \subseteq C_n$ such that

- (1) $(\bigcup_{n < \omega} F_\alpha^n)^- \cap A_\alpha = \emptyset$.
- (2) $\chi(F_\alpha^n, \omega^*) \leq |\alpha| \cdot \omega$ for each $n < \omega$,
- (3) if $\kappa < \alpha$ and $n < \omega$, then $F_\alpha^n \subseteq F_\kappa^n$.

Let $U \subseteq \omega^*$ be an open F_σ which witnesses the fact that A_0 is an R -set. Define $E = \{n < \omega : C_n \setminus \bar{U} \neq \emptyset\}$ and for each $n \in E$ choose a nonempty clopen $E_n \subseteq C_n \setminus \bar{U}$. For all $n \notin E$, pick a point $t_n \in C_n \cap U$. Since A_0 is an R -set, $A_0 \cap \{t_n : n \notin E\}^- = \emptyset$. Consequently, we can find for any $n \notin E$ a clopen neighborhood E_n of t_n such that $E_n \subseteq C_n$ and $(\bigcup_{n \notin E} E_n)^- \cap A_0 = \emptyset$. For each $n < \omega$ define $F_0^n = E_n$. By construction, $(\bigcup_{n \notin E} E_n)^- \cap A_0 = \emptyset$ and since ω^* is an F -space, $(\bigcup_{n \in E} E_n)^- \cap A_0 \subseteq (\bigcup_{n \in E} E_n)^- \cap \bar{U} = \emptyset$. Consequently, the F_0^n 's are as required.

Suppose that we have completed the construction for all $\mu < \alpha < c$. Put $G_n = \bigcap_{\mu < \alpha} F_\mu^n$ for all $n < \omega$ and observe that $\chi(G_n, \omega^*) \leq |\alpha| \cdot \omega$. Let $U \subseteq \omega^*$ be an open F_σ which witnesses the fact that A_α is an R -set. Put $E = \{n < \omega : G_n \cap \bar{U} \neq \emptyset\}$ and for each $n \in E$ let $\{V_\rho^n : \rho < |\alpha| \cdot \omega\}$ be a neighborhood

basis for G_n . For each $n \in E$ and $\rho < |\alpha| \cdot \omega$ pick a point in $V_\rho^n \cap U$ and let Z be the set of points obtained in this way. Then $|Z| < c$ and therefore there is a clopen neighborhood C of A_α which misses \bar{Z} . Define $F_\alpha^n = G_n$ if $n \notin E$ and $F_\alpha^n = G_n \setminus C$ if $n \in E$. An easy check, again using the fact that ω^* is an F -space, shows that everything is defined properly.

For each $n < \omega$ take a point $x_n \in \bigcap_{\alpha < c} F_\alpha^n$. Then $\{x_n : n < \omega\}$ is as required. \square

We need one more lemma.

3.3.6. LEMMA. *Let $f \in \omega^\omega$ be finite to one. If $p \in \omega^*$ is an R -point, then $\beta f^{-1}(\{p\})$ is an R -set.*

PROOF. Observe that $\beta f : \beta\omega \rightarrow \beta\omega$ is open and that $\beta f(\omega^*) \subseteq \omega^*$. If $p \notin \beta f(\beta\omega)$, then there is nothing to prove, so, without loss of generality, f is onto. Let $U \subseteq \omega^*$ be an open F_σ which witnesses the fact that p is an R -point. Since βf is open, $Z = \beta f^{-1}(\{p\}) \subseteq \bar{V} \setminus V$, where $V = \beta f^{-1}(U)$. Notice that V is an open F_σ of ω^* since $\beta f^{-1}(\omega^*) = \omega^*$. It is clear that if $F \in [V]^{<c}$, then $\bar{F} \cap Z = \emptyset$, whence Z is an R -set. \square

We now come to the main result of this section.

3.3.7. THEOREM. *For each R -point $x \in \omega^*$ there is a point $y \in \omega^*$ such that $x \leq y$ but x and y are \leq -incomparable.*

PROOF. Let $f : \omega \rightarrow \omega$ be such that $|f^{-1}(\{n\})| = \omega$ for all $n < \omega$. For each $n < \omega$, let $\{E_n^i : i < \omega\}$ be a family of countably many pairwise disjoint (faithfully indexed) nonempty clopen subsets of $\beta f^{-1}(\{n\}) \cap \omega^*$. By Theorem 3.3.5 and Lemma 3.3.6 we may pick for all $i, n < \omega$ a point $x_n^i \in E_n^i$ such that $\{x_n^i : i, n < \omega\}^- \cap \bigcup \{\beta g^{-1}(\{x\}) : g \in \omega^\omega \text{ is finite to one}\} = \emptyset$. For each $i < \omega$, let $S_i = \{x_n^i : n < \omega\}$. Observe that $\beta f(S_i) = \omega$ which implies that $\bar{S}_i \cap \beta f^{-1}(\{x\}) \neq \emptyset$. If $i \neq j$ then, since ω^* is an F -space, $\bar{S}_i \cap \bar{S}_j = \emptyset$ and this implies that

$$|\{x_n^i : i, n < \omega\}^- \cap \beta f^{-1}(\{x\})| \geq \omega,$$

Therefore $\{x_n^i : i, n < \omega\}^- \cap \beta f^{-1}(\{x\})$ contains a countably infinite relatively discrete set, which is C^* -embedded in ω^* by Theorem 1.5.2, and which therefore has the property that its closure has cardinality 2^c (Lemma 3.1.2(c)). Since $\{p \in \omega^* : p \leq x\} \leq c$, we can therefore find a point $y \in \beta f^{-1}(\{x\}) \setminus (\{p \in \omega^* : p \leq x\} \cup \bigcup \{\beta g^{-1}(\{x\}) : g \in \omega^\omega \text{ is finite to one}\})$. It is clear that y is as required. \square

Since, by Lemma 3.3.4, R -points in ω^* exist, we have therefore obtained, the following.

3.3.8. COROLLARY. *There are points $p, q \in \omega^*$ such that $p \leq q$ but p and q are \leq -incomparable.*

3.3.9. REMARK. It might come as a surprise that the proof in this section has not very much in common with the proof in Section 3.1 that \leq -incomparable points in $\beta\omega$ exist. I do not know whether Corollary 3.3.8 can also be obtained by the method of Section 3.1. Notice however that both methods have an important fact in common, namely that beforehand things have been arranged so that a transfinite induction of length ϵ was possible.

3.3.10. REMARK. In Section 4.5 we will compare the orders \leq, \subseteq and \leq with one another.

3.4. Applications of the Rudin–Keisler order

In this section we will give a surprisingly general nonhomogeneity result. This will allow us to give another proof that ω^* is not homogenous.

3.4.1. THEOREM. *Let X be an infinite compact space in which all countable discrete subspaces are C^* -embedded. Then X is not homogeneous.*

PROOF. Since X contains a countable discrete subspace, for convenience assume that $\omega \subseteq X$. The assumptions on X then imply that $\bar{\omega} = \beta\omega$. By Theorem 3.1.6 there are points $p, q \in \beta\omega$ which are \leq -incomparable. We claim there is no homeomorphism $h : X \rightarrow X$ with $h(p) = q$. Striving for a contradiction, assume that such an h exists.

Let $\{U_n : n < \omega\}$ be a family of open subsets of X such that

- (1) $n \in U_n \subseteq \bar{U}_n \subseteq X \setminus \omega^*$,
- (2) if $n \neq m$, then $\bar{U}_n \cap \bar{U}_m = \emptyset$.

Put $E = \{n < \omega : h(n) \notin \bigcup_{m < \omega} U_m \cup \bar{\omega}\}$.

Case 1: $q \in \bar{h(E)}$. Since $h(E) \cup \omega$ is clearly a discrete subset of X , and since $h(E) \cap \omega = \emptyset$, the assumptions on X imply that $\bar{h(E)} \cap \bar{\omega} = \emptyset$, which is impossible since $q \in \bar{\omega}$.

Put $F = \{n < \omega : h(n) \in \omega^*\}$.

Case 2: $q \in \bar{h(F)}$. Then $p \in \bar{F}$ and we may conclude that $p \subseteq q$ and consequently, by Lemma 3.2.1, $p \leq q$. This is a contradiction.

Put $G = \{n < \omega : h(n) \in \bigcup_{m < \omega} U_m\}$.

Case 3: $q \in \bar{h(G)}$. Define a function $f : \omega \rightarrow \omega$ by

$$\begin{cases} f(k) = 0 & \text{if } k \notin G, \\ f(k) = n & \text{if } k \in G \text{ and } h(k) \in U_n. \end{cases}$$

An easy check shows that $\beta f(p) = q$, i.e. $q \leq p$, which is also a contradiction.

Since $E \cup F \cup G = \omega$ and $p \in \bar{\omega}$, $q = h(p) \in \overline{h(E)} \cup \overline{h(F)} \cup \overline{h(G)}$. We therefore have derived a contradiction. \square

3.4.2. COROLLARY. *No compact infinite F -space is homogeneous. In particular, ω^* is not homogeneous.*

PROOF. By using the same technique as in the proof of Theorem 1.5.2, the reader can easily check that every countable subspace of an F -space is C^* -embedded. \square

Notes for Section 3

The Rudin–Keisler order on $\beta\omega$ was defined by KATĚTOV [1961] and independently, by M.E. RUDIN [1966] and KEISLER [1967]. Theorem 3.1.1 is due to Katětov, M.E. Rudin and Keisler. Lemma 3.1.2(a) and (b) are well-known. The proof of Lemma 3.1.2(a) is due to SIERPIŃSKI [1928]. The proof of Lemma 3.1.2(b) from 3.1.2(a) is new and was suggested to me by Charley Mills. Lemma 3.1.2(c) is due to HAUSDORFF [1936]. Lemma 3.1.4 is well-known and also follows from the fact that 2^c is separable. Lemma 3.1.5 and Theorem 3.1.6 are due to KUNEN [1972]. In fact, Kunen proves that there are c pairwise \leq -incomparable points. Recently, SHELAH and R.E. RUDIN [1978] showed that there even exist 2^c pairwise \leq -incomparable points. The Rudin–Frolík order on ω^* was defined by M.E. RUDIN [1966] and FROLÍK [1967a]. See also M.E. RUDIN [1971]. Theorem 3.2.3 is due to FROLÍK [1967a]. Under CH, it was earlier shown by W. RUDIN [1956]. Lemma 3.3.2 is due to KUNEN [1978]. All other results in section 3.3 were taken from VAN MILL [1981a]. Theorem 3.4.1 was formulated in COMFORT [1977]. The method of proof used in Theorem 3.4.1 is due to FROLÍK [1967b]. Much of the material presented in this chapter can also be found in COMFORT and NEGREPONTIS [1974].

For some recent information concerning the Rudin–Frolík order, see BUKOVSKÝ & BUTKOVICOVÁ [1981].

4. Weak P -points and other points in ω^*

We have seen that, under CH, there are P -points and non P -points in ω^* , whence ω^* is not homogeneous, see section 1.7. However, in section 2.7 we saw that this nonhomogeneity proof in ZFC did not work. In sections 3.2 and 3.4 we gave proofs in ZFC that ω^* is not homogeneous, but these proofs ‘only’ showed that ω^* is not homogeneous but not *why* it is not homogeneous. The aim of this section is to present several ‘special’ points in ω^* , thus giving a ‘real’ proof that ω^* is not homogeneous.

4.1. A technical result

The aim of this section is to prove a technical result which enables us later to construct several special points in ω^* .

4.1.1. DEFINITION. Let X be a compact extremally disconnected space, and let $\mathcal{C} = \{C_n : n < \omega\}$ be a sequence of nonempty, faithfully indexed, pairwise disjoint, clopen subsets of X and put $Z = X \setminus \cup \mathcal{C}$. In addition, let $f : Z \rightarrow Y$ be a continuous surjection and let $B \subseteq Z$ be closed.

If $1 \leq n < \omega$, an indexed family $\{A_i : i \in I\}$ of clopen subsets of X is *precisely n -linked* w.r.t. $\langle B, f \rangle$ if for all $\sigma \in [I]^n$,

$$f\left(\bigcap_{i \in \sigma} A_i \cap B\right) = Y,$$

but for all $\sigma \in [I]^{n+1}$, $\bigcap_{i \in \sigma} A_i \cap Z = \emptyset$.

An indexed family $\{A_n : i \in I, 1 \leq n < \omega\}$ of clopen subsets of X is a *linked system* w.r.t. $\langle B, f \rangle$, if for each n , $\{A_n : i \in I\}$ is precisely n -linked w.r.t. $\langle B, f \rangle$, and for each n and i , $A_{in} \subseteq A_{i,n+1}$. An indexed family $\{A_{in}^j : i \in I, 1 \leq n < \omega, j \in J\}$ is an *I by J independent linked family* w.r.t. $\langle B, f \rangle$ if for each $j \in J$, $\{A_{in}^j : i \in I, 1 \leq n < \omega\}$ is a linked system w.r.t. $\langle B, f \rangle$, and:

$$f\left(\bigcap_{j \in \tau} \left(\bigcap_{i \in \sigma_j} A_{in_j}^j\right) \cap B\right) = Y,$$

whenever $\tau \in [J]^{<\omega}$, and for each $j \in \tau$, $1 \leq n_j < \omega$ and $\sigma_j \in [I]^{n_j}$.

If $f : \omega \rightarrow \omega$ is a function, let $\bar{f} = \beta f \upharpoonright \omega^*$. In Definition 4.1.1, let $X = \beta\omega$ and, for all $n < \omega$, let $C_n = \{n\}$. We then have the following important lemma.

4.1.2. LEMMA. *There is a finite to one function $\pi : \omega \rightarrow \omega$ and a c by c independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle \omega^*, \bar{\pi} \rangle$.*

PROOF. Let $S = \{\langle k, g \rangle : k \in \omega \ \& \ g \in \mathcal{P}\mathcal{P}(k)^{\mathcal{P}(k)}\}$. Identify S and ω and define $\pi : S \rightarrow \omega$ by $\pi(\langle k, g \rangle) = k$. It is clear that π is finite to one. For all $X, Y \in \mathcal{P}(\omega)$ and $n < \omega$, define

$$A_{Xn}^Y = \{\langle k, g \rangle \in S : |g(Y \cap k)| \leq n \ \& \ X \cap k \in g(Y \cap k)\}.$$

It is easily seen that the family

$$\{E_{Xn}^Y : X \in \mathcal{P}(\omega), 1 \leq n < \omega, Y \in \mathcal{P}(\omega)\},$$

where E_{Xn}^Y is the closure of A_{Xn}^Y in βS , is as required. \square

4.1.3. DEFINITION. Let X be a space. A closed subspace $A \subseteq X$ is called κ -OK provided that for each sequence $\{U_n : n < \omega\}$ of neighborhoods of A , there is a sequence $\{V_\alpha : \alpha < \kappa\}$ of neighborhoods of A such that for each $n \geq 1$ and $\alpha_1 < \alpha_2 < \dots < \alpha_n < \kappa$,

$$\bigcap_{1 \leq i \leq n} V_{\alpha_i} \subseteq U_n.$$

Observe that the property of being κ -OK gets stronger as κ gets bigger. A point $x \in X$ is called a κ -OK point if $\{x\}$ is a κ -OK set of X .

4.1.4. DEFINITION. Let X be a space and let $\mathcal{U} = \{U_n : n < \omega\}$ be a family of open subsets of X . A closed subset $Z \subseteq \overline{\bigcup \mathcal{U}} \cup \mathcal{U}$ is called nice w.r.t. \mathcal{U} provided that for each neighborhood V of Z the set $\{n < \omega : V \cap U_n = \emptyset\}$ is finite.

We now come to the main result of this section.

4.1.5. THEOREM. Let X be a compact extremally disconnected space of weight c and let $\mathcal{C} = \{C_n : n < \omega\}$ be a sequence of nonempty, faithfully indexed, pairwise disjoint, clopen subsets of X and put $Z = \overline{\bigcup \mathcal{C}} \cup \mathcal{C}$. If $A \subseteq Z$ is nice w.r.t. \mathcal{C} and if Y is a continuous image of ω^* , then there is a closed set $B \subseteq A$ which is a c -OK set of Z and which admits an irreducible surjection on Y .

PROOF. Since $\overline{\bigcup \mathcal{C}}$ is clopen in X , without loss of generality, $\overline{\bigcup \mathcal{C}} = X$. Define $f : \bigcup \mathcal{C} \rightarrow \omega$ by $f(x) = n$ iff $x \in C_n$ and let $\beta f : X \rightarrow \beta\omega$ be its Stone extension. Since X is an F -space, $\overline{\bigcup \mathcal{C}} = \beta(\bigcup \mathcal{C})$. Let $\pi : \omega \rightarrow \omega$ be the finite to one function of Lemma 4.1.2 and let $\{E_{\alpha n}^\beta : \alpha < c, 1 \leq n < \omega, \beta < c\}$ be the c by c independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle \omega^*, \bar{\pi} \rangle$ of Lemma 4.1.2. In addition, let $g : \omega^* \rightarrow Y$ be a continuous surjection.

Define $h : Z \rightarrow Y$ by $h = g \circ \bar{\pi} \circ (\beta f \upharpoonright Z)$ and observe that the family

$$\{A_{\alpha n}^\beta : \alpha < c, 1 \leq n < \omega, \beta < c\},$$

where $A_{\alpha n}^\beta = \beta f^{-1}(E_{\alpha n}^\beta)$, is an independent linked family w.r.t. $\langle A, h \rangle$. For this one only needs to verify that $\beta f(A) = \omega^*$, and this is easy. Let $\{Z_\mu : \mu < c \ \& \ \mu \text{ is even}\}$ enumerate the family of all clopen subsets of X and let $\{\{S_{\mu n} : n < \omega\} : \mu < c \ \& \ \mu \text{ is odd}\}$ enumerate all sequences of nonempty clopen subsets of X satisfying

$$S_{\mu, n+1} \subseteq S_{\mu n} \setminus \bigcup_{i \leq n} C_i.$$

Furthermore, assume that each sequence is listed cofinally often. By induction on μ we construct F_μ and K_μ so that:

- (1) $F_\mu \subseteq A$ is closed, $K_\mu \subseteq c$, and $\{A_{\alpha n}^\beta : \alpha < c, 1 \leq n < \omega, \beta \in K_\mu\}$ is an independent linked family w.r.t. $\langle F_\mu, h \rangle$,
- (2) $K_0 = 2^\omega$ and $F_0 = A$;
- (3) $\nu < \mu$ implies $F_\nu \supseteq F_\mu$ and $K_\mu \subseteq K_\nu$,
- (4) if μ is a limit ordinal, $F_\mu = \bigcap_{\nu < \mu} F_\nu$ and $K_\mu = \bigcap_{\nu < \mu} K_\nu$,
- (5) for each μ , $K_\mu \setminus K_{\mu+1}$ is finite,
- (6) if μ is even, either $F_{\mu+1} \subseteq Z_\mu$ or $h(F_{\mu+1} \cap Z_\mu) \neq Y$,
- (7) if μ is odd and $F_\mu \subseteq \bigcap_{n < \omega} S_{\mu n}$, then there are clopen neighborhoods $D_{\mu\alpha}$ of $F_{\mu+1}$ for $\alpha < c$ such that for all $n \geq 1$ and $\alpha_1 < \alpha_2 < \dots < \alpha_n < c$, there is an $m < \omega$ such that

$$(D_{\mu\alpha_1} \cap \dots \cap D_{\mu\alpha_n}) \setminus S_{\mu n} \subseteq \bigcup_{i \leq m} C_i.$$

Fix $\mu < c$ and assume that F_ν, K_ν have been constructed for all $\nu \leq \mu$. We will construct $F_{\mu+1}$ and $K_{\mu+1}$.

Suppose first that μ is even and define $T = F_\mu \cap Z_\mu$. If

$$\{A_{\alpha n}^\beta : \alpha < c, 1 \leq n < \omega, \beta \in K_\mu\}$$

is an independent linked family w.r.t. $\langle T, h \rangle$, we put $F_{\mu+1} = T$ and $K_{\mu+1} = K_\mu$. If not, then

$$h\left(F_\mu \cap Z_\mu \cap \bigcap_{\beta \in \tau} \left(\bigcap_{\alpha \in \sigma_\beta} A_{\alpha n_\beta}^\beta\right)\right) \neq Y$$

for some $\tau \in [K_\mu]^{<\omega}$, $n_\beta < \omega$ and $\sigma_\beta \in [c]^{n_\beta}$. Then let $K_{\mu+1} = K_\mu \setminus \tau$, and let

$$F_{\mu+1} = F_\mu \cap \bigcap_{\beta \in \tau} \left(\bigcap_{\alpha \in \sigma_\beta} A_{\alpha n_\beta}^\beta\right).$$

Clearly, $F_{\mu+1}$ and $K_{\mu+1}$ are as required.

If μ is odd and there is an $n < \omega$ such that $F_\mu \setminus S_{\mu n} \neq \emptyset$, put $F_{\mu+1} = F_\mu$ and $K_{\mu+1} = K_\mu$. In case $F_\mu \subseteq \bigcap_{n < \omega} S_{\mu n}$, fix $\beta \in K_\mu$ and let $K_{\mu+1} = K_\mu \setminus \{\beta\}$. For each $\alpha < c$, define

$$D_{\mu\alpha} = \left(\bigcup_{1 \leq n < \omega} A_{\alpha n}^\beta \cap S_{\mu n}\right)^-,$$

and put $F_{\mu+1} = \bigcap_{\alpha < c} D_{\mu\alpha} \cap F_\mu$. We claim that $F_{\mu+1}$ and the sequence $\langle D_{\mu\alpha} : \alpha < c \rangle$ are as required.

First observe that each $D_{\mu\alpha}$ is clopen since it is the closure of an open set in X . To verify condition (7), let $\alpha_1 < \alpha_2 < \dots < \alpha_n < c$ and put

$$T = (D_{\mu\alpha_1} \cap \dots \cap D_{\mu\alpha_n}) \setminus S_{\mu n}.$$

If $n = 1$, then clearly $T = \emptyset$, since $D_{\mu\alpha_1} \subseteq S_{\mu 1}$. Therefore, assume that $n > 1$.

Claim. $T \subseteq A_{\alpha_1, n-1}^\beta \cap \dots \cap A_{\alpha_n, n-1}^\beta$.

Take $x \in D_{\mu\alpha_1} \cap \dots \cap D_{\mu\alpha_n}$ and assume that

$$x \in \bigcap_{1 \leq i \leq n} A_{\alpha_i k_i}^\beta \cap S_{\mu k_i},$$

where $k_{i_0} \geq n$ for some $1 \leq i_0 \leq n$. Since $S_{\mu k_{i_0}} \subseteq S_{\mu n}$ it follows that $x \notin T$. Next suppose that

$$x \in \bigcap_{1 \leq i \leq n} A_{\alpha_i k_i}^\beta \cap S_{\mu k_i}$$

where $k_i < n$ for all $1 \leq i \leq n$. Since $A_{\alpha_i k_i}^\beta \subseteq A_{\alpha_i, n-1}^\beta$ for all $1 \leq i \leq n$, this implies that $x \in \bigcap_{1 \leq i \leq n} A_{\alpha_i, n-1}^\beta$. We therefore conclude that

$$\begin{aligned} T &= (D_{\mu\alpha_1} \cap \dots \cap D_{\mu\alpha_n}) \cap (X \setminus S_{\mu n}) \\ &= \left(\bigcap_{1 \leq i \leq n} \left(\bigcup_{1 \leq k < \omega} A_{\alpha_i k}^\beta \cap S_{\mu k} \right) \cap (X \setminus S_{\mu n}) \right)^- \\ &\subseteq \bigcap_{1 \leq i \leq n} A_{\alpha_i, n-1}^\beta. \end{aligned}$$

This implies that for some $m < \omega$ we have that $T \subseteq \bigcup_{i \leq m} C_i$, since the $\{A_{\alpha_i, n-1}^\beta : 1 \leq i \leq n\}$ are precisely $(n - 1)$ -linked.

Finally, to verify condition (1), observe that $D_{\mu\alpha} \supseteq S_{\mu n} \cap A_{\alpha n}^\beta$ for each $n < \omega$.

Now put $B = \bigcap_{\mu < c} F_\mu$. We claim that B is as required. By (1), $h(F_\mu) = Y$ for all $\mu < c$ and therefore, by compactness and by (3), $h(B) = Y$. By (7), B is a c -OK set of Z and it therefore suffices to prove that $h \upharpoonright B$ is irreducible. If $B' \subseteq B$ is a proper closed set, then for some even $\mu < c$, $B' \subseteq Z_\mu$ and $B \setminus Z_\mu \neq \emptyset$. Then, by (6), $h(F_{\mu+1} \cap Z_\mu) \neq Y$. Since $B' \subseteq F_{\mu+1} \cap Z_\mu$, we conclude that $h(B') \neq Y$, consequently, $h \upharpoonright B$ is irreducible. \square

4.2. A compactification of ω

We will show that there is a compactification $\gamma\omega$ of ω such that $\gamma\omega \setminus \omega$ is not separable but yet satisfies the countable chain condition. This compactification we need in the next section to construct several special points in ω^* .

Let $P = \{f \in \omega^\omega : 0 \leq f(n) \leq n + 1 \text{ for each } n < \omega\}$ and $N = \{f \upharpoonright n : f \in P \text{ and } n < \omega\}$. Define $T = \{\pi \in N^\omega : \text{dom}(\pi(n)) = n + 1 \text{ for each } n < \omega\}$. For each $s \in N$, let $C_s = \{t \in N : s \subseteq t\}$ and for each $\pi \in T$ put

$$C_\pi = \bigcup_{n < \omega} C_{\pi(n)}.$$

Observe that $N \setminus C_\pi$ is infinite for each π . Let \mathcal{B} be the smallest Boolean subalgebra of $\mathcal{P}(N)$ containing $\mathcal{A} = \{C_\pi : \pi \in T\} \cup \{N \setminus C_\pi : \pi \in T\}$. Notice that $\{\{s\} : s \in N\} \cup \{C_s : s \in N\} \subseteq \mathcal{B}$. Let $\gamma\omega$ denote the Stone space of \mathcal{B} . Clearly, $\gamma\omega$ is a compactification of the countable discrete space $\{\{B \in \mathcal{B} : s \in B\} : s \in N\}$ which we identify with ω . Put $X = \gamma\omega \setminus \omega$.

4.2.1. LEMMA. *X is not separable.*

PROOF. Let $\{p_n : n < \omega\}$ be countably many free ultrafilters on \mathcal{B} . For each $n < \omega$, there exists $\pi(n)$ with $\text{dom}(\pi(n)) = n + 1$ such that $C_{\pi(n)} \in p_n$. Simply observe that $N = \{s \in N : \text{dom}(s) \leq n\} \cup \{C_s : \text{dom}(s) = n + 1\}$ for each $n < \omega$. Consequently, $\{p \in X : N \setminus C_\pi \in p\}$ is a nonempty open set of X disjoint from $\{p_n : n < \omega\}$. \square

A family of sets is called *linked* provided that each subfamily of cardinality at most 2 has nonempty intersection. Call a family of sets σ -*linked* provided that it is the union of countably many linked subfamilies. It is obvious that a space having a σ -linked base is ccc.

4.2.2. LEMMA. *X has a σ -linked base.*

PROOF. It suffices to show that $\{B \in \mathcal{B} : |B| = \omega\} = \bigcup_{n \in \omega} \mathcal{B}_n$ such that for each n every two members of \mathcal{B}_n have infinite intersection. To this end, for each $j \in \omega$ and for each $s \in N$ with $2j - 1 \leq \text{dom } s$, define

$$\mathcal{B}(j, s) = \left\{ B \in \mathcal{B} : \exists K \in [T]^{<\omega} \text{ and } L \in [T]^j \text{ with } s \in \bigcap_{\pi \in K} C_\pi \cap \bigcap_{\pi \in L} N \setminus C_\pi \in [B]^\omega \right\}.$$

Since for each $B \in \mathcal{B}$ with $|B| = \omega$, there exists a set D which is a finite intersection of elements of \mathcal{A} , with $D \in [B]^\omega$ and since any infinite subset of N contains elements of arbitrarily large domain, it follows that

$$\{B \in \mathcal{B} : |B| = \omega\} = \bigcup \{ \mathcal{B}(j, s) : j \in \omega, s \in N, \text{ and } 2j - 1 \leq \text{dom } s \}.$$

Fix an index j and $s \in N$ with $2j - 1 \leq \text{dom } s$. If $\{B_0, B_1\} \subseteq \mathcal{B}(j, s)$, then there exist

$K_i \in [T]^{<\omega}$ and $L_i \in [T]^j$ such that for each $i < 2$,

$$s \in D_i = \bigcap_{\pi \in K_i} C_\pi \cap \bigcap_{\pi \in L_i} N \setminus C_\pi \in [B_i]^\omega.$$

We now define, by induction on $\text{dom } s \leq n$, an $h \in P$ such that

$$\{h \upharpoonright n : \text{dom } s \leq n\} \subseteq D_0 \cap D_1.$$

Stage $\text{dom } s$: Let $h \upharpoonright \text{dom } s = s$. Then $h \upharpoonright \text{dom } s \in D_0 \cap D_1$. Assume we have defined $h \upharpoonright n$ for some $\text{dom } s \leq n$ such that $h \upharpoonright n \in D_0 \cap D_1$.

Stage $n+1$: Define $h \upharpoonright n+1$ to be some sequence in N of domain $n+1$ that extends $h \upharpoonright n$ and such that $h \upharpoonright n+1 \notin \{\pi(n) : \pi \in L_0 \cup L_1\}$. This is possible because there are $n+2$ sequences in N of domain $n+1$ that extend $h \upharpoonright n$ and $|L_0 \cup L_1| \leq 2j < \text{dom } s + 2 \leq n+2$. Then $h \upharpoonright n+1 \in D_0 \cap D_1$. \square

4.3. Weak P -points in ω^*

In this section we will show that ω^* contains at least two types of weak P -points. Let X be a space. A subset $F \subseteq X$ is called a *weak P -set* provided that $F \cap \bar{D} = \emptyset$ for any countable $D \subseteq X \setminus F$.

4.3.1. LEMMA. *Let X be a space and let $S \subseteq X$. Then*

- (a) *if S is ω_1 -OK then S is a weak P -set of X , and*
- (b) *if S is κ -OK, where $\text{cf}(\kappa) \geq \omega_1$, and S is not a P -set, then $c(X) \geq \kappa$.*

PROOF. Let $F = \{t_n : n < \omega\} \subseteq X \setminus S$ be any sequence. Since S is ω_1 -OK, we can find a collection $\{U_\xi : \xi < \omega_1\}$ of neighborhoods of S such that for all $\xi_1 < \xi_2 < \dots < \xi_n < \omega_1$, we have that

$$(*) \quad \bigcap_{1 \leq i \leq n} U_{\xi_i} \subseteq X \setminus \{t_n\}.$$

If $U_\xi \cap F \neq \emptyset$ for all $\xi < \omega_1$, then there are an uncountable $A \subseteq \omega_1$ and an $n < \omega$ such that $t_n \in \bigcap_{\xi \in A} U_\xi$. But this obviously contradicts (*). For (b), let $F_n \subseteq \bar{F}_n \subseteq X \setminus S$ ($n < \omega$) be a sequence of open sets in X such that

$$S \cap \left(\overline{\bigcup_{n < \omega} F_n} \setminus \bigcup_{n < \omega} \bar{F}_n \right) \neq \emptyset.$$

Choose a family $\{U_\xi : \xi < \kappa\}$ of neighborhoods of S such that for all $\xi_1 < \xi_2 < \dots < \xi_n < \kappa$, $\bigcap_{1 \leq i \leq n} U_{\xi_i} \subseteq X \setminus \bar{F}_n$. Since $\text{cf}(\kappa) \geq \omega_1$, there have to be a set $A \in [\kappa]^\kappa$

and an $n < \omega$ such that $U_\xi \cap F_n \neq \emptyset$ for all $\xi \in A$. Then

$$\mathcal{B} = \{U_\xi \cap F_n : \xi \in A\}$$

is a family of κ open subsets of X such that any intersection of n of them has empty intersection. By transfinite induction, for each $\xi < \kappa$ we will define a maximal subfamily $\mathcal{G}_\xi \subseteq \mathcal{B}$ such that $\bigcap \mathcal{G}_\xi \neq \emptyset$ and $\mathcal{G}_\xi \neq \mathcal{G}_\eta$ for all $\eta < \xi < \kappa$. If \mathcal{G}_η has been defined for all $\eta < \xi < \kappa$, then take

$$B \in \mathcal{B} \setminus \bigcup_{\eta < \xi} \mathcal{G}_\eta$$

Such a B exists since $|\mathcal{G}_\eta| \leq n - 1$ for all $\eta < \xi$. Then let \mathcal{G}_ξ be any maximal subfamily of \mathcal{B} which contains B and has nonempty intersection. The family $\{\bigcap \mathcal{G}_\xi : \xi < \kappa\}$ consists of κ pairwise disjoint nonempty open subsets of X , whence $c(X) \geq \kappa$. \square

4.3.2. COROLLARY. *Let $x \in X$ be ω_1 -OK. Then x is a weak P -point of X .*

We now come to the main result of this section.

4.3.3. THEOREM. *Let $A = \{x \in \omega^* : x \text{ is } c\text{-OK}\}$ and $B = \{x \in \omega^* : x \text{ is a weak } P\text{-point and } x \in \bar{C} \setminus C \text{ for some } C \subseteq \omega^* \text{ satisfying the ccc}\}$. Then $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$.*

PROOF. That $A \neq \emptyset$ follows directly from Theorem 4.1.4 and that $A \cap B = \emptyset$ is a consequence of Lemma 4.3.1(b). It remains to show that $B \neq \emptyset$. To this end, let X be the ccc nonseparable remainder of ω constructed in section 4.2. Observe that the one point compactification $a(\omega \times X)$ of $\omega \times X$, is a continuous image of ω^* , whence, by Theorem 4.1.5, there is a closed c -OK set $Z \subseteq \omega^*$ and irreducible surjection $f : Z \rightarrow a(\omega \times X)$. For each $n < \omega$, put $Z_n = f^{-1}(\{n\} \times X)$. Observe that by irreducibility of f , we have that $(\bigcup_{n < \omega} Z_n)^- = Z$. Let $\pi : \omega \times X \rightarrow X$ be the projection and let \mathcal{U} be a maximal disjoint family of separable clopen subsets of X . Since X is ccc but not separable, $(\bigcup \mathcal{U})^- \neq X$ and consequently we can pick a nonempty clopen $C \subseteq X$ which misses $\bigcup \mathcal{U}$. Observe that C is nowhere separable. For each countable $D \subseteq \bigcup_{n < \omega} Z_n$, let $\{C_n(D) : n < \omega\}$ be a maximal disjoint family of nonempty clopen subsets of C such that

$$\bigcup_{n < \omega} C_n(D) \cup \overline{(\pi(f(D)))} \cap C = \emptyset$$

and define

$$F(D) = \bigcup_{n < \omega} f^{-1}\left(\bigcup_{i \leq n} \{i\} \times C_i(D)\right).$$

Define $F = \bigcap \{ \overline{F(D)} : D \in [f^{-1}(\omega \times X)]^{\leq \omega} \}$. It is easily seen that F is nice w.r.t. $\{f^{-1}(\{n\} \times X) : n < \omega\}$. Since X is an F -space, being closed in ω^* , and since, by irreducibility of f , Z is ccc, we may conclude that Z is extremally disconnected (Lemma 1.2.2). By Theorem 4.1.4, there is a point $x \in S = Z \setminus \bigcup_{n < \omega} Z_n$ which belongs to F and which is a c -OK point of S .

Claim. If $D \in [\bigcup_{n < \omega} Z_n]^{\leq \omega}$, then $x \notin \overline{D}$.

By construction, $F(D)$ is a clopen subspace of $\bigcup_{n < \omega} Z_n$ which misses D . Since Z is extremally disconnected,

$$\overline{F(D)} \cap \left(\overline{\bigcup_{n < \omega} Z_n \setminus F(D)} \right) = \emptyset.$$

Since $x \in \overline{F(D)}$, we conclude that $x \notin \overline{D}$.

The claim now can be used to prove quite easily that x is a weak P -point of ω^* . \square

4.3.4. COROLLARY. ω^* contains weak P -points.

Notice that Theorem 4.3.3 proves once again that ω^* is not homogeneous. This proof is totally different from the previous ones, since we found an easy to state topological property shared by some but not all points in ω^* .

A space X is called *first order homogeneous* provided that no property which can be expressed in first order language distinguishes points of X . It is clear that any homogeneous space is first order homogeneous, but not conversely. It was shown by VAN DOUWEN & VAN MILL [1981a] that c -OK points can be used to show that ω^* is not first order homogeneous.

4.4. Some other points of interest

In this section we will continue our search for 'special' points in ω^* . In Theorem 4.3.3 we constructed two types of weak P -points in ω^* . It is natural to ask whether every point $x \in \omega^*$ which is a limit point of some countable subset of $\omega^* \setminus \{x\}$ is also a limit point of some countable discrete subspace of $\omega^* \setminus \{x\}$. Our first result is that the answer to this question is in the negative.

4.4.1. THEOREM. *There is a point $x \in \omega^*$ such that x is a limit point of some countable subset of $\omega^* \setminus \{x\}$, but not of any countable discrete subset of $\omega^* \setminus \{x\}$.*

PROOF. It is clear that the one point compactification S of $\omega \times [0, 1]$ is a continuous image of ω^* . Therefore, by Theorem 4.1.4, there is a c -OK set $T \subseteq \omega^*$ which can be mapped by an irreducible map, say f , onto S . For all $n < \omega$, put $T_n = f^{-1}(\{n\} \times [0, 1])$.

Claim. For each $n \geq 1$ there is a family \mathcal{F}_n of closed subsets of $[0, 1]$ such that

- (1) \mathcal{F}_n has the n -intersection property,
- (2) if $D \subseteq [0, 1]$ is nowhere dense, then there is an $F \in \mathcal{F}_n$ with $F \cap D = \emptyset$.

This was shown in Lemma 1.9.2.

For each $n < \omega$, let $\mathcal{G}_n = \{f^{-1}(\{n\} \times F) : F \in \mathcal{F}_{n+1}\}$. Observe that G_n has the $(n + 1)$ -intersection property and that for each nowhere dense set $D \subseteq T_n$ there is a $G \in \mathcal{G}_n$ with $G \cap D = \emptyset$. (This is obvious since $f(D)$ is nowhere dense in $\{n\} \times [0, 1]$.) Put

$$G = \bigcap \{ \bar{A} : A \subseteq f^{-1}(\omega \times [0, 1]) \text{ and } A \cap T_n \in \mathcal{G}_n \text{ for all } n < \omega \}.$$

It is easily seen that $G \subseteq T \setminus \bigcup_{n < \omega} T_n$ and that G is nice w.r.t. the sequence $\{T_n : n < \omega\}$.

By similar arguments as in the proof of Theorem 4.3.3 we may conclude that T is extremally disconnected and by an appeal to Theorem 4.1.5 we can find a point $x \in G$ which is a c -OK point of $T \setminus \bigcup_{n < \omega} T_n$. It is easily seen that the point x is as required. The details of checking this out are left to the reader. \square

Let us now pose a rather innocent question. Does every point in ω^* have character c in ω^* ? Under CH, this is obviously true. However, under \neg CH there can be points in ω^* which have character ω_1 , KUNEN [1972]. The following now directly comes to mind: can all points in ω^* be of character less than c ? The answer to this question is in the negative. In Lemma 3.3.4 we showed that there is an R -point in ω^* , and R -points obviously have character c in ω^* . Let us give a somewhat easier proof than the one in Lemma 3.3.4, that points of character c in ω^* exist.

4.4.2. THEOREM. *There is a point $x \in \omega^*$ such that $\chi(x, \omega^*) = c$.*

PROOF. Let $\{(A_\alpha^0, A_\alpha^1) : \alpha < c\}$ be a family of pairs of disjoint subsets of ω^* such as in Lemma 3.1.2(b). Take any point x in the intersection

$$\bigcap \{ \bar{A}_\alpha^0 \cap \omega^* : \alpha < c \} \cap \bigcap \left\{ C : C \subseteq \omega^* \text{ is clopen and } \exists D \in [c]^\omega \right. \\ \left. \text{such that } \omega^* \setminus C \subseteq \bigcap_{\alpha \in D} \bar{A}_\alpha^0 \right\}.$$

By using similar arguments as in the proof of Theorem 2.1.1, it follows that x has character c in ω^* . \square

In the proof of Theorem 2.1.1, we constructed a Parovičenko space T such that $\pi(x, T) = c$ for all $x \in T$. In view of the above result, is therefore quite natural to ask whether the above result can be strengthened to the statement that there is a

point $x \in \omega^*$ such that $\pi(x, \omega^*) = c$. This is impossible. BELL & KUNEN [1980] show it to be consistent with $c = \omega_{\omega_1}$ that each point $x \in \omega^*$ has π -character $\omega_1(<c)$. However, the following is true in ZFC.

4.4.3. THEOREM. *There is a point $x \in \omega^*$ with $\pi(x, \omega^*) \geq \text{cf}(c)$.*

PROOF. Let $\{A_\beta^\alpha : \alpha, \beta < c\}$ be a c by c independent matrix of clopen subsets of ω^* (Lemma 3.3.2) and let $\{C_\alpha : \alpha < c\}$ enumerate the family of all clopen subsets of ω^* . For all $\alpha, \kappa < c$ there is at most one $\beta < c$ such that $C_\kappa \subseteq A_\beta^\alpha$. It is therefore easy to pick for each $\alpha < c$ an element $f(\alpha) < c$ such that for all $\beta < \alpha$,

$$C_\beta \not\subseteq A_{f(\alpha)}^\alpha.$$

Take any point $x \in \bigcap_{\alpha < c} A_{f(\alpha)}^\alpha$. We claim that $\pi(x, \omega^*) \geq \text{cf}(c)$. If $\pi(x, \omega^*) < \text{cf}(c)$, then there is an $\alpha < c$ such that the family $\{C_\beta : \beta < \alpha\}$ constitutes a π -basis for x . But $A_{f(\alpha)}^\alpha$ is a neighborhood of x which does not contain any C_β for all $\beta < \alpha$. \square

Since ω^* is an F -space, each countable subspace of ω^* is C^* -embedded in ω^* (Theorem 1.5.2). If $2^{\omega_1} = c$, then $\beta\omega_1$ can be embedded in ω^* since $\beta\omega_1$ is extremally disconnected and has weight c (Theorem 1.4.7). Therefore, under $2^{\omega_1} = c$, ω^* contains subspaces of cardinality ω_1 that are C^* -embedded. Having this in mind, it is quite natural to ask whether all subspaces of ω^* of cardinality ω_1 can be C^* -embedded. We will show that this is not the case. As usual, a P -space is a space in which all G_δ 's are open.

4.4.4. THEOREM. *Let X be a P -space of weight at most c . Then X can be embedded in ω^* .*

PROOF. We may assume that $X \subseteq 2^c$ (here 2^c denotes the Cantor cube of weight c). Take $p \in 2^c$. The map $g_p : 2^c \rightarrow 2^c$ defined by $g_p(x) = x + p$ lifts to a map $eg_p : E(2^c) \rightarrow E(2^c)$ ($E(2^c)$ is the projective cover of 2^c , see Section 0). The homeomorphism eg_p will be called h_p for short. Let π be the canonical irreducible surjection from $E(2^c)$ onto 2^c , i.e. π is defined by

$$\{\pi(u)\} = \bigcap \{\bar{U} : U \in u\}.$$

Take a point $u_0 \in \pi^{-1}(\mathbf{0})$, where $\mathbf{0}$ denotes the identity of 2^c . If $p \in X$, let $u_p = h_p(u_0)$. Observe that

$$\pi(u_p) = \pi(h_p(u_0)) = g_p(\pi(u_0)) = g_p(\mathbf{0}) = p,$$

whence $u_p \in \pi^{-1}(p)$.

Let U be a regular open subset of 2^c . We can find a countable subset $D \subseteq c$ and

a regular open subset $U' \subseteq 2^D$ such that if $\pi_D: 2^c \rightarrow 2^D$ denotes the projection, then $\pi_D^{-1}(U') = U$ (uses the fact that 2^c is ccc, JUHÁSZ [1980]).

Claim. If $p \upharpoonright D = q \upharpoonright D$ for $p, q \in X$, then $U \in u_p$ iff $U \in u_q$.

Indeed, simply observe that $U \in u_p$ iff $U + p \in u_0$ iff $U + p + q \in u_q$ iff $U \in u_q$.

Now, let $P = \{u_p : p \in X\}$. We claim that $\pi \upharpoonright P : P \rightarrow X$ is a homeomorphism. For convenience, put $f = \pi \upharpoonright P$. Then f is clearly one to one, onto and continuous. It therefore suffices to show that f is open. Let U be a regular open subspace of 2^c and let U' and D be as above. The set $\tilde{U} = \{u \in E(2^c) : U \in u\}$ is a basic open subset of $E(2^c)$, so we only need to show that $f(\tilde{U})$ is open in X . Take $p \in f(\tilde{U})$ arbitrarily. Define $Z = \{q \in X : p \upharpoonright D = q \upharpoonright D\}$. By the claim, $Z \subseteq f(\tilde{U})$. Observe that $Z = \pi_D^{-1}(p \upharpoonright D) \cap X$, whence Z is a G_δ in X . Since G_δ 's in X are open, and since $p \in Z$, we conclude that $f(\tilde{U})$ is a neighborhood of p .

We conclude that X can be embedded in $E(2^c)$. Since $E(2^c)$ is separable, it has weight c , and therefore, by Theorem 1.4.7, it embeds in $\beta\omega$. Since $\beta\omega$ embeds in ω^* , we are done. \square

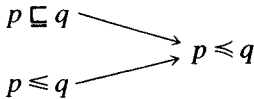
4.4.5. COROLLARY. *There is a point $x \in \omega^*$ and a (relatively) discrete sequence $\{x_\alpha : \alpha < \omega_1\} \subseteq \omega^* \setminus \{x\}$, such that each neighborhood of x contains all but countably many of the x_α 's.*

PROOF. There is clearly a P -space of cardinality ω_1 and containing precisely one nonisolated point. Now apply Theorem 4.4.4. \square

4.4.6. REMARK. Observe that the proof of Theorem 4.4.4 actually shows that if X is a P -space, then βX can be embedded in the Čech–Stone compactification of some discrete space.

4.5. Partial orderings on $\beta\omega$, II

In Section 3 we defined three ‘partial’ orders on $\beta\omega$, namely \sqsubseteq , \sqsubset and \leq . We observed that the following relations hold:



(see Lemma 3.2.1 and the definition of \leq). In 3.3.8 we showed that there are points $p, q \in \omega^*$ with $p \leq q$ but p and q are \leq -incomparable. We begin by establishing a similar result for the order \sqsubseteq .

4.5.1. THEOREM. *There is a finite to one function $\pi : \omega \rightarrow \omega$ such that for all $x \in \omega^*$ there is a c -OK point $y \in \omega^*$ with $\bar{\pi}(y) = x$.*

PROOF. We will be brief. Let $\pi : \omega \rightarrow \omega$ be the finite to one function of Lemma

4.1.2 and let $\mathcal{A} = \{A_{\alpha n}^\beta : \alpha < c, 1 \leq n < \omega, \beta < c\}$ be the c by c independent linked family of clopen subsets of $\beta\omega$ w.r.t. $\langle \omega^*, \bar{\pi} \rangle$ given by Lemma 4.1.2. Take $x \in \omega^*$ arbitrarily. Since \mathcal{A} is independent w.r.t. $\langle \omega^*, \bar{\pi} \rangle$, \mathcal{A} is also an independent linked family w.r.t. $\langle \bar{f}^{-1}(x), g \rangle$, where $g : \omega^* \rightarrow \{0\}$ maps ω^* onto 0. By using precisely the same technique as in the proof of Theorem 4.1.4, we can construct a c -OK point $y \in \omega^*$ which belongs to $\bar{f}^{-1}(x)$. \square

4.5.2. COROLLARY. ω^* contains 2^c c -OK points.

PROOF. This is clear since $|\omega^*| = 2^c$, Lemma 31.2(c). \square

4.5.3. COROLLARY. There are points $p, q \in \omega^*$ with $p \leq q$, and consequently $p \sqsubseteq q$, but p and q are \sqsubseteq -incomparable.

PROOF. By Theorem 4.3.3, there is a c -OK point $p \in \omega^*$. An application of Theorem 4.5.1 gives us a c -OK point $q \in \omega^*$ with $p \leq q$. Since p and q are weak P -points (Corollary 4.3.2), p and q are obviously \sqsubseteq -incomparable. \square

Since $p \leq q$ whenever $p \sqsubseteq q$, the question naturally arises whether $p \sqsubseteq q$ implies that $p \leq q$. We will show that this is not the case.

4.5.4. THEOREM. There are points $p, q \in \omega^*$ with $p \sqsubseteq q$, and consequently $p \leq q$, but p and q are \leq -incomparable.

PROOF. Let $\{C_n : n < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subsets of ω^* . For each $n < \omega$, let $\{E_m^n : m < \omega\}$ be a sequence of pairwise disjoint nonempty clopen subspaces of C_n . In addition, let $p \in \omega^*$ be an arbitrarily chosen R -point (Lemma 3.3.4). Let $G = \{f \in \omega^\omega : f \text{ is finite to one}\}$. For all $f \in G$ put $A_f = \bar{f}^{-1}(\{p\})$. By Lemma 3.3.6, each A_f is an R -set of ω^* . By Theorem 3.3.5 for all $n, m < \omega$ we can pick a point $x_m^n \in E_m^n$ such that

$$\{x_m^n : n, m < \omega\}^- \cap \bigcup_{f \in G} A_f = \emptyset.$$

For each $n < \omega$, put $Z_n = \{x_m^n : n, m < \omega\}^-$. Observe that $Z_n \approx \beta\omega$ since ω^* is an F -space and that consequently $|Z_n| = 2^c$, Lemma 3.1.2(c). For each $n < \omega$, let $\langle q_\alpha^n : \alpha < 2^c \rangle$ enumerate Z_n . We choose the enumeration to be most economical, i.e. each point of Z_n occurs precisely once in the sequence $\langle q_\alpha^n : \alpha < 2^c \rangle$. Observe that this implies that if $\alpha < \beta < 2^c$, then

$$(*) \quad \{q_\alpha^n : n < \omega\}^- \cap \{q_\beta^n : n < \omega\}^- = \emptyset$$

(use that $\bigcup_{n < \omega} C_n$ is C^* -embedded in ω^* , Theorem 1.5.2). For each $\alpha < 2^c$, define

$g_\alpha : \omega \rightarrow \omega^*$ by

$$g_\alpha(n) = q_\alpha^n.$$

Then (*) implies that the set $\{\beta g_\alpha(p) : \alpha < 2^c\}$ has cardinality 2^c . Since $|\{x \in \omega^* : x \leq p\}| \leq c$, we can therefore find an $\alpha < 2^c$ such that $q = \beta g_\alpha(p) \notin \{x \in \omega^* : x \leq p\}$. Then q is as required. \square

Notes for Section 4

The notion of a κ -OK point is due to KUNEN [1978]. Theorem 4.1.5 for the special case $X = \beta\omega$, $Z = \omega^*$, $A = \omega^*$ and $Y = \{0\}$ is due to KUNEN [1978]. Theorem 4.1.5 is implicit in VAN MILL [1981b] and was subsequently partly generalized in VAN MILL [1982]. The ccc nowhere separable remainder of ω described in Section 4.2 is due to BELL [1981]. Interestingly, this compactification is also an important step in the proof of the main result of VAN MILL [1982]. Lemma 4.3.1 is due to KUNEN [1978]. That the set A of Theorem 4.3.3 is nonempty is due to Kunen and that the set B of Theorem 4.3.3 is nonempty is due to VAN MILL [1981b]. Corollary 4.3.4 is due to KUNEN [1978] and for generalizations see VAN MILL [1979a], [1981b], [1982] and DOW [1982]. Theorem 4.4.1 is due to VAN MILL [1981b] and Theorem 4.4.2 to POSPIŠIL [1939]. The proof of Theorem 4.4.2 presented here was taken from KUNEN [1974]. Theorem 4.4.3 is due to BELL & KUNEN [1981]. Corollary 4.4.5 is due independently to BALCAR, SIMON & VOJTÁS [1981], KUNEN and SHELAH. Theorem 4.4.4 is due to VAN DOUWEN (unpublished), but the proof presented here is due to DOW & VAN MILL [1982]. Our proof of Theorem 4.4.4 differs from van Douwen's proof, but both proofs have in common that they are based on the technique of Balcar, Simon, Vojtás, Kunen and Shelah. All other results in this chapter are new.

We have seen that there are many 'special' points in ω^* . In VAN MILL [1981b] it is shown that there are at least 16 definable types in ω^* . Call a space π -homogeneous provided that all nonempty open subspaces have the same π -weight. Define

$$A_1 = \{x \in \omega^* : \exists \text{ countable discrete } D \subseteq \omega^* \setminus \{x\} \text{ with } x \in \bar{D}\}.$$

$$A_2 = \{x \in \omega^* : \exists \text{ countable, dense in itself, } \pi\text{-homogeneous subset } D \subseteq \omega^* \setminus \{x\} \text{ of countable } \pi\text{-weight such that } x \in \bar{D}\},$$

$$A_3 = \{x \in \omega^* : \exists \text{ countable, dense in itself, } \pi\text{-homogeneous subset } D \subseteq \omega^* \setminus \{x\} \text{ of } \pi\text{-weight } \omega_1 \text{ such that } x \in \bar{D}\}.$$

$$A_4 = \{x \in \omega^* : \exists \text{ locally compact, ccc, nowhere separable } D \subseteq \omega^* \setminus \{x\} \text{ with } x \in \bar{D}\}.$$

By using similar ideas as developed in this section it can be shown that for all subsets $F \subseteq \{1, 2, 3, 4\}$ the set

$$\bigcap_{i \in F} A_i \setminus \bigcup_{i \notin F} A_i$$

is nonempty. This gives 16 definable types of points in ω^* . For details, see VAN MILL [1981b].

5. Remarks

The reader will undoubtedly have noticed that we did not discuss several important facts about $\beta\omega$. For example, we did not say anything about normality in $\beta\omega$. There are several simple proofs that for any $p \in \omega^*$, the spaces $\beta\omega \setminus \{p\}$ and $\omega^* \setminus \{p\}$ are not normal under CH. However, for years there has not been made significant progress in this area of $\beta\omega$. We don't know that $\omega^* \setminus \{p\}$ is not normal for any $p \in \omega^*$ without the help of some set theoretic hypothesis. It is known however, that for some $x \in \omega^*$ the space $\omega^* \setminus \{x\}$ is not normal. The best result of this type is, as far as I know, due to BŁASZCZYK & SZYMAŃSKI [1980a]. They showed that if $x \in \omega^*$ is a limit point of some countable discrete subset of ω^* , then $\omega^* \setminus \{x\}$ is not normal.

What else is there to say about $\beta\omega$? Consider the following question: is $\beta\omega$ homeomorphic to $(\beta\omega)^2$? The answer is of course: NO! It is easy to see that $(\beta\omega)^2$ is not extremally disconnected. Make the question a little bit less trivial: is $(\beta\omega)^2$ homeomorphic to $(\beta\omega)^3$? This question is easy to state, but the answer to the question is not simple at all. VAN DOUWEN [1982] showed that $(\beta\omega)^n \approx (\beta\omega)^m$ iff $n = m$, for all $n, m \geq 1$. The list of interesting results about $\beta\omega$ seems endless.

Let X be a space which is dense in itself. Define

$$n(X) = \min\{\kappa : X \text{ can be covered by } \kappa \text{ nowhere dense sets}\}.$$

This number is called the Nořak number of X . It is clear that if $n(\omega^*) > c$, then ω^* contains P -points. Therefore, in Shelah's model in which there are no P -points, $n(\omega^*) \leq c$. Observe that $n(\omega^*) \geq \omega_2$. The Nořak number $n(\omega^*)$ can be almost anything you want, for details, see BALCAR, PELANT & SIMON [1980] and also HECHLER [1978].

In this paper we have restricted our attention to the space $\beta\omega$. The reader should however realize that many of the results we obtained can be generalized to higher cardinals with proofs that are essentially identical. For example the Rudin-Keiler order can without any problem be defined for higher cardinals and the proof we gave for the existence of \leq -incomparable points in $\beta\omega$ can be copied

to prove without extra difficulty that there are \leq -incomparable uniform ultrafilters on any infinite cardinal κ . However, there are also results that exclusively only work for ω . For example, the Rudin–Frolík order on ω^* cannot even be defined for higher cardinals.

Open problems

The following problems are unsolved as far as I know. It is recognized that a few of the problems listed below may be inadequately worded, be trivial or be known. Of many of the problems it is unknown who asked the problem. For that reason we do not credit anybody for posing a certain problem. The following mathematicians (with addresses listed in the AMS-MAA Combined Membership List) are sources of continuing information on many of the problems and their background: B. Balcar, W.W. Comfort, E.K. van Douwen, N. Hindman, K. Kunen, J. van Mill, M.E. Rudin and R.G. Woods.

1. Is ω^* homeomorphic to ω_1^* ? (No if MA.)
2. Are there points $p, q \in \omega^*$ such that if $f: \omega \rightarrow \omega$ is any finite to one map, then $\beta f(p) \neq \beta f(q)$? (Yes if MA.)
3. Are $\omega^* \setminus \{p\}$ and $\beta\omega \setminus \{p\}$ nonnormal for any $p \in \omega^*$? (Yes if MA.)
4. Is there a model in which there are no P -points and no Q -points in ω^* ?
5. Is there a model in which every point in ω^* is an R -point?
6. Is there a ccc closed P -set in ω^* ? (Yes if CH.)
7. Let X be a compact space that can be mapped onto ω^* . Is X non-homogeneous? (Yes if X has weight at most c .)
8. Is the autohomeomorphism group of ω^* algebraically simple? (Yes is consistent.)
9. Is there an extremally disconnected, normal, locally compact space that is not paracompact? (Yes if MA + \neg CH or if there is a weakly compact cardinal.)
10. Is every first countable compactum a continuous image of ω^* ?
11. Which spaces can be embedded in $\beta\omega$?
12. Is there a separable closed subspace of ω^* which is not a retract of $\beta\omega$? (Yes if CH.)
13. Let $(*)$ denote the statement that every Parovičenko space is coabsolute with ω^* . Is $(*)$ equivalent to CH? (It is known that $c < 2^{\omega_1}$ implies $\neg(*)$.)

14. Let X be the Stone space of the reduced measure algebra of $[0, 1]$. Is it consistent that X is not a continuous image of ω^* ?
15. Let $(**)$ denote the statement that every compact zero-dimensional F -space of weight \mathfrak{c} can be embedded in ω^* . Is $(**)$ equivalent to CH? (It is known that CH implies $(**)$ but $\text{MA} + \mathfrak{c} = \omega_2$ implies $\neg(**)$.)
16. Is it consistent that there is a compact basically disconnected space of weight \mathfrak{c} that cannot be embedded in $\beta\omega$? (Such an example cannot be the Čech-Stone compactification of a P -space.)
17. Is there a $p \in \omega^*$ such that $\omega^* \setminus \{p\}$ is not C^* -embedded in ω^* ? (Yes if CH.)
18. Assume MA. Are there $P_{\mathfrak{c}}$ -points $p, q \in \omega^*$ which are not of the same type, i.e. for which $h(p) \neq q$ for any autohomeomorphism h of ω^* ?
19. Is it true that for all $p \in \omega^*$ there is a $q \in \omega^*$ such that p and q are \leq -incomparable?
20. Is every subspace of ω^* strongly zero-dimensional?
21. Is there a point $p \in \omega^*$ such that every compactification of $\omega \cup \{p\}$ contains a copy of $\beta\omega$?
22. Is there a point $p \in \omega^*$ for which there is a compactification of $\omega \cup \{p\}$ that does not contain a copy of $\beta\omega$? (Yes if MA.)
23. Let D be any nowhere dense subset of ω^* . Is D a \mathfrak{c} -set, i.e. is there a disjoint family \mathcal{A} of \mathfrak{c} open sets in ω^* such that $D \subseteq \bar{A}$ for all $A \in \mathcal{A}$? (Yes if $|D| = 1$.)
24. Is there a point $p \in \omega^*$ such that if $f: \omega \rightarrow \omega$ is any map, then either $\beta f(p) \in \omega$ or $\beta f(p)$ has character \mathfrak{c} in $\beta\omega$? (Yes if MA.)

Remarks added in August 1982. Murray Bell has constructed a consistent example of a compact space X of weight \mathfrak{c} which is first countable in all but one point and which in addition is not a continuous image of ω^* . This gives a partial answer to Question 10. Alan Dow showed that if $\text{cf}(\mathfrak{c}) = \omega_1$, then all Parovičenko spaces are coabsolute. This solves Question 13 in the negative.

Remark added in May 1983. Andrzej Szymański has recently constructed, under MA, a separable closed subspace of ω^* which is not a retract of $\beta\omega$ (this concerns Question 12).

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