Topology and its Applications 15 (1983) 59–64
North-Holland Publishing Company

A COMPACT F-SPACE NOT CO-ABSOLUTE WITH $\beta\mathbb{N}-\mathbb{N}$

Jan VAN MILL
Vrije Universiteit, De Boelelaan 1007, Amsterdam, The Netherlands

Scott W. WILLIAMS
State University of New York at Buffalo, Buffalo, NY 14214, USA

Received 28 August 1980
Revised 15 May 1981 and 27 April 1982

We show that if every Parovičenko space of weight $\mathfrak{c}$ is co-absolute with $\beta\mathbb{N}-\mathbb{N}$, then $\mathfrak{c}<2^{\aleph_1}$.

AMS(MOS) Subj. Class. (1980): 54G05, 54D40, 54A35, 06E99

F-space absolute
$\beta\mathbb{N}-\mathbb{N}$ Boolean algebra Continuum Hypothesis

1. Introduction

It will be convenient to call a space X a *Parovičenko* space if
(α) X is a zero-dimensional compact space without isolated points,
(β) every two disjoint open F_σ-sets have disjoint closures, and
(γ) every nonempty G_δ-set in X has non-empty interior.
Compact spaces satisfying (β) are usually called *F-spaces*, while spaces satisfying (γ) are called *almost-P spaces*. Examples of F-spaces are the extremally disconnected spaces. Examples of almost-P spaces are η_α-sets (and their compactifications). Examples of compact F-almost-P (Parovičenko) spaces are all spaces of the form $X^*=\beta X-X$, where X is a locally compact realcompact (respectively, zero-dimensional) space [6, 7].

It is well-known that under CH, the continuum hypothesis, all Parovičenko spaces of weight $\mathfrak{c}$ are homeomorphic [9]. The converse of this result is true, i.e., if all Parovičenko spaces of weight $\mathfrak{c}$ are homeomorphic, then CH is true [4]. The standard example of a Parovičenko space of weight $\mathfrak{c}$ is $\mathbb{N}^*$, where $\mathbb{N}$ is the discrete space of natural numbers; however, more examples can be produced using spaces of the form $(K\times\mathbb{N})^*$, where K is a compact zero-dimensional space of weight at most $\mathfrak{c}$ (e.g. K equal to the Cantor set or $\mathbb{N}^*$).

The *absolute* (see [10] or [16] for surveys) of a regular space X is the unique (up to homeomorphism) extremally disconnected space $\mathscr{E}(X)$, which can be mapped by a perfect irreducible map onto X. Spaces X and Y are called *co-absolute* when $\mathscr{E}(X)$ and $\mathscr{E}(Y)$ are homeomorphic. R.G. Woods first considered "Parovičenko-like" characterizations of the co-absolute of $\mathbb{N}^*$, [14, 15]. These were recently improved by Broverman and Weiss, [2], who presented

1.1. Theorem. CH *implies every Parovičenko space of π-weight at most $\mathfrak{c}$ is co-absolute with $\mathbb{N}^*$.*

Broverman and Weiss also show that their result is not a theorem in ZFC by proving

1.2. Theorem. *Assume $\aleph_1 < \mathfrak{c}$ and that every Parovičenko space of weight $\mathfrak{c}$ is co-absolute with $\mathbb{N}^*$. Then $\mathfrak{c} < 2^{\mathfrak{c}}$.*

The purpose of this paper is to present a partial converse to 1.1 which is simultaneously an improvement of 1.2. Specifically we establish, through two examples, the following

1.3. Theorem. *Assume that every Parovičenko space of weight $\mathfrak{c}$ is co-absolute with $\mathbb{N}^*$. Then $\mathfrak{c} < 2^{\aleph_1}$.*

2. The first example

Throughout this paper, $\mathfrak{c} = 2^{\aleph_0}$ and $w(X)$ denotes the weight of a space X. The intersection of at most $\aleph_1$ open sets of a space will be called a $G\aleph_1$-set. Observe that a compact zero-dimensional space X possesses precisely $w(X)$ many clopen sets; moreover, since each closed (= compact) subset has a neighborhood base of clopen sets, X has at most $w(X)^{\aleph_1}$ closed $G\aleph_1$-sets.

2.1. Lemma. *Suppose that X is compact. Then there is a Parovičenko space Ω_X and a continuous surjection $\varphi_X : \Omega_X \to X$ satisfying*

(1) *$w(\Omega_X) \leq w(X)^{\aleph_1}$, and*

(2) *if G is a non-empty $G\aleph_1$-set, then $\varphi_X^{-1}(G)$ has non-empty interior in Ω_X.*

Proof. Since each compact space is a continuous image of compact zero-dimensional space of the same weight, we assume, without loss of generality, X is zero-dimensional. For convenience, put $\kappa = w(X)^{\aleph_1}$. From the above observation, we may choose $A \subseteq X$ having cardinality at most κ and intersection each non-empty closed $G\aleph_1$-set of X. Topologizing $Y = (X \times \{0\} \cup A \times \{1\})$ as in the Alexandrov double (see [5, p. 173]) transforms Y into a compact zero-dimensional space of weight at most κ with $A \times \{1\}$ open and discrete. Let $p : Y \to X$ be the natural projection. If G is a non-empty $G\aleph_1$-set of X, then G contains a non-empty closed $G\aleph_1$-set H. By construction, $H \cap A \neq \emptyset$ which implies $p^{-1}(H)$, and hence $p^{-1}(G)$, has non-empty interior in Y.

Define $\Omega_X = (Y \times \mathbb{N}^*)$, let $\pi: Y \times \mathbb{N} \to Y$ be the natural projection, and define $\varphi_X = p \circ (\beta\pi \restriction \Omega_X)$, where $\beta\pi$ is the Stone extension of π to $\beta(Y \times \mathbb{N})$. Since $Y \times \mathbb{N}$ is a σ-compact zero-dimensional space, it is strongly zero-dimensional and has at most $w(Y)^{\aleph_0} = \kappa^{\aleph_0} = \kappa$ clopen sets. Consequently, $\beta(Y \times \mathbb{N})$ is a zero-dimensional space of weight at most κ. By [6] and [7] we may therefore conclude Ω_X and φ_X are as advertised. □

We employ the previous lemma in the construction of the following.

2.2. Example. *There is a Parovičenko space S of weight at most $2^{\aleph_1}$ such that each non-empty $G\aleph_1$-set of S has non-empty interior.*

Proof. By recursion, for each ordinal $\lambda < \omega_2$ we will construct a Parovičenko space S_λ and for each $\nu < \lambda$ a continuous surjection $f_{\lambda\nu}: S_\lambda \to S_\nu$ subject to the restrictions

(1) $w(S_\lambda) \leq 2^{\aleph_1}$.

(2) If G is a non-empty $G\aleph_1$-set, then $f_{\lambda\nu}^{-1}(G)$ has non-empty interior in S_λ.

(3) If $\nu < \mu < \lambda$, then $f_{\mu\nu} \circ f_{\lambda\mu} = f_{\lambda\nu}$.

Let $S_0 = \mathbb{N}^*$ and suppose $\kappa < \omega_2$ is an ordinal for which everything has been constructed for all $\lambda < \kappa$. We put

$$X = \varprojlim(S_\lambda, f_{\lambda\nu}, \kappa) \quad \text{and} \quad g_\nu = \varprojlim(f_{\lambda\nu}, \kappa) \qquad \text{for each } \nu < \kappa.$$

Observe that $w(X) \leq \aleph_1 \cdot 2^{\aleph_1} = 2^{\aleph_1}$ and that X is compact and zero-dimensional. Define $S_\kappa = \Omega_X$ (from Lemma 2.1). In addition, for each $\nu < \kappa$ define $f_{\kappa\nu} = g_\nu \circ \varphi_X$. It is clear that our recursion hypothesis is satisfied.

Now define $S = \varprojlim (S_\lambda, f_{\lambda\nu}, \omega_2)$ and for each $\nu < \omega_2$ define $f_\nu = \varprojlim (f_{\lambda\nu}, \omega_2)$. First observe that if $C \subseteq S$ is clopen, then there is $\lambda < \omega_2$ and a clopen $K \subseteq S_\lambda$ such that $f_\lambda^{-1}(K) = C$. This readily implies that S has all the required properties. For let F, F' be disjoint open F_σ-sets. Applying the observation above (and compact zero-dimensional), we may find λ, $\lambda' < \omega_2$ and open F_σ-sets E and E' of S_λ and $S_{\lambda'}$ respectively, such that $f_\lambda^{-1}(F) = E$ and $f_{\lambda'}^{-1}(F') = E'$. Without loss of generality, $\lambda' \leq \lambda$. Then, $f_{\lambda\lambda'}^{-1}(F')$ and F are two disjoint open F_σ-sets of the Parovičenko space S_λ, and hence, they have disjoint closures in S_λ. Now (3) implies E and E' have disjoint closures in S. So we conclude S is an F-space. Similarly, using (2) (and the fact the inverse system is "increasing" of length $\omega_2 > \omega_1$), the reader can easily check that each non-empty $G\aleph_1$-set of S has non-empty interior. □

3. The second example

Given a space X, the *Novák number*, $n(X)$, is defined (see [1] for studies of $n(\mathbb{N}^*)$) by

$$n(X) = \inf\{\text{cardinals } \kappa : X \text{ can be covered by } \kappa \text{ nowhere dense sets}\}.$$

Analogously to the weak Lindelöf degree of a space X, we define the *weak Nověak number*, $wn(X)$, by

$$wn(X) = \inf\{|\mathscr{D}|: \mathscr{D} \subseteq \mathbb{P}(X), \bigcup\mathscr{D} \text{ is dense in } X, \text{ and each } D \in \mathscr{D} \text{ is nowhere dense in } X\}.$$

Observe that whenever X has no isolated points, the density of X is not smaller than $wn(X)$.

3.1. Example. *There is a Parovičenko space T of weight $\mathfrak{c}$ such that $wn(T) \leq \aleph_1$.*

Proof. Let W be the one-point compactification of the discrete space of cardinality $\aleph_1$, and ∞ be the non-isolated point of W. Let $M = {}^{\omega_1}W$ have the Tychonov product topology and define $T = (\mathbb{N} \times M)^*$. We claim T is as required. Indeed, T is clearly a Parovičenko space of weight at most $w(M)^{\aleph_0} = \aleph_1^{\aleph_0} = \mathfrak{c}$ (see the argument in the last paragraph of 2.1). Since each almost-P space without isolated points contains a family of $\mathfrak{c}$ pairwise-disjoint open sets, it follows that $w(T) = \mathfrak{c}$.

For each ordinal $\alpha \in \omega_1$, let $\pi_\alpha: M \to W$ be the projection onto the αth coordinate. So if $w_n \in W - \{\infty\}$ for each $n \in \mathbb{N}$, then

$$\{(n, \pi_\alpha^{-1}(\{w_n\})): n \in \mathbb{N}\}$$

is a clopen set of $\mathbb{N} \times M$ disjoint from $\mathbb{N} \times \pi_\alpha^{-1}(\{\infty\})$. Put

$$D_\alpha = \overline{\mathbb{N} \times \pi_\alpha^{-1}(\{\infty\})} - (\mathbb{N} \times M),$$

where closure is taken in $\beta(\mathbb{N} \times M)$. It is easily seen that each D_α is a nowhere dense closed subspace of T. That $wn(T) \leq \aleph_1$ will follow once we show $\bigcup\{D_\alpha: \alpha \in \omega_1\}$ is dense T.

Suppose that C is a non-compact clopen subset of $\mathbb{N} \times M$. It will be sufficient to find an $\alpha \in \omega_1$ such that in $\beta(\mathbb{N} \times M)$,

$$\bar{C} \cap D_\alpha - (\mathbb{N} \times M) \neq \emptyset.$$

To this end we first let

$$N = \{n \in \mathbb{N}: C \cap \{n\} \times M \neq \emptyset\}$$

and

$$C_n = C \cap \{n\} \times M \quad \text{if } n \in N.$$

For $n \in N$, C_n is clopen in $\{n\} \times M$, so we can find a finite subset $F_n \subseteq \omega_1$ such that $\nu \in \omega_1 - F_n$ implies

$$(\{n\} \times \pi_\nu^{-1}(\{\infty\})) \cap C_n \neq \emptyset.$$

Take $\alpha \in \omega_1 - \bigcup\{F_n: n \in N\}$ arbitrarily. Further, choose

$$w_n \in (\{n\} \times \pi_\alpha^{-1}(\{\infty\}) \cap C_n.$$

Then in $\beta(\mathbb{N}\times M)$ each limit point of $\{w_n : n\in N\}$ is easily determined to be an element of $\bar{C}\cap D_\alpha$. $\square$

4. The proof of Theorem 1.3 and remarks

First we observe that the weak Novák number is a co-absolute invariant property.

4.1. Lemma. *For any space X, $wn(\mathscr{E}(X)) = wn(X)$.*

Proof. This is easy, for given a closed continuous surjection $f: Y\to Z$ that is also irreducible [i.e., $f(F)$ is a proper closed subset of Z whenever F is a proper closed subset of Y], $D\subseteq Z$ is dense (respectively, nowhere dense) iff $f^{-1}(D)$ is dense (nowhere dense) in Z. $\square$

We will now give the proof of our main result, Theorem 1.3. To this end we show $wn(T) < wn(S)$ which implies, according to Lemma 4.1, that $\mathscr{E}(S)$ and $\mathscr{E}(T)$ are not homeomorphic.

Suppose κ is a regular ordinal, $\omega_0\leqslant\kappa\leqslant\omega_2$, and suppose $\{G_\lambda : \lambda\in\kappa\}$ is a family of open dense sets of S. Since every non-empty $G\aleph_1$-set of S has non-empty interior, we may construct, via compactness and a Baire category type argument, a family $\{C_\lambda : \lambda\in\kappa\}$ of non-empty clopen sets of S such that

$$C_\lambda \subseteq \bigcap\{G_\lambda\cap C_\mu : \mu<\lambda\}\quad \text{for each } \lambda\in\kappa.$$

Again by compactness, $\bigcap\{C_\lambda : \lambda\in\kappa\}\neq\emptyset$. It is now clear that

$$wn(T)\leqslant\aleph_1<\aleph_1<wn(S).\quad\square$$

4.2. Remarks. (1) The reader should observe the translation of the results in this paper to Boolean algebraic terminology. First, a pair of compact spaces are co-absolute iff they have isomorphic regular-open set algebras. Second, a Parovičenko space is characterized as the Stone space of a weakly countably complete, ω-closed, atomless Boolean algebra, see [3] or [8]. Through such translations the following improvements of 1.1 announced in [11] (with a totally different proof than that in [2]) exists:

CH *implies that X and $\mathbb{N}^*$ are co-absolute whenever X is a compact almost-P space of π-weight $\mathfrak{c}$ without isolated points.*

We do not know whether the converse is true.

(2) The machine used in [2] to prove 1.2 appears new; however, the authors did not realize that the resulting Boolean algebra is always the so-called $\mathfrak{c}$-homogeneous-universal Boolean algebra, which exists iff $\mathfrak{c} = 2^{\mathfrak{c}}$, [3]. A somewhat similar but much more complicated machine was subsequently constructed by the second author of this paper in order to give the first proof of 1.3 [12].

(3) We feel, as do the authors of [2], that the converse to 1.1 is a theorem. Attempts to prove this have failed so far. The reader should be aware of the following curious but related results from [13]. Let K be the Tychonov product of $\aleph_1$ many two-point discrete spaces and put $Z = (\mathbb{N} \times K)^*$. Then $\mathbb{N}^*$ and Z are co-absolute iff X^* and $\mathbb{N}^*$ are co-absolute whenever X is locally compact, realcompact, non-compact, and of weight at most $\mathfrak{c}$. Further, "N^* and Z are co-absolute" is consistent with $\aleph_1 < \mathfrak{c}$.

References

[1] B. Balcar, J. Pelant, P. Simon, The space of ultrafilters on N covered by nowhere dense sets, Fund. Math. 110 (1980) 11–24.

[2] S. Broverman and W. Weiss, Spaces co-absolute with $\beta N - N$, Topology Appl. 12 (1981) 127–133.

[3] W. Comfort and S. Negrepontis, The Theory of Ultrafilters, Grund. Math. Wiss. Bd. 211 (Springer-Verlag, Berlin, 1974).

[4] E. van Douwen and J. van Mill, Parovičenko's characterization of $\beta\omega - \omega$ implies CH, Proc. AMS 72 (1978) 539–541.

[5] R. Engelking, General Topology (PWN-Polish Sci. Pub., Warsaw, 1977).

[6] N. Fine and L. Gillman, Extensions of continuous functions in βN, Bull. AMS 66 (1960) 376–381.

[7] L. Gillman and M. Henriksen, Rings of continuous functions in which every finitely generated ideal is principle, Trans. AMS 82 (1956) 366–391.

[8] J. van Mill, An introduction to $\beta\omega$, to appear.

[9] I. Parovičenko, A universal bicompact of weight $\aleph$, Sov. Math. Dohl. 4 (1963) 592–595.

[10] V. Ponomarev and L. Šapiro, Absolutes of topological spaces and their continuous maps, Russian Math. Surv. 31 (1976) 138–154.

[11] S. Williams, An application of trees to topology, Topology Proc. 3 (1978) 523–525.

[12] S. Williams, Some compact F-spaces of continuous weight, pre-print.

[13] S. Williams, Trees, Gleason spaces and co-absolutes of $\beta N - N$, Trans. AMS, to appear.

[14] R.G. Woods, A Boolean algebra of regular closed subsets of $\beta X - X$, Trans. AMS 154 (1971) 23–36.

[15] R.G. Woods, Co-absolutes of remainders of Stone–Čech compactifications, Pacific J. Math. 37 (1971) 545–560.

[16] R.G. Woods, A survey of absolutes of topological spaces, Topological Structures 2, Math. Centre Tracts 116 (Amsterdam, 1979) 323–362.