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A *continuum* is a compact connected Hausdorff space.

A continuum is *decomposable* if it can be represented as a union of two of its proper subcontinua; otherwise, it is *indecomposable*.

A continuum is *locally a bundle* of arcs if there exists a compact totally disconnected space X such that every point has a neighbourhood homeomorphic to $X \times (0,1)$.

It is rather easy to construct continua that are locally a bundle of arcs. Let X be a space consisting of two sequences with limit points, say

$$X = \{a_n\colon n \text{ is an integer}\} \cup \{0,3\},$$

where $a_n = \frac{1}{n}$ if n is a positive integer and

$$a_n = 3 + \frac{1}{n-1} \text{ if } n \text{ is a negative integer or zero.}$$

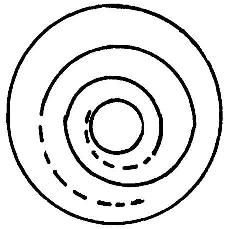
Let f be a homeomorphism from X onto itself defined by

$$f(0) = 0,$$

$$f(a_n) = a_{n+1},$$

$$f(3) = 3.$$

The space $X \times I/f$ obtained from the product $X \times I$, where I is the closed unit interval, by identifying for each $x \in X$ the point $\langle x,0 \rangle$ with $\langle f(x),1 \rangle$ is a planar continuum (see the picture).



Remark 1. There exists in the plane an indecomposable continuum that is locally homeomorphic to $C \times (0,1)$, where C denotes the Cantor set.

In Bing's paper [1], on pages 222 and 223 a description is given of an example of such an indecomposable plane continuum Y. Every proper subcontinuum of Y is an arc and Y is locally homogeneous (i.e., for each pair of points p, q of Y there are arbitrarily small homeomorphic open subsets N_p, N_q containing p, q, respectively). Hence, Y is also locally a bundle $C \times (0,1)$.

We are indebted to F. Burton Jones for referring us to this example.

The next two theorems describe continua that are locally a bundle of arcs.

Theorem 1 [3, p. 29]. *Let* X *be a compact totally disconnected Hausdorff space. If a continuum* K *is locally a bundle* $X \times (0,1)$, *then* K *can be obtained as a quotient of the product* $X \times I$ *by identifying for each* $x \in X$ *the points* $\langle x,e \rangle$ *and* $h\langle x,e \rangle$, *where* h *is an involution with no fixed points defined on* $X \times \{0,1\}$.

Theorem 2 [2, Corollary on page 552]. *Let* X *be a compact Hausdorff totally disconnected and dense in itself space. If a homeomorphism* f *from* X *onto itself is such that for some* x *in* X *the set* $\{f^n(x)$: n *is an integer*$\}$ *is dense in* X, *then the space* $X \times I/f$ *obtained from the product* $X \times I$ *by identifying for each* $x \in X$ *the points* $\langle x,0 \rangle$ *with* $\langle f(x),1 \rangle$, *is an indecomposable continuum.*

To insure the existence of homeomorphisms described in the preceding theorem, we use the following:

Theorem 3. *Let* P *and* Q *be closed and nowhere dense subsets of the Cantor set* C, *and let* h *be a homeomorphism from* P *onto* Q. *Then there exists an extension* h' *of* h *such that* h' *is a homeomorphism from* C *onto itself while moreover for some* c *of* C *the set* $\{(h')^n(c)$: n *is an integer*$\}$ *is a dense subset of* C.

The first proof of this theorem was published in [4]. Based on the well-known idea of a shift-mapping, we give a short proof of the theorem.

Proof. Let us observe that because $P \cup Q$ is a nowhere dense subset of C, then there exists a nowhere dense closed subset D of C, which is homeomorphic to C and which is such that $P \cup Q$ is nowhere dense in D. By the theorem of Ryll-Nardzewski [7, Corollary 2, p. 186] there exists a homeomorphism from D onto itself that is an extension of h.

Thus, without the loss of generality, we can assume that P = Q.

Let Z denote the set of integers and let C^Z be the product of countably many copies of the Cantor set.

Let i_P be an embedding of the set P into C^Z defined by

$$i_P(p) = (\cdots, h^{-1}(p), p, h(p), h^2(p), \cdots),$$

where p is the zero-coordinate, $h(p)$ is the 1-coordinate of $i_P(p)$, and so on.

Because we assume that $P = Q$, the function i_P is also an embedding of Q.

Let s be the shift-mapping from C^Z onto itself defined by $s((c_n)_{n\in Z}) = (c_{n+1})_{n\in Z}$.

Observe that $s(i_P(p)) = i_P(h(p))$. By the already mentioned theorem of Ryll-Nardzewski there exists a homeomorphism f_P from C onto C^Z such that $f_{P|P} = i_P$ (see the diagram).

$$\begin{tikzcd}
 & C^Z \arrow[rr, "s"] & & C^Z & \\
C \arrow[ur, dashed, "f_P"] \arrow[r, phantom, "\supseteq"] & P \arrow[u, hook, "i_P"'] \arrow[rr, "h"] & & Q \arrow[u, hook, "i_P"] \arrow[r, phantom, "\subseteq"] & C \arrow[ul, dashed, "f_P"']
\end{tikzcd}$$

The homeomorphism $h' = f_P^{-1} \cdot s \cdot f_P$ is an extension of h.

Let A be a countable dense subset of C^Z, i.e., $A = \{(d_n^i)_{n\in Z}$: i is a positive integer$\}$. Let d be a point of C^Z such that for any finite subset $(x_1, \cdots, x_m)$ of $\{d_n^i$: i is a positive integer and $n \in Z\}$ there is a $k \in Z$ such that $d_{k+j} = x_j$ for all $1 \le j \le m$.

The orbit $\{(h')^k(f_P^{-1}(d))$: k is an integer$\}$ is dense in C.

Question 1. Does there exist a planar indecomposable continuum obtained from $C \times I$ by identifying for each $x \in C$ the point $\langle x,0\rangle$ with $\langle f(x),1\rangle$, for some homeomorphism from C onto itself?

Ch. L. Hagopian proved in [6] that an indecomposable continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs.

R. D. Anderson pointed out that there is a continuum on the surface of a torus that is indecomposable, has only arcs as proper subcontinua, is locally a bundle $C \times (0,1)$ and is obtained from the product $C \times I$ by identifying for each $x \in C$ the point $\langle x,0\rangle$ with $\langle f(x),1\rangle$, for some homeomorphism f from the Cantor set C onto itself. This continuum is not a solenoid.

The Anderson example can be described in the following way.

Let $C = (\{e^{it}: t \in R\} \times \{0\}) \cup (\{e^{in}: n \text{ is an integer}\} \times \{1\})$, where $i = \sqrt{-1}$. Induce a topology on C by taking the following sets as a basis: $[(\{e^{it}: r \leq t < s\} \times \{0,1\}) \cup \{\langle e^{is},1\rangle\}] \cap C$, where r and s are reals such that $e^{ir} = e^{in}$ and $e^{is} = e^{ik}$ for some integers n and k. It is easy to see that C with this topology is the Cantor set. Define a homeomorphism f from C onto itself by $f\langle e^{it},j\rangle = \langle e^{i(t+1)},j\rangle$. The orbit of every point is dense and the space $C \times I/f$ is an indecomposable continuum.

Question 2. Let $\mathcal{M}$ be the family of all continua that are indecomposable, locally a bundle $C \times (0,1)$, have only arcs as proper subcontinua, are not solenoids and are obtained from $C \times I$ by identifying $C \times \{0\}$ with $C \times \{1\}$ under some homeomorphism. Does the family $\mathcal{M}$ contain 2^{ω} non-homeomorphic continua?

Question 3. In [5], an example was found of a homogeneous indecomposable and non-metrizable continuum that has only arcs as proper subcontinua. Are there 2^{ω} non-homeomorphic continua of this type?

Question 4. Let K be a homogeneous indecomposable circle-like continuum that has only arcs as proper subcontinua. Is the continuum K metrizable (and hence a solenoid)?

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