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PERFECT IMAGES OF ZERO-DIMENSIONAL SEPARABLE METRIC SPACES

BY

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ABSTRACT. Let $\mathbf{Q}$ denote the rationals, $\mathbf{P}$ the irrationals, $\mathbf{C}$ the Cantor set and $\mathbf{L}$ the space $\mathbf{C}-\{p\}$ (where $p\in\mathbf{C}$). Let $f:X\to Y$ be a perfect continuous surjection. We show: (1) If $X\in\{\mathbf{Q},\mathbf{P},\mathbf{Q}\times\mathbf{P}\}$, or if f is irreducible and $X\in\{\mathbf{C},\mathbf{L}\}$, then Y is homeomorphic to X if Y is zero-dimensional. (2) If $X\in\{\mathbf{P},C,\mathbf{L}\}$ and f is irreducible, then there is a dense subset S of Y such that $f\mid f^{\leftarrow}[S]$ is a homeomorphism onto S. However, if Z is any σ-compact nowhere locally compact metric space then there is a perfect irreducible continuous surjection from $\mathbf{Q}\times\mathbf{C}$ onto Z such that each fibre of the map is homeomorphic to $\mathbf{C}$.

§ 1. **Introduction and known results.** Internal characterizations of the metric spaces $\mathbf{Q}$, $\mathbf{C}$, $\mathbf{L}$, $\mathbf{Q}\times\mathbf{C}$, and $\mathbf{P}$ have long been known. Sierpinski [Si] characterized $\mathbf{Q}$, Brouwer [B] characterized $\mathbf{C}$ and $\mathbf{L}$, and Alexandroff and Urysohn [AU] characterized $\mathbf{Q}\times\mathbf{C}$ and $\mathbf{P}$. More recently the first-named author has derived an internal characterization of $\mathbf{Q}\times\mathbf{P}$ [vM]. We summarize these characterizations in the following theorem. (If $\mathcal{P}$ is a topological property then a space X is said to be nowhere locally $\mathcal{P}$ if no point of X has a neighborhood with $\mathcal{P}$).

1.1. THEOREM. *Let X be a zero-dimensional separable metric space. Then:*

(a) *X is homeomorphic to $\mathbf{Q}$ iff X is countable and nowhere locally compact.*

(b) *X is homeomorphic to $\mathbf{C}$ iff X is compact and has no isolated points.*

(c) *X is homeomorphic to $\mathbf{L}$ iff X is locally compact, non-compact, and has no isolated points.*

(d) *X is homeomorphic to $\mathbf{Q}\times\mathbf{C}$ iff X is nowhere locally compact, nowhere locally countable, and σ-compact.*

(e) *X is homeomorphic to $\mathbf{P}$ iff X is nowhere locally compact and topologically complete (a space is topologically complete if it is a G_δ-subset of its Stone-Cech compactification).*

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(f) *X is homeomorphic to* $\mathbf{Q}\times\mathbf{P}$ *iff X is nowhere σ-compact, nowhere topologically complete, and is a countable union of closed topologically complete subspaces.*

One consequence of 1.1 is that certain products of two zero-dimenional separable metric spaces are homeomorphic to one of the spaces characterized in 1.1. The following matrix, whose rows and columns are indexed by spaces, summarizes the situation; in the Xth row and Yth column is listed a homeomorph of the product space $X\times Y$. All listings are immediate corollaries of 1.1.

	Q	C	L	P	Q×C	Q×P
Q	Q	Q×C	Q×C	Q×P	Q×C	Q×P
C	Q×C	C	L	P	Q×C	Q×P
L	Q×L	L	L	P	Q×L	Q×P
P	Q×P	P	P	P	Q×P	Q×P
Q×C	Q×C	Q×C	Q×C	Q×P	Q×C	Q×P
Q×P	Q×P	Q×P	Q×P	Q×P	Q×P	Q×P

Figure 1.

A *map* is a continuous surjection. A *perfect map* is a closed map such that point inverses are compact subspaces of the domain. We will make use of the following well-known properties of perfect maps; see for example [D], Chapter 11, or problems $3X$ and $3Y$ of [E].

1.2. THEOREM. *Let $\mathcal{P}$ be one of compactness, local compactness, σ-compactness, and topological completeness. Let $f:X\to Y$ be a perfect map. Then X has $\mathcal{P}$ iff Y has $\mathcal{P}$.*

A subset A of a topological space X is regular closed if $A=\mathrm{cl}_X(\mathrm{int}_X A)$. Let $\mathcal{R}(X)$ denote the Boolean algebra of regular closed subsets of X. The following theorem is well-known, see, for example, [Sik, §1, 20]

1.3. THEOREM. *$\mathcal{R}(X)$ is a complete Boolean algebra under the following operations.*

(i) $A\le B$ *iff* $A\subseteq B$

(ii) $\bigvee_\alpha A_\alpha=\mathrm{cl}_X[\bigcup_\alpha A_\alpha]$

(iii) $\bigwedge_\alpha A_\alpha=\mathrm{cl}_X\,\mathrm{int}_X[\bigcap_\alpha A_\alpha]$

(iv) $A'=\mathrm{cl}_X(X-A)$

We assume the reader is familiar with the theory of Stone spaces of Boolean algebras (see [Sik]), but we summarize it briefly. If $\mathcal{A}$ is a Boolean algebra let $S(\mathcal{A})$ denote the set of ultrafilters on $\mathcal{A}$. If $A\in\mathcal{A}$ let $\lambda(A)=\{\alpha\in S(\mathcal{A}):A\in\alpha\}$. Then $\{\lambda(A):A\in\mathcal{A}\}$ is a base for a topology τ on $S(\mathcal{A})$. With this topology, $S(\mathcal{A})$ is a compact zero-dimensional Hausdorff space and $A\to\lambda(A)$ is a Boolean algebra isomorphism from $\mathcal{A}$ onto the set of clopen subsets of $S(\mathcal{A})$. The space $(S(\mathcal{A}),\tau)$ is called the *Stone space* of $\mathcal{A}$.

A closed map f from X onto Y is called *irreducible* if $f[B]\neq Y$ whenever B is a proper closed subset of X. Perfect irreducible mappings have the following well-known properties; see 2.3 of [Wo] for a proof of 1.4(b).

1.4. LEMMA. *Let X and Y be regular Hausdorff spaces and let $f:X\to Y$ be a perfect irreducible map. Then*:

(a) *If S is dense in Y, $f^{\leftarrow}[S]$ is dense in X.*

(b) *The correspondence $A\to f[A]$ is a Boolean algebra isomorphism from $\mathcal{R}(X)$ onto $\mathcal{R}(Y)$.*

(c) *The correspondence $x\to f(x)$ is a bijection from the isolated points of X onto the isolated points of Y.*

The next lemma is a simple generalization of Lemma 1 of [Str].

1.5. LEMMA. *Let X be a regular Hausdorff space and let $\mathcal{A}$ be a subalgebra of $\mathcal{R}(X)$ that is a basis for the closed subsets of X. Let $E_{\mathcal{A}}X=\{\alpha\in S(\mathcal{A}):\bigcap\{A:A\in\alpha\}\neq\phi\}$, regarded as a subspace of $S(\mathcal{A})$. Let $\pi:E_{\mathcal{A}}X\to X$ be defined by: $\pi(\alpha)=\bigcap\{A:A\in\alpha\}$. Then π is a well-defined perfect irreducible map onto X.*

§2. **Perfect images of Baire spaces.** In this section we characterize perfect, and perfect irreducible, zero-dimensional images of the Baire spaces $\mathbf{C}$, $\mathbf{L}$, and $\mathbf{P}$. These characterizations are obtained as corollaries of more general results.

Let κ be an infinite cardinal. As in [T], we say that a space Y is κ-Baire if the intersection of fewer than κ dense open subsets of Y is dense in Y. Thus Baire spaces are just $\aleph_1$-Baire spaces.

Let $w(X)$ denote the weight of a space X, i.e. the least cardinal occurring as the cardinality of a base for the open sets of X. Let κ^+ denote the smallest cardinal greater than κ. If $A\subset X$, bd_xA denotes the boundary of A in X.

2.1. LEMMA. *Let X and Y be regular Hausdorff spaces and let $f:X\to Y$ be a perfect irreducible map. Suppose that Y is a $w(X)^+$-Baire space. Then there is a dense subspace S of Y such that $f\restriction f^{\leftarrow}[S]:f^{\leftarrow}[S]\to S$ is a homeomorphism. Also, $f^{\leftarrow}[S]$ is dense in X and $|f^{\leftarrow}(p)|=1$ for each $p\in S$.*

Proof. Let $M=\{y\in Y:|f^{\leftarrow}(y)|>1\}$, and let $\mathcal{B}$ be an open base for X of cardinality $w(X)$. If $y\in M$ choose x and z to be distinct points of $f^{\leftarrow}(y)$. Choose $B(y)\in\mathcal{B}$ such that $x\in\operatorname{int}_X\operatorname{cl}_X B(y)$ and $z\notin\operatorname{cl}_X B(y)$. Then

$$\begin{aligned}y&\in f[\operatorname{cl}_X B(y)]\cap f[\operatorname{cl}_X[X-B(y)]\\&=f[\operatorname{cl}_X B(y)]\cap\operatorname{cl}_Y[Y-f[\operatorname{cl}_X B(y)]]\quad\text{(by 1.4(b))}\\&=bd_Yf[\operatorname{cl}_X B(y)].\end{aligned}$$

Thus $M\subseteq\bigcup\{bd_Yf[\operatorname{cl}_X B(y)]:y\in M\}$. Hence M is contained in the union of no more than $w(X)$ closed nowhere dense subsets of Y. As Y is a $w(X)^+$-Baire space, $Y\backslash M$ is dense in Y. By 1.4(a) $f^{\leftarrow}[Y\backslash M]$ is dense in X, and obviously $f \upharpoonright f^{\leftarrow}[Y\backslash M]$ is a homeomorphism onto $Y\backslash M$ (it is closed as f is). □

2.2. THEOREM. (a) *A perfect zero-dimensional image of* $\mathbf{P}$ *is homeomorphic to* $\mathbf{P}$.

(b) *Let X be one of* $\mathbf{C}$, $\mathbf{L}$, *or* $\mathbf{P}$. *Let $f:X\to Y$ be a perfect irreducible surjection. Then there is a dense subset S of Y such that $f \upharpoonright f^{\leftarrow}[S]:f^{\leftarrow}[S]\to S$ is a homeomorphism. If Y is zero-dimensional then Y is homeomorphic to X.*

Proof. (a) This follows from 1.1(e) and 1.2.

(b) That Y is homeomorphic to X follows from 1.1(b), 1.1(c), 1.2, and 1.4(c). The remaining assertion follows from 2.1 and the fact that $\mathbf{C}$, $\mathbf{L}$, and $\mathbf{P}$ are $\aleph_1$-Baire spaces of weight $\aleph_0$. □

We note in passing that 2.1 has interesting applications to spaces other than metric spaces. Let $\beta\mathbf{N}$ denote the Stone-Čech compactification of the countable discrete space $\mathbf{N}$. It is known (see, for instance, [Wa]) that if the continuum hypothesis is assumed then $\beta\mathbf{N}\backslash\mathbf{N}$ is an $\aleph_2$-Baire space of weight $\aleph_1$. Hence by 2.1 if f is a perfect irreducible map from $\beta N\backslash N$ onto itself, there is a dense subspace S of $\beta N\backslash N$ such that $f \upharpoonright S$ is a homeomorphism from S onto $f[S]$.

§3. **Perfect images of $\mathbf{Q}\times\mathbf{C}$.** The principal new result of this paper is the following theorem.

3.1. THEOREM. *Let X be a σ-compact nowhere locally compact metric space. Then there exists a perfect irreducible map $f:\mathbf{Q}\times\mathbf{C}\to X$ such that for each $p\in X$, $f^{\leftarrow}(p)$ is homeomorphic to* $\mathbf{C}$.

Before proving 3.1, we state (and sometimes prove) a series of technical lemmas.

3.2. LEMMA (1.2 of [PW]). *If X is a metric space without isolated points and if C is a closed nowhere dense subset of X, then there exists $A\in\mathcal{R}(X)$ such that $C\subset bd_X A$.*

3.3. LEMMA. *Let X be a metric space without isolated points and let $A\in\mathcal{R}(X)$. If $C\subset X$ is nowhere dense and closed then there exist H and K in $\mathcal{R}(X)$ such that:*

(1) $H\vee K=A$
(2) $H\wedge K=\phi$
(3) $C\cap A\subset bd_X H\cap bd_X K$.

Proof. Since $C\cap A$ is a closed nowhere dense subset of the metric space A, by 3.2 there exists $H\in\mathcal{R}(A)$ such that $C\cap A\subset bd_A H$. If $K=\mathrm{cl}_A(A\backslash H)$ then $K\in\mathcal{R}(A)$ and $bd_A K=bd_A H$. Since $H\in\mathcal{R}(A)$ and $A\in\mathcal{R}(X)$ it follows readily that $H\in\mathcal{R}(X)$; similarly for K. Hence (1) and (2) hold. It is easy to check that $bd_A H\subset bd_X H$; hence (3) holds. □

3.4. Lemma. *Let X be a metric space without isolated points and let $\mathscr{B}$ be a Boolean subalgebra of $\mathscr{R}(X)$. If $\mathscr{C}$ is a family of closed nowhere dense sets of X then there is a Boolean subalgebra $\mathscr{B}'$ of $\mathscr{R}(X)$ containing $\mathscr{B}$ so that for all $B \in \mathscr{B}$ and $C \in \mathscr{C}$ there are F, $G \in \mathscr{B}'$ so that*

(1) $F \vee G = B$
(2) $F \wedge G = \phi$
(3) $C \cap B \subset bd_X F \cap bd_X G$.

Moreover, $\mathscr{B}'$ can be chosen so that $|\mathscr{B}'| \leq \max\{|\mathscr{B}|, |\mathscr{C}|\}$.

Proof. For each $B \in \mathscr{B}$ and $C \in \mathscr{C}$ use 3.3 to choose $F(B, C)$, $G(B, C) \in \mathscr{R}(X)$ such that $F(B, C) \vee G(B, C) = B$, $F(B, C) \wedge G(B, C) = \phi$, and $C \cap B \subset bd_X F(B, C) \cap bd_X G(B, C)$. Let $\mathscr{B}'$ be the subalgebra of $\mathscr{R}(X)$ generated by $\mathscr{B} \cup \{F(B, C), G(B, C) : B \in \mathscr{B}, C \in \mathscr{C}\}$. $\square$

3.5. Definition. Let X be a metric space without isolated points and let $\mathscr{B}_0 \subset \mathscr{R}(X)$ be a countable subalgebra of $\mathscr{R}(X)$ that forms a basis for the closed sets of X. Let $\mathscr{C}$ be a family of closed and nowhere dense subsets of X. Inductively define Boolean subalgebras $\mathscr{B}_n(\mathscr{C}) \subset \mathscr{R}(X)$ by

(1) $\mathscr{B}_0(\mathscr{C}) = \mathscr{B}_0$
(2) $\mathscr{B}_{n+1}(\mathscr{C}) = (\mathscr{B}_n(\mathscr{C}))'$,

where $(\mathscr{B}_n(\mathscr{C}))'$ is as in Lemma 3.4. Put $\mathscr{B}(\mathscr{C}) = \bigcup_{n<\omega} \mathscr{B}_n(\mathscr{C})$ and observe that $\mathscr{B}(\mathscr{C})$ is a Boolean subalgebra of $\mathscr{R}(X)$.

Proof of 3.1. Let $X = \bigcup_{n<\omega} C_n$ where the C_n's are compact and nowhere dense. Put $\mathscr{C} = \{C_n : n < \omega\}$. In addition, let $\mathscr{B}$ be a countable basis for the closed subsets of X which is a Boolean subalgebra of $\mathscr{R}(X)$. Let $\mathscr{A} = \mathscr{B}(\mathscr{C})$ (see the preceding definition). Notice that $\mathscr{A}$ is countable. Let $E_{\mathscr{A}}X$ and π be as in 1.5.

Since X is σ-compact and nowhere locally compact, and since π is a perfect map, $E_{\mathscr{A}}X$ is also σ-compact and nowhere locally compact by 1.2. Evidently $E_{\mathscr{A}}X$ is a separable zero-dimensional metric space. Hence to show that $E_{\mathscr{A}}X$ is homeomorphic to $\mathbf{Q} \times \mathbf{C}$ it suffices by 1.1(d) to show that each non-empty open set of $E_{\mathscr{A}}X$ is uncountable. Let V be a non-empty open subset of $E_{\mathscr{A}}X$. As π is irreducible $X \backslash \pi[E_{\mathscr{A}}X \backslash V] \neq \phi$. If $p \in X \backslash \pi[E_{\mathscr{A}}X \backslash V]$ then $\pi^{\leftarrow}(p) \subset V$. Thus to show that $E_{\mathscr{A}}X$ is homeomorphic to $\mathbf{Q} \times \mathbf{C}$ it suffices to show that $\pi^{\leftarrow}(x_0)$ is uncountable for each $x_0 \in X$. As $\pi^{\leftarrow}(x_0)$ is a compact metric subspace of $E_{\mathscr{A}}X$, this is equivalent to showing that $\pi^{\leftarrow}(x_0)$ contains a Cantor space. We will in fact show that $\pi^{\leftarrow}(x_0)$ is a Cantor set.

There exists $n \in \omega$ such that $x_0 \in C_n$. Since $\pi^{\leftarrow}(x_0)$ is a compact zero-dimensional separable metric space, we only need to show that $\pi^{\leftarrow}(x_0)$ contains no isolated points. Suppose, to the contrary, that α is an isolated point of

$\pi^{\leftarrow}(x_0)$. Then we can find $A\in\mathcal{A}$ so that $\lambda(A)\cap\pi^{\leftarrow}(x_0)=\{\alpha\}$. Since $\mathcal{A}=\bigcup_{n<\omega}\mathcal{B}_n(\mathcal{C})$ there is an $m\in\omega$ so that $A\in\mathcal{B}_m(\mathcal{C})$. Notice that $x_0\in C_n\cap A$. Hence, by construction, there are $F, G\in\mathcal{B}_{m+1}(\mathcal{C})\subset\mathcal{A}$ so that

(1) $F\vee G=A$
(2) $F\wedge G=\phi$
(3) $A\cap C_n\subset bd_x F\cap bd_X G.$

Since $F\vee G=A$, without loss of generality, $F\in\alpha$, which implies that $G\notin\alpha$ since $F\wedge G=\phi$. We claim that $\lambda(G)\cap\pi^{\leftarrow}(x_0)\neq\phi$ which is a contradiction since

$$\lambda(G)\cap\pi^{\leftarrow}(x_0)\subset(\lambda(A)-\{\alpha\})\cap\pi^{\leftarrow}(x_0)=\phi.$$

Define $\mathcal{F}$ to be $\{B\in\mathcal{B}: x_0\in\mathrm{int}_X B\}\cup\{G\}$. Since $x_0\in A\cap C_n\subset bd_X G\subset G$ and since $G\in\mathcal{A}$, the family $\mathcal{F}$ is a subfamily of $\mathcal{A}$ whose finite subfamilies have non-empty infima in $\mathcal{A}$. Hence $\mathcal{F}$ can be extended to an ultrafilter β on $\mathcal{A}$. As $\mathcal{B}$ is a base for the closed sets of X, $\beta\in\pi^{\leftarrow}(x_0)$; since $\beta\in\lambda(G)$ we have derived the desired contradiction. □

3.7. Corollary. *There is a perfect irreducible map from* $\mathbf{Q}\times\mathbf{C}$ *onto* $\mathbf{Q}$.

§4. **Perfect images of non-Baire spaces.** In this section we consider perfect, and perfect irreducible, images of $\mathbf{Q}$ and $\mathbf{Q}\times\mathbf{P}$. Our results for perfect images are similar to those in 2.2 (except for $\mathbf{Q}\times\mathbf{C}$), but those for perfect irreducible images are quite different from the analogous results in 2.2.

4.1. Theorem. *A perfect zero-dimensional image of* $\mathbf{Q}(\mathbf{Q}\times\mathbf{P})$ *is homeomorphic to* $\mathbf{Q}(\mathbf{Q}\times\mathbf{P})$.

Proof. This follows from 1.1(a), 1.1(f), and 1.2. □

4.2. Example. Let $f:\mathbf{Q}\times\mathbf{C}\to\mathbf{Q}$ be the perfect irreducible map provided in 3.7. We may assume that $|f^{\leftarrow}(q)|=c$ for each $q\in\mathbf{Q}$. Let $1_{\mathbf{P}}$ be the identity map on $\mathbf{P}$. Then $f\times 1_{\mathbf{P}}:\mathbf{Q}\times\mathbf{C}\times\mathbf{P}\to\mathbf{Q}\times\mathbf{P}$ is perfect (see [D]) and irreducible. As noted in Fig. 1, $\mathbf{Q}\times\mathbf{C}\times\mathbf{P}$ is homeomorphic to $\mathbf{Q}\times\mathbf{P}$. Obviously $|f^{\leftarrow}(x)|=c$ for each $x\in\mathbf{Q}\times\mathbf{P}$, so there is no dense subset S of $\mathbf{Q}\times\mathbf{P}$ such that $f\mid f^{\leftarrow}[S]$ is a homeomorphism onto S. This contrasts with 2.2.

4.3. Example. Let $X=\mathbf{Q}\times 2$ with the topology induced by the lexicographic ordering on X. Evidently X is homeomorphic to $\mathbf{Q}$. Let $\pi:X\to\mathbf{Q}$ be defined by $\pi((q,i))=q$ $(q\in\mathbf{Q}, i=1,2)$. It is easily seen that π is a perfect irreducible surjection such that $|\pi^{\leftarrow}(x)|=2$ for each $x\in\mathbf{Q}$.

It is known that if f is a continuous surjection from a space H onto a space K and if S is a dense subspace of H such that $f\restriction S:S\to f[S]$ is a homeomorphism, then $f[H\backslash S]=K\backslash f[S]$ (see 6.11 of [GJ]). Thus if $y\in f[S]$ then $|f^{\leftarrow}(y)|=1$. As $|\pi^{\leftarrow}(q)|=2$ for each $q\in\mathbf{Q}$, there is no dense subset S of $\mathbf{Q}$ such that $\pi\restriction S$ is a homeomorphism from S onto $\pi[S]$. This concludes the example.

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