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0. Introduction

All spaces are completely regular and X^* denotes $\beta X - X$.

The point $x \in X$ is called a *P-point* whenever $x \notin \overline{F}$ for each F_σ F of X which does not contain x. It is known that ω^* contains P-points under CH (cf. Rudin [R]); however, Shelah (see [M] or [W]) showed that it is consistent with the usual axioms of set theory that there are no P-points in ω^*. The point $x \in X$ is called a *weak P-point* whenever $x \notin \overline{F}$ for each countable $F \subset X - \{x\}$. Clearly each P-point is a weak P-point. Recently, Kunen [K_2] showed that there are 2^{2^ω} points in ω^* which are weak P-points but not P-points. In this note we generalize this result.

0.1. *Theorem. Let* X *be a compact infinite* F-*space without isolated points of weight* 2^ω *in which each nonempty* G_δ *has nonempty interior. Then there are* 2^{2^ω} *points in* X *which are weak* P-*points but not* P-*points.*

The condition that each nonempty G_δ in X has nonempty interior is essential of course, since no separable space without isolated points can have weak P-points (we don't know whether the theorem is true for compact nowhere separable F-spaces). Such F-spaces cannot have weak P-points, but they might have points which are not limit points of countable discrete sets. We have the following partial answer.

0.2. *Theorem. Let* E *be the projective cover of a compact space which is a product of at most* ω_1 *spaces of countable* π*-weight. Then there is an* $x \in E$ *such that* $x \notin \overline{D}$ *for each countable discrete* $D \subset E - \{x\}$.

Let us notice that under CH such points exist in each compact F-space of weight 2^ω ([vM$_2$]).

Let (*) denote the innocent statement that there is a compactification $\gamma\omega$ of ω such that $\gamma\omega-\omega$ is ccc but not separable. It is known (cf. section 5) that CH implies (*). I conjecture that (*) is true in ZFC. It is certainly worthwhile to try to solve this conjecture since a positive answer would imply that the following theorem is true in ZFC.

0.3. *Theorem. Assume* (*) *and let* X *be a compact infinite* F*-space without isolated points in which each nonempty* G_δ *has nonempty interior. Then there are* 2^{2^ω} *points in* X *which are weak* P*-points but not* P*-points.*

In particular, the theorem is true under CH. I have also been able to prove that the theorem is true under $2^\omega = 2^{\omega_1}$.

I am indebted to Evert Wattel for some helpful suggestions and to Charley Mills for reading a preliminary version of this note.

1. Independent Matrices

An ordinal is the set of smaller ordinals and a cardinal is an initial ordinal. Whenever X is a set and κ is a cardinal we define (as usual)

$$[X]^{\kappa} = \{A \subset X: |A| = \kappa\}$$

and

$$[X]^{<\kappa} = \{A \subset X: |A| < \kappa\}$$

respectively.

Let X be a space. An indexed family $\{A^i_j: i \in I, j \in J\}$ of subsets of X is called a J *by* I *independent matrix* if

- each A^i_j is the closure of some nonempty open F_σ in X;
- whenever $j_0, j_1 \in J$ are distinct and $i \in I$ then $A^i_{j_0} \cap A^i_{j_1} = \emptyset$;
- for each finite $F \subset I$ and $\phi: F \to J$ we have that $\cap\{A^{\alpha}_{\phi(\alpha)}: \alpha \in F\} \neq \emptyset$.

(This concept, in a slightly different form, is due to Kunen.)

An F-space is a space in which each cozero-set is C*-embedded. It is well-known, and easy to prove, that a normal space X is an F-space iff any two disjoint open F_σ's of X have disjoint closures. This fact will be used frequently without explicit reference throughout the remaining part of this note.

The following Proposition is the key to our construction. I am indebted to Evert Wattel for pointing out to me that my original proof was unnecessarily complicated and for allowing me to publish his proof here.

1.1. *Proposition. Let* X *be a compact infinite* F-*space without isolated points in which every nonempty* G_δ *has nonempty interior. Then each nonempty open subset of* X *contains an* ω_1 *by* ω_1 *independent matrix for* X.

Proof. Wellorder $\omega_1 \times \omega_1$ by $<^*$ in order type ω_1. For each $\langle\alpha,\beta\rangle \in \omega_1 \times \omega_1$ let

$$Q\langle\alpha,\beta\rangle = \{\langle\gamma,\delta\rangle : \langle\gamma,\delta\rangle \leq^* \langle\alpha,\beta\rangle\}.$$

$$\mathcal{F}\langle\alpha,\beta\rangle = \{\phi : \phi \text{ is a function with } |\phi| < \omega \text{ and } \phi \subset Q\langle\alpha,\beta\rangle\}, \text{ and}$$

$$\mathcal{G}\langle\alpha,\beta\rangle = \{\phi \in \mathcal{F}\langle\alpha,\beta\rangle : \langle\alpha,\beta\rangle \in \phi\}.$$

Notice that $\emptyset \in \mathcal{F}\langle\alpha,\beta\rangle - \mathcal{G}\langle\alpha,\beta\rangle$ for each $\langle\alpha,\beta\rangle \in \omega_1 \times \omega_1$. A function $\psi \in \mathcal{F}\langle\alpha,\beta\rangle$ is called an extension of $\phi \in \mathcal{F}\langle\gamma,\delta\rangle$ whenever $\langle\gamma,\delta\rangle \leq^* \langle\alpha,\beta\rangle$ and $\psi \cap Q\langle\gamma,\delta\rangle = \phi$. Without loss of generality $\langle 0,0\rangle$ is the first element of $\omega_1 \times \omega_1$. For the sake of notational simplicity we write $\max \emptyset = \langle 0,0\rangle$.

We will now construct for each $\langle\alpha,\beta\rangle \in \omega_1 \times \omega_1$ a collection of nonempty closed G_δ's $\{S(\langle\alpha,\beta\rangle,\phi) : \phi \in \mathcal{F}\langle\alpha,\beta\rangle\}$ and a collection of nonempty open F_σ's $\{U(\langle\alpha,\beta\rangle,\phi) : \phi \in \mathcal{G}\langle\alpha,\beta\rangle\}$ such that

(1) $\{S(\langle\alpha,\beta\rangle,\phi) : \phi \in \mathcal{F}\langle\alpha,\beta\rangle\}$ is a disjoint collection;

(2) if $\langle\gamma,\delta\rangle <^* \langle\alpha,\beta\rangle$ and $\phi \in \mathcal{F}\langle\alpha,\beta\rangle$ extends $\psi \in \mathcal{F}\langle\gamma,\delta\rangle$ then $S(\langle\alpha,\beta\rangle,\phi) \subset S(\langle\gamma,\delta\rangle,\psi)$;

(3) if $\phi \in \mathcal{G}\langle\alpha,\beta\rangle$ and $\psi = \phi - \{\langle\alpha,\beta\rangle\}$ then $S(\langle\alpha,\beta\rangle,\phi) \subset U(\langle\alpha,\beta\rangle,\phi) \subset U(\langle\alpha,\beta\rangle,\phi)^- \subset S(\langle\gamma,\delta\rangle,\psi)$ whenever $\max \psi \leq^* \langle\gamma,\delta\rangle <^* \langle\alpha,\beta\rangle$;

(4) if $\phi \in \mathcal{G}\langle\alpha,\beta\rangle$ and $\psi = \phi - \{\langle\alpha,\beta\rangle\}$ then $U(\langle\alpha,\beta\rangle,\phi)^- \cap S(\langle\alpha,\beta\rangle,\psi) = \emptyset$.

Let $S(\langle 0,0\rangle,\emptyset)$ and $S(\langle 0,0\rangle,\{\langle 0,0\rangle\})$ be any two nonempty disjoint closed G_δ's. In addition, let $U(\langle 0,0\rangle,\{\langle 0,0\rangle\})$ be any nonempty open F_σ of X the closure of which is properly contained in $\operatorname{int} S(\langle 0,0\rangle,\{\langle 0,0\rangle\})$. Now suppose that we have completed the construction for all $\langle\gamma,\delta\rangle <^* \langle\alpha,\beta\rangle$. For each $\phi \in \mathcal{F}\langle\alpha,\beta\rangle - \mathcal{G}\langle\alpha,\beta\rangle$ put

$$T(\langle\alpha,\beta\rangle,\phi) = \bigcap\{S(\langle\gamma,\delta\rangle,\phi)\colon \max\phi \leq^* \langle\gamma,\delta\rangle <^* \langle\alpha,\beta\rangle\}.$$

Observe that by (1) and (2)

$$\{T(\langle\alpha,\beta\rangle,\phi)\colon \phi \in \mathcal{J}\langle\alpha,\beta\rangle - \mathcal{G}\langle\alpha,\beta\rangle\}$$

is a disjoint collection of nonempty closed G_δ's. Put

$$\mathcal{H}\langle\alpha,\beta\rangle = \{\phi \in \mathcal{J}\langle\alpha,\beta\rangle\colon \phi \cup \{\langle\alpha,\beta\rangle\} \in \mathcal{G}\langle\alpha,\beta\rangle\}.$$

For each $\phi \in \mathcal{J}\langle\alpha,\beta\rangle - \mathcal{H}\langle\alpha,\beta\rangle$ define $S(\langle\alpha,\beta\rangle,\phi) = T(\langle\alpha,\beta\rangle,\phi)$. In addition, for each $\phi \in \mathcal{H}\langle\alpha,\beta\rangle$ choose disjoint nonempty G_δ's $Z_0(\phi)$, $Z_1(\phi) \subset \operatorname{int} T(\langle\alpha,\beta\rangle,\phi)$. In addition, let $U(\phi)$ be an open F_σ such that

$$Z_1(\phi) \subset U(\phi) \subset U(\phi)^- \subset \operatorname{int} T(\langle\alpha,\beta\rangle,\phi) - Z_0(\phi).$$

Define

$$\begin{cases} S(\langle\alpha,\beta\rangle,\phi) = Z_0(\phi), \\ S(\langle\alpha,\beta\rangle,\phi \cup \{\langle\alpha,\beta\rangle\}) = Z_1(\phi). \\ U(\langle\alpha,\beta\rangle,\phi \cup \{\langle\alpha,\beta\rangle\}) = U(\phi). \end{cases}$$

It is trivial that our inductive hypotheses are satisfied.

For each $\langle\alpha,\beta\rangle \in \omega_1 \times \omega_1$ now put

$$U\langle\alpha,\beta\rangle = \bigcup\{U(\langle\alpha,\beta\rangle,\phi)\colon \phi \in \mathcal{G}\langle\alpha,\beta\rangle\}.$$

Now let ϕ be a finite subset of $\omega_1 \times \omega_1$ such that no two elements of ϕ have their first entries equal. Let $\langle\alpha,\beta\rangle$ be the maximal member of ϕ. Then

$$\bigcap\{U\langle\gamma,\delta\rangle\colon \langle\gamma,\delta\rangle \in \phi\} \supset S(\langle\alpha,\beta\rangle,\phi),$$

since, $U\langle\gamma,\delta\rangle \supset U(\langle\gamma,\delta\rangle,\phi \cap Q\langle\gamma,\delta\rangle) \supset S(\langle\gamma,\delta\rangle,\phi \cap Q\langle\gamma,\delta\rangle) \supset S(\langle\alpha,\beta\rangle,\phi)$, according to (2) and (3), and since $S(\langle\alpha,\beta\rangle,\phi) \neq \emptyset$, also

$$\bigcap\{U\langle\gamma,\delta\rangle\colon \langle\gamma,\delta\rangle \in \phi\} \neq \emptyset.$$

Suppose that $\langle\alpha,\gamma\rangle <^* \langle\alpha,\beta\rangle$. We claim that $U\langle\alpha,\gamma\rangle \cap U\langle\alpha,\beta\rangle = \emptyset$. Let us assume, to the contrary, that $U\langle\alpha,\gamma\rangle \cap U\langle\alpha,\beta\rangle \neq \emptyset$, say

$$U(\langle\alpha,\gamma\rangle,\phi) \cap U(\langle\alpha,\beta\rangle,\psi) \neq \emptyset$$

for certain $\phi \in \mathcal{G}\langle\alpha,\gamma\rangle$ and $\psi \in \mathcal{G}\langle\alpha,\beta\rangle$. Define $\psi' = \psi \cap Q\langle\alpha,\gamma\rangle$. Notice that $\langle\alpha,\gamma\rangle \notin \psi'$. So $\phi \neq \psi'$ and consequently

$$U(\langle\alpha,\gamma\rangle,\phi) \cap S(\langle\alpha,\gamma\rangle,\psi') = \emptyset.$$

Put $\psi'' = \psi - \{\langle\alpha,\beta\rangle\}$. Then

$$U(\langle\alpha,\beta\rangle,\psi) \subset S(\max \psi'',\psi'') \subset S(\langle\alpha,\gamma\rangle,\psi')$$

by (2) and (3). We conclude that $U(\langle\alpha,\gamma\rangle,\phi) \cap U(\langle\alpha,\beta\rangle,\psi) = \emptyset$, a contradiction.

Finally, since every $U(\langle\alpha,\beta\rangle,\phi)$ is an F_σ and since $\mathcal{G}\langle\alpha,\beta\rangle$ is at most countable, each $U\langle\alpha,\beta\rangle$ is itself and open F_σ. So $\overline{U\langle\alpha,\gamma\rangle} \cap \overline{U\langle\alpha,\beta\rangle} = \emptyset$ for all $\langle\alpha,\gamma\rangle <^* \langle\alpha,\beta\rangle$. We conclude that $\{\overline{U\langle\alpha,\beta\rangle}: \langle\alpha,\beta\rangle \in \omega_1 \times \omega_1\}$ is an ω_1 by ω_1 independent matrix. The same proof shows that actually such a matrix can be chosen in any nonempty open subset of X.

1.2. *Remark*. In the sequel we will only need the existence of an ω by ω_1 independent matrix.

1.3. *Question*. Let X be a compact F-space without isolated points in which each nonempty G_δ has nonempty interior. Is there a 2^ω by 2^ω independent matrix for X?

2. Nice Filters

Let X be a normal space. A closed *filterbase* on X is a collection of closed subsets of X which is closed under finite intersections and which does not contain the empty set. The closed *filter* generated by the filterbase $\mathcal{F}$ is the collection $\{A \subset X: A = \overline{A}\ \&\ \exists\ F \in \mathcal{F}: F \subset A\}$. A closed *ultrafilter* is a maximal filter. The points X* are identified with the nonprincipal closed ultrafilters on X. So

$A \in p$ means $p \in cl_{\beta X}A$.

Let X be the topological sum of countably many nonempty compact spaces, say $X_n (n < \omega)$. A closed filter $\mathcal{F}$ on X is called *nice* provided that for each $F \in \mathcal{F}$ the set

$$\{n < \omega: F \cap X_n = \emptyset\}$$

is finite, while in addition $\cap\mathcal{F} = \emptyset$.

It is clear that no nice filter is an ultrafilter, in fact each nice filter can be extended to at least 2^{2^ω} ultrafilters.

2.1. *Lemma. Let* X *be a compact* F*-space without isolated points in which each nonempty* G_δ *has nonempty interior. For each* $n < \omega$ *let* Z_n *be a nonempty closed* G_δ *in* X *such that* $Z_n \cap (\cup\{Z_i: i \neq n\})^- = \emptyset$ *and put* $Z = (\cup\{Z_n: n < \omega\})^- - \cup\{Z_n: n < \omega\}$. *Then there is a nice filter* $\mathcal{F}$ *on* $\cup\{Z_n: n < \omega\}$ *such that*

(α) each $F \in \mathcal{F}$ *is the closure (in* $\cup\{Z_n: n < \omega\}$*) of some open* F_σ *in* X*;*

(β) for each countable $D \subset X - Z$ *there is some* $F \in \mathcal{F}$ *such that* $D \cap F = \emptyset$ *(hence* $\overline{D} \cap \overline{F} = \emptyset$*).*

Proof. For each $n < \omega$ let $\{A_\beta^m(n): m < \omega, \beta < \omega_1\}$ be an ω by ω_1 independent matrix for X each element of which is contained in int Z_n (Proposition 1.1). Let $D \subset X - Z$ be countable. Observe that for each $n < \omega$ and $m < \omega$ the family

$$\{\overline{A_\beta^m(n)}: \beta < \omega_1\}$$

is pairwise disjoint. Consequently, for each $n < \omega$ and $m < \omega$ we can find an index $\beta(n,m;D) < \omega_1$ such that

$$D \cap (A^m_{\beta(n,m;D)}(n))^- = \emptyset.$$

For each $n < \omega$ put

$$U_n(D) = \bigcup_{k=0}^{n} A^k_{\beta(n,k;D)}(n).$$

Notice that $U_n(D) \subset Z_n$ and that $D \cap \overline{U_n(D)} = \emptyset$. Define

$$U(D) = \bigcup_{n<\omega} U_n(D)$$

and let $\mathcal{D} = \{D \subset X - Z: |D| \leq \omega\}$.

Claim. $\overline{U(D)} \cap \overline{D} = \emptyset$ *for each* $D \in \mathcal{D}$. *Moreover, for each finite* $\mathcal{E} \subset \mathcal{D}$ *there is an* $i < \omega$ *such that* $Z_j \cap \bigcap\{U(E): E \in \mathcal{E}\} \neq \emptyset$ *for each* $j \geq i$.

It is clear that $\overline{U(D)} \cap D = \emptyset$ which implies that $\overline{U(D)} \cap \overline{D} = \emptyset$ since D is countable and X is an F-space. The second clause of the claim is trivial.

Now let $\mathcal{F}$ be the closed filter generated by $\{\overline{U(D)} \cap \bigcup_{n<\omega} Z_n: D \in \mathcal{D}\}$. Then $\mathcal{F}$ is as required.

3. Extending Nice Filters to OK-Points

For technical reasons we slightly change Kunen's [K_2] concept of an OK-point. Let X be a normal space. In this note, a point $p \in X^*$ is called κ-OK provided that for each sequence $\{U_n: n < \omega\}$ of neighborhoods of p in X^* there are $A_\alpha \in p(\alpha < \kappa)$ such that for each $n \geq 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa$:

$$\bigcap_{1\leq i\leq n} \mathrm{cl}_{\beta X} A_{\alpha_i} \cap X^* \subset U_n.$$

Observe that the property of being κ-OK gets stronger as κ gets bigger.

3.1. *Lemma. Let* X *be a locally compact and* σ*-compact space and let* $p \in X^*$ *be* ω_1*-OK. Then* p *is a weak* P*-point of* X^*.

Proof. Let $F \subset X^* - \{p\}$ be countable. Since p is

ω_1-OK there is an $A \in p$ such that $cl_{\beta X}A \cap F = \emptyset$ (with precisely the same technique as in [K_2, 1.3]). Put $Y = X \cup F$. Then Y is σ-compact, hence normal, and since F and A are both closed in Y there is a Urysohn map $f\colon Y \to [0,1]$ such that $f[F] = 0$ and $f[A] = 1$. Let $\beta f\colon \beta Y \to [0,1]$ be the Stone extension of f. Since $\beta X = \beta Y$ ([GJ, 6.7]) and since $\beta f(p) = 1$ we conclude that $p \notin cl_{\beta X}F$ (this type of argument is due to Negrepontis [N]).

We are going to treat X like Kunen treated ω, so we have to make appropriate adaptations of Kunen's definitions.

3.2. *Definition.* Let $\mathcal{F}$ be a closed filter on X and assume that no $F \in \mathcal{F}$ is compact.

If $1 \leq n < \omega$, an indexed family $\{A_i\colon i \in I\}$ of closed subsets of X is *precisely* n-*linked w.r.t.* $\mathcal{F}$ if for all $\sigma \in [I]^n$ and $F \in \mathcal{F}$, $\bigcap_{i\in\sigma} A_i \cap F$ is not compact, but for all $\sigma \in [I]^{n+1}$, $\bigcap_{i\in\sigma} A_i$ is compact.

An indexed family $\{A_{in}\colon i \in I, 1 \leq n < \omega\}$ is a *linked system w.r.t.* $\mathcal{F}$ if for each n, $\{A_{in}\colon i \in I\}$ is precisely n-linked w.r.t. $\mathcal{F}$, and for each n and i, $A_{in} \subset A_{i,n+1}$.

An indexed family $\{A^j_{in}\colon i \in I, 1 \leq n < \omega, j \in J\}$ is an I *by* J *independent linked family w.r.t.* $\mathcal{F}$ if for each $j \in J$, $\{A^j_{in}\colon i \in I, 1 \leq n < \omega\}$ is a linked system w.r.t. $\mathcal{F}$, and:

$$\bigcap_{j\in\tau} \Big(\bigcap_{i\in\sigma_j} A^j_{in_j} \Big) \cap F$$

is not compact, whenever $\tau \in [J]^{<\omega}$, and for each $j \in \tau$, $1 \leq n_j < \omega$ and $\sigma_j \in [I]^{n_j}$ and $F \in \mathcal{F}$.

(All these definitions are copied from Kunen [K_2]).

3.3. *Lemma. There is a* 2^ω *by* 2^ω *independent linked family in* ω *w.r.t. the filter of cofinite sets.*

For a proof of this Lemma see [K_2, 2.2].

3.4. *Theorem. Let* X *be the topological sum of countably many compact nonempty spaces of weight at most* 2^ω, *say* $X_n (n < \omega)$. *Then each nice filter* $\mathcal{F}$ *on* X *can be extended to a closed ultrafilter* p *which is* 2^ω-OK.

Proof. Let $\{Z_\mu : \mu < 2^\omega \;\&\; \mu \text{ is even}\}$ enumerate all nonempty closed G_δ's of X (there are clearly only 2^ω closed G_δ's). Let $\{\langle C_{\mu n} : n < \omega\rangle : \mu < 2^\omega \;\&\; \mu \text{ is odd}\}$ enumerate all sequences of closed nonempty G_δ's satisfying

$$C_{\mu,n+1} \subset \text{int } C_{\mu n} \cap \bigcup_{i \geq n} X_i$$

for each $n < \omega$. Furthermore we assume that each sequence is listed cofinally often. Finally, let $\{A^\beta_{\alpha n} : \alpha < 2^\omega, 1 \leq n < \omega, \beta < 2^\omega\}$ be an independent linked family of ω with respect to the cofinite filter.

By induction on μ we construct $\mathcal{F}_\mu$ and K_μ so that

1) $\mathcal{F}_\mu$ is a closed filter on X, $K_\mu \subset 2^\omega$, and $\{\bigcup\{X_i : i \in A^\beta_{\alpha n}\} : \alpha < 2^\omega, 1 \leq n < \omega, \beta \in K_\mu\}$ is an independent linked family w.r.t. $\mathcal{F}_\mu$;
2) $K_0 = 2^\omega$ and $\mathcal{F}_0 = \mathcal{F}$;
3) $\nu < \mu$ implies $\mathcal{F}_\nu \subset \mathcal{F}_\mu$ and $K_\nu \supset K_\mu$;
4) if μ is a limit ordinal, $\mathcal{F}_\mu = \bigcup_{\nu<\mu} \mathcal{F}_\nu$ and $K_\mu = \bigcap_{\nu<\mu} K_\nu$;
5) for each μ, $K_\mu - K_{\mu+1}$ is finite;
6) if μ is even, either $Z_\mu \in \mathcal{F}_\mu$ or some $F \in \mathcal{F}_\mu$ misses Z_μ;
7) if μ is odd and each $C_{\mu n} \in \mathcal{F}_\mu$, then there are

$D_{\mu\alpha} \in \mathcal{J}_{\mu+1}$ for $\alpha < 2^\omega$ such that for all $n \geq 1$ and all $\alpha_1 < \alpha_2 < \cdots < \alpha_n < 2^\omega$, $(D_{\mu\alpha_1} \cap \cdots \cap D_{\mu\alpha_n}) - C_{\mu n}$ has compact closure.

Notice that since $\mathcal{J}$ is a nice filter, the collection $\{\cup\{X_i: i \in A^\beta_{\alpha n}\}: \alpha < 2^\omega, 1 \leq n < \omega, \beta < 2^\omega\}$ is indeed an independent linked family w.r.t. $\mathcal{J}$.

That this construction can be carried out follows by precisely the same argumentation as in Kunen [K_2, the proof of theorem 3.1]. The only place where the proof, modulo some obvious adaptations, is different, is at the end, namely in the case that μ is odd and each $C_{\mu n} \in \mathcal{J}_\mu$. Now the "refinement system" for $\langle C_{\mu n}: n < \omega\rangle$ must be defined as follows:

$$D_{\mu\alpha} = \bigcup_{1 \leq n < \omega} \cup \{X_i: i \in A^\beta_{\alpha n}\} \cap (C_{\mu n} - \operatorname{int} C_{\mu, n+1}).$$

For details we refer to [K_2, 3.1]. Now let p be the ultrafilter generated by $\cup_\mu \mathcal{J}_\mu$. Then p is as required.

3.5. *Proof of Theorem* 0.1. We only show that some $p \in X$ is a weak P-point but not a P-point; one can find 2^{2^ω} such points by combining our proof with the argument in [K_2, 0.1].

Let $Z_n (n < \omega)$ be a sequence of nonempty closed G_δ's of X such that

$$Z_n \cap (\cup\{Z_i: i \neq n\})^- = \emptyset.$$

Put $Y = \bigcup_{n<\omega} Z_n$ and observe that Y is C*-embedded in X, or, equivalently, $\beta Y = \overline{Y}$. Let $\mathcal{J}$ be a nice filter for Y as described in Lemma 2.1. By Theorem 3.4, $\mathcal{J}$ can be extended to a closed ultrafilter $p \in Y^*$ which is 2^ω-OK. Let $D \subset X - \{p\}$ be countable. Take a neighborhood U of p in X which misses $D \cap (\overline{Y} - Y)$ (Lemma 3.1). By Lemma 2.1 some $F \in \mathcal{F}$ misses $D - Y^*$; hence,

$$\overline{F} \cap (D - Y^*)^- = \emptyset.$$

Since $p \in \overline{F}$ we can find a neighborhood V of p which misses $D - Y^*$. Then $U \cap V$ does not intersect D. Hence p is a weak P-point; clearly p is not a P-point.

3.6. *Remark*. In fact, Theorem 0.1 can be generalized. With the same proof it follows that each compact F-space X of weight 2^{ω} which can be mapped onto a compact F-space without isolated points in which each nonempty G_{δ} has nonempty interior contains a weak P-point.

3.7. *Remark*. Notice that Theorem 0.1 implies that each compact F-space of weight 2^{ω} in which each nonempty G_{δ} has nonempty interior contains a weak P-point.

4. Remote Points

The point $x \in X^*$ is called a *remote point* of X provided that

$$x \notin cl_{\beta X}A$$

for each nowhere dense set $A \subset X$. Van Douwen [vD, 4.2] and Chae and Smith [CS] have shown that each nonpseudocompact space of countable π-weight[1] has a remote point. Subsequently, van Mill [vM_1] showed that, more generally, each nonpseudocompact space which is a product of at most ω_1 spaces of countable π-weight has a remote point. For more

[1]A π-basis $\mathcal{B}$ for X is a collection of nonempty open subsets of X such that each nonempty open set in X contains an element of $\mathcal{B}$. The π-weight, $\pi(X)$, is $\omega.\min\{|\mathcal{B}|: \mathcal{B}$ is a π-basis for $X\}$.

information concerning remote points, see [vD], [vDvM], [CS], [KvMM], [vM_1], [Wo].

4.1. *Theorem. Let* X *be a locally compact normal non-pseudocompact space which is a product of at most* ω_1 *spaces of countable* π*-weight. Then* X *has a remote point* x *which is also a* 2^ω-OK *point.*

Proof. The "remote filter" $\mathcal{F}$ for X constructed in [vM_1] is defined on a discrete sequence of compact subspaces while this filter in addition is nice. Now apply Theorem 3.4.

For each space X let EX be the unique extremally disconnected space which admits a perfect irreducible map onto X. The space EX is called the *projective cover* of X (for a beautiful survey on projective covers, see Woods [Wo]).

4.2. *Proof of Theorem* 0.2. The theorem is trivial in case E has an isolated point, so assume that E has no isolated points. Theorem 4.1 implies that there is a sequence $\{C_n: n < \omega\}$ of pairwise disjoint nonempty clopen sets such that

$$C = \bigcup_{n<\omega} C_n$$

has a remote point x which is also a 2^ω-OK point (observe that whenever $f: Z_0 \to Z_1$ is perfect and irreducible and Z_0 is normal, that $|\beta f^{-1}(p)| = 1$ for each remote point p of Z_1 (βf is the Stone extension of f)). We claim that $x \notin \overline{D}$ for each countable discrete $D \subset E - \{x\}$. This is trivial however, since x is 2^ω-OK and $\overline{D} \cap C$ is nowhere dense in C for each countable discrete $D \subset E - (\overline{C} - C)$.

5. Proof of Theorem 0.3

Let us recall that (*) denotes the statement that there is a compactification $\gamma\omega$ of ω such that $\gamma\omega - \omega$ is ccc and not separable.

5.1. *Lemma.* CH *implies* (*) *and* (*) *implies that there is a compactification* $\gamma\omega$ *of* ω *such that* $\gamma\omega - \omega$ *is ccc and nowhere separable.*

Proof. By Tall [T, Ex. 7.5], the Stone space of the Boolean algebra of Lebesgue measurable subsets of [0,1] modulo the nullsets is a compact extremally disconnected ccc nonseparable space of weight 2^{ω}. Under CH, each compact space of weight at most 2^{ω} is a continuous image of ω^*, or, equivalently, is the remainder of some compactification of ω (cf. Parovičenko [P]). Hence CH implies (*).

Now assume that $b\omega$ is a compactification of ω such that $b\omega - \omega$ is ccc and not separable. Let $\mathcal{U}$ be a maximal disjoint family of nonempty separable open subsets of $b\omega - \omega$. Then $|\mathcal{U}| \leq \omega$ and $\cup\mathcal{U}$ is not dense. Let V be a nonempty open F_σ of $b\omega - \omega$ such that $\overline{V} \cap \overline{\cup\mathcal{U}} = \emptyset$. Now let $\gamma\omega$ be the quotient space one obtains from $b\omega$ by collapsing $(b\omega - \omega) - V$ to a single point.

For each space X let RO(X) be the Boolean algebra of regular open subsets of X. It is clear that $|RO(X)| \leq w(X)^{c(X)}$, where w(X) and c(X) denote the weight and cellularity of X. If $f\colon X \to Y$ is a closed irreducible[2] surjection then $f\#\colon RO(X) \to RO(Y)$ defined by

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[2]A continuous surjection $f\colon X \to Y$ is called *irreducible* whenever $f[A] \neq Y$ for each proper closed set $A \subset X$.

$$f\#(U) = Y - f[X - U]$$

clearly is a Boolean isomorphism, hence $|RO(X)| = |RO(Y)| \leq w(Y)^{c(Y)}$. This observation will be used in the remaining part of this section.

5.2. *Proof of Theorem* 0.3. For each $n < \omega$ let Z_n be a nonempty closed G_δ of X such that

$$Z_n \cap (\cup\{Z_i: i \neq n\})^- = \emptyset.$$

Let $\{E_n: n < \omega\}$ be a partition of ω in countably many infinite sets. For each $n < \omega$ let $\mathcal{F}_n$ be a nice filter on $\cup\{Z_i: i \in E_n\}$ as described in Lemma 2.1. For each $n < \omega$ put

$$F(n) = \bigcap_{F \in \mathcal{F}_n} \overline{F}.$$

Notice that $F(n) \cap F(m) = \emptyset$ whenever $n \neq m$ and that $\bigcup_{n<\omega} F(n)$ is C*-embedded in X. Define $f: \bigcup_{n<\omega} Z_n \to \omega$ by

$$f(x) = n \iff x \in Z_n;$$

let $f_n = f \restriction \bigcup_{i \in E_n} Z_i$. Let βf_n be the Stone extension of $f_n (n < \omega)$. Since $\bigcup_{n<\omega} Z_n$ is C*-embedded in X,

$$\beta(\cup\{Z_i: i \in E_n\}) = (\cup\{Z_i: i \in E_n\})^-$$

for each $n < \omega$. Put

$$S(n) = (\cup\{Z_i: i \in E_n\})^- - \cup\{Z_i: i \in E_n\}$$

$(n < \omega)$. Then clearly $\beta f_n[S(n)] = E_n^* \approx \omega^*$. Since $\mathcal{F}_n$ is a nice filter we also have that

$$\beta f_n[F(n)] = E_n^*.$$

By (*), let Y be some ccc nowhere separable remainder of a compactification of ω (cf. Lemma 5.1). For each $n < \omega$ let g_n map E_n^* onto Y and let h_n be the composition of $\beta f_n \restriction S(n)$ and g_n. Notice that $h_n[F(n)] = Y$ for each $n < \omega$. For each

$n < \omega$ let $Y(n) \subset F(n)$ be closed such that $h_n \upharpoonright Y(n) \to Y$ is an irreducible surjection. Then $|RO(Y(n))| = |RO(Y)| \leq w(Y)^{c(Y)} \leq (2^\omega)^\omega = 2^\omega$. We conclude that $Y(n)$ has weight 2^ω.

For each countable subset G of $\bigcup_{n<\omega} S(n)$ let $\{U_n(G): n < \omega\}$ be a maximal pairwise disjoint collection of nonempty regular closed sets of Y none of which intersects $(\bigcup_{n<\omega} h_n[G \cap S(n)])^-$. Define

$$L(G) = \bigcup_{n<\omega} (h_n^{-1}[\bigcup_{i\in n} U_i(G)] \cap Y(n)).$$

Notice that $L(G)$ is a closed subset of $\bigcup_{n<\omega} S(n)$ and that $L(G)$ does not intersect the closure (in $\bigcup_{n<\omega} S(n)$) of G. Also,

$$\mathcal{L} = \{L(G): G \text{ is a countable subset of } \bigcup_{n<\omega} S(n)\}$$

is centered and the filter $\mathcal{L}'$ generated by $\mathcal{L}$ is nice. By Theorem 3.4 $\mathcal{L}'$ can be extended to an ultrafilter p which is 2^ω-OK. Since clearly $\bigcup_{n<\omega} Y(n)$ is C*-embedded in X, p is a point of X. We claim that p is a weak P-point of X.

Let $H \subset X - \{p\}$ be countable. Put $Z = (\bigcup_{i<\omega} Z_i)^- - \bigcup_{i<\omega} Z_i$ and let

$$H_0 = H - Z.$$

For each $n < \omega$ there is some $G_n \in \mathcal{F}_n$ such that $G_n \cap H_0 = \emptyset$. By construction of the filters $\mathcal{F}_n$, and since X is an F-space,

$$(\bigcup_{n<\omega} G_n)^- \cap \overline{H}_0 = \emptyset.$$

Since $p \in (\bigcup_{n<\omega} G_n)^-$ this shows that $p \notin \overline{H}_0$. Now, notice that Z is an F-space, being a closed subspace of the compact F-space X, and that each $S(n)$ is a clopen subspace of Z; consequently

$$(\bigcup_{n<\omega} S(n))^- \cap \overline{E} = \emptyset$$

for each countable $E \subset Z - (\bigcup_{n<\omega} S(n))^-$. We conclude that

$$p \notin (H \cap (Z - (\bigcup_{n<\omega} S(n))^-))^-.$$

Now let

$$H_1 = H \cap \bigcup_{n<\omega} S(n).$$

By construction of $\mathcal{L}$ some $L \in \mathcal{L}$ misses the closure (in $\bigcup_{n<\omega} S(n)$) of H_1. Therefore

$$\bar{L} \cap \bar{H}_1 = \emptyset,$$

since disjoint closed sets in $\bigcup_{n<\omega} S(n)$ have disjoint closures in X. This shows that $p \notin \bar{H}_1$. Define

$$H_2 = H \cap ((\bigcup_{n<\omega} S(n))^- - \bigcup_{n<\omega} S(n).$$

If $H_2' = H_2 - (\bigcup_{n<\omega} Y(n))^-$, then by precisely the same technique as in Lemma 3.1 it follows that

$$\overline{H_2'} \cap (\bigcup_{n<\omega} Y(n))^- = \emptyset;$$

we conclude that $p \notin \bar{H}_2'$. Finally, put

$$H_3 = H \cap ((\bigcup_{n<\omega} Y(n))^- - \bigcup_{n<\omega} Y(n)).$$

Then $p \notin \bar{H}_3$ since p is 2^ω-OK. We conclude that $p \notin \bar{H}$, i.e. p is a weak P-point of X. By construction, p is not a P-point.

By making an appropriate adaptation of [K_2, 0.1] one can find 2^{2^ω} weak P-points which are not P-points.

5.3. *Remark*. By a somewhat different and more technical comstruction I have proved that Theorem 0.3 is true if one assumes that $2^\omega = 2^{\omega_1}$. In addition, Theorem 0.3 is true under the following hypothesis.

(**) for each compact ccc space X of weight 2^ω there is a closed P-set[3] in ω^* which can be mapped onto X.

As (*), (**) is true if CH.

5.4. *Question.* Is one of (*), (**) true in ZFC?

5.5. *Question.* Let X be a compact infinite F-space without isolated points in which each nonempty G_δ has nonempty interior. Are there $|X|$ weak P-points in X?

6. Remarks

It is shocking that the answer to a question as simple as:

> is there in ZFC a compactification $\gamma\omega$ of ω such that $\gamma\omega - \omega$ is ccc and not separable[4]

is unknown. The easiest way of solving this question would be to construct a compact ccc nonseparable space of weight ω_1, since each compact space of weight ω_1 is the remainder of some compactification of ω ([P]). However, under MA + $\neg$CH, each compact ccc space of weight less than 2^ω is separable ([T, Theorem 1.4 (a)]), which blocks this attempt (this was brought to my attention by Eric van Douwen).

Let us finally notice that Kunen has asked whether one can delete "F-space" from the hypotheses of our results.

6.1. *Question* (Kunen). Let X be a compact space of weight 2^ω in which each nonempty G_δ has nonempty interior.

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[3] A subset $P \subset X$ is called a *P-set* whenever $P \cap \overline{F} = \emptyset$ for each $F_\sigma F$ of X which misses P.

[4] The author offers a bottle of Jenever (Dutch gin) for the first valid solution of this problem.

Is there a weak P-point in X?

It is clear that this is true under CH.

Remarks added in August 1980: Our question whether a ccc nonseparable growth of ω exists was answered, in the affirmative, by Bell [B]. Theorems 0.1 and 0.3 were generalized by Dow and van Mill [DvM] who showed, using Bell's result, that each compact nowhere ccc F-space contains a weak P-point. Subsequently, Dow [D] showed that each compact F-space of weight greater than 2^{ω} contains a weak P-point. For more generalizations, see [vM_1], [vM_3].

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