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0. Introduction

The motivation for this paper partially arose from the observation that the function $m: I^3 \to I$, where I denotes the closed unit interval [0,1], defined by $m(x,y,z)$ = the middle of x, y and z, is continuous. This function induces a similar function $m_\infty: Q^3 \to Q$ on the Hilbert cube $Q(= I^\infty)$ as follows:

$$m_\infty(x,y,z)_n = \text{the middle of } x_n, y_n \text{ and } z_n.$$

This function has among others the following algebraic property

$$(*) m_\infty(x,x,y) = m_\infty(x,y,x) = m_\infty(y,x,x) = x$$

for all $x,y \in Q$. Since having a function with this property is clearly a retraction invariant it follows that every (compact) AR has such a function. A ternary operation μ on a space X which satisfies (*) will be called from now on a *mixer*. Hence we can reformulate the above observation by saying that every AR has a mixer and the question arises whether every (metrizable) continuum with a mixer is an AR. Note that the above mixer and its "retracts" are also symmetric, i.e. $m_\infty(x,y,z) = m_\infty(x,z,y) = \cdots$. As this condition can be avoided in all of our results we did not include it in our definition.

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A class of spaces with a very "geometric" mixer is the class of all triple-convex subspaces of the Hilbert cube. A subset $X \subset Q$ is called *triple-convex* whenever $m_\infty[X^3] = X$, where m_∞ is defined as above (cf. van Mill & Wattel [6]). It has been proved by van Mill [4] that every compact connected triple-convex subset of Q is an AR, a result which indicates that the answer to the above question might well be in the affirmative. Unfortunately not every AR is realizable as a triple-convex subset of Q. A. Szymański has observed that each compact connected triple-convex subset of Q is even a local AR, so that, for example, the two dimensional AR having the singularity of Mazurkiewicz, described by Borsuk ([1], p. 152), is not realizable as a triple-convex subset of Q. This observation makes our question interesting even in the finite dimensional case.

We will prove that each continuum with a mixer is C^∞ and LC^∞, in particular, such a continuum is locally connected. As a consequence, a finite dimensional continuum X is an AR iff X has a mixer. We were unable to solve our problem in the infinite dimensional case. However, we will show that a contractible continuum with a mixer is EC which shows that for obtaining a counterexample to our question an example like Borsuk's ([1], p. 124) famous contractible and locally contractible compactum which is not an AR is of no help.

All spaces in consideration are compact metric.

1. Spaces with a Mixer are C^∞ and LC^∞

In this section we show that each continuum with a mixer is C^∞ and LC^∞. This allows us to give an internal characterization of finite dimensional AR's.

1.1. *Lemma. Let* X *be a continuum with a mixer. Then* X *is locally connected.*

Proof. Let $U \subset X$ be open and let K be a component of U. We will show that K is open.

Let $\mu: X^3 \to X$ be a mixer and take $x \in K$. Then

$$\mu^{-1}[U] \supset \mu^{-1}(x) \supset (\{x\} \times \{x\} \times X) \cup (\{x\} \times X \times \{x\}) \cup (X \times \{x\} \times \{x\}),$$

and by compactness there is a neighborhood V of x such that

$$\mu^{-1}[U] \supset (V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V).$$

Since clearly $(V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V)$ is connected we conclude that

$$\mu[(V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V)] \subset K$$

and consequently $x \in V \subset K$. Therefore, K is open.

1.2. *Lemma. Let* X *be a compact space, and let* μ: $X^3 \to X$ *be a mixer. If* x_n, y_n, z_n $(n \in \mathbf{N})$ *are points of* X *such that the sequences* $(x_n)_{n\in\mathbf{N}}$ *and* $(y_n)_{n\in\mathbf{N}}$ *both converge to* $a \in X$, *then the sequence* $(\mu(x_n,y_n,z_n))_{n\in\mathbf{N}}$ *converges to* a.

Proof. Let U be a neighborhood of a. As in the proof of the preceding lemma we can find a neighborhood V of a with

$$(V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V) \subset \mu^{-1}[U].$$

Let $n_o \in \mathbf{N}$ be such that $x_n, y_n \in V$ for all $n \geq n_o$. For such n, the triple (x_n,y_n,z_n) is in the left hand set above, whence

$$\mu(x_n,y_n,z_n) \in U$$

for each $n \geq n_o$.

We can now prove our first main theorem.

1.3. *Theorem. Let* X *be a continuum with a mixer. Then* X *is* C^∞ *and* LC^∞.

Proof. We will only show that X is LC^∞. The proof that X is C^∞ is similar, though easier. Let $\mu: X^3 \to X$ be a mixer. By Lemma 1.1, X is locally path connected (LC^0) by our assumption on metrizability. Let $n \geq 1$, and let U be a neighborhood of $x \in X$. As above, we can find a neighbor-hood V of x with

$$(V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V) \subset \mu^{-1}[U].$$

Let $f: S^n \to V$ be a map. We use the standard representations

$$S^n = \{(x_0,\cdots,x_n) \in \mathbf{R}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\},$$

$$B^{n+1} = \{(x_0,\cdots,x_n) \in \mathbf{R}^{n+1} : \sum_{i=0}^{n} x_i^2 \leq 1\}.$$

Let $u \in B^{n+1}$ be defined by

$$u_0 = 1; \quad u_i = 0 \text{ for } 1 \leq i \leq n.$$

For each $v \in B^{n+1}$ the equation

$$\sum_{i=0}^{n-1} v_i^2 + y^2 = 1$$

has exactly two solutions $y = g_1(v) \geq 0$ and $y = g_2(v) \leq 0$ each depending continuously on v.
For each $v \in B^{n+1}\backslash\{u\}$ the line through u and v meets $S^n\backslash\{u\}$ in exactly one point $g_3(v)$ depending continuously on v. We put $g_3(u) = u$ for convenience.

This leads us to a map

$$g = (g_1,g_2,g_3): B^{n+1} \to (S^n)^3$$

which is continuous in all points $v \neq u$. Define $\overline{f}: B^{n+1} \to X$ as the composition

$$B^{n+1} \xrightarrow{g} (S^n)^3 \to X^3 \xrightarrow{\mu} X,$$

where the map in the middle is (f,f,f). Then $\overline{f}$ extends f since for each $\nu \in S^n$, two points out of $g_1(\nu)$, $g_2(\nu)$, $g_3(\nu)$ equal ν. Also, $\overline{f}$ is continuous in each point $\nu \in B^{n+1}\setminus\{u\}$. Let $(a_n)_{n\in\mathbf{N}}$ be a sequence in $B^{n+1}\setminus\{u\}$ converging to u. Then $fg_1(a_n)$ and $fg_2(a_n)$ both converge to $f(u)$, whence by Lemma 1.2

$$\overline{f}(a_n) = \mu(fg_1(a_n), fg_2(a_n), fg_3(a_n))$$

converges to $f(u)$. This proves continuity of $\overline{f}$.

Finally note that for each $\nu \in S^n$ we have $f(\nu) \in V$. By the construction of V, we find that $\overline{f}[B^{n+1}] \subset U$. Hence, X is LC^n for each $n \geq 1$.

Observe that the above extension $\overline{f}$ of f is obtained through a constant (i.e. not depending on f) procedure on B^{n+1}, resulting into a map which is then composed with f.

1.4. *Corollary. Let* X *be a finite dimensional continuum. Then* X *is an* AR *iff it admits a mixer.*

Proof. This is a direct consequence of Theorem 1.3 and Borsuk ([1], p. 122).

2. EC Structures

A *local equiconnecting function* (cf. Fox [3]) for a space Y is a map $\lambda: U \times I \to Y$, where U is a neighborhood of the diagonal in $Y \times Y$, such that $\lambda(y_0,y_1,i) = y_i$ $(i \in \{0,1\})$, and $\lambda(y,y,t) = y$, for every $y_0,y_1,y \in Y$, $t \in I$. An *equiconnecting function* for a space Y is a local equiconnecting function the domain of which is $Y \times Y \times I$. We say that Y is EC (LEC) if it admits an equiconnecting function (a local equiconnection function).

2.1. *Theorem. Let* X *be a continuum with a mixer. If* X *is contractible then* X *admits an equiconnecting function.*

Proof. Let $H: X \times I \to X$ be a homotopy which is the identity at stage 0 and which is constant at stage 1. In addition, let μ be a mixer. Define an equiconnecting function λ by

$$\lambda(x_1,x_2,t) = \begin{cases} \mu(x_1,x_2,H(x_1,2t)) & (t \le 1/2) \\ \mu(x_1,x_2,H(x_2,2-2t)) & (t \ge 1/2). \end{cases}$$

The check that λ is indeed an equiconnecting function is left to the reader.

As a corollary we obtain that Borsuk's [1] contractible and locally contractible compactum which is not an AR does not have a mixer since Dugundji [2] has shown that this space is not LEC.

The contractibility condition in Theorem 2.1 is an unpleasant limitation. In van Mill & van de Vel [5] this condition will be weakened considerably. There we will show, using the idea of the proof of Theorem 2.1, that whenever X has a (local) mixer and has an open cover by sets contractible within X then X is LEC (also for non-compact X).

Let μ be a mixer on X. If $a,b \in X$ are distinct such that $\mu(a,b,x) = x$ for each $x \in X$, then we say that a and b are *endpoints* for μ, and we say μ is a *mixer with endpoints*.

2.2 *Theorem. Each* AR *has a mixer with endpoints and each continuum having a mixer with endpoints is contractible, hence even* EC.

Proof. Let X be a non-degenerate AR and embed X in Q in such a way that both $(0,0,\cdots)$ and $(1,1,\cdots)$ belong to X. Let $r\colon Q \to X$ be a retraction and let m_∞ be as in the introduction. Define $\mu\colon X^3 \to X$ by

$$\mu(x,y,z) = rm_\infty(x,y,z).$$

Clearly μ is a mixer with endpoints.

Now let $\mu\colon X^3 \to X$ be a mixer with endpoints, say $\mu(a,b,x) = x$ for each $x \in X$ and distinct $a,b \in X$. By Lemma 1.1 X is a Peano continuum, so X is path-connected. Fix $c \in X - \{a,b\}$ and let $f_0\colon I \to X$ be a path with $f_0(0) = a$ and $f_0(1) = c$ and let $f_1\colon I \to X$ be a path with $f_1(0) = b$ and $f_1(1) = c$. Now define $H\colon X \times I \to X$ by

$$H(x,t) = \mu(f_0(t),f_1(t),x).$$

It is easily seen that H is a contraction. By using Theorem 2.1 we find that X is even EC.

3. Remarks

An important construction related to a mixer $\mu\colon X^3 \to X$ is to obtain neighborhoods $V \subset U \subset X$ with

$$(V \times V \times X) \cup (V \times X \times V) \cup (X \times V \times V) \subset \mu^{-1}[U].$$

The above results can be adapted for "local" mixers provided one requires such a map to be defined on a neighborhood of the diagonal of X^3 containing enough small sets of the above type. This will be investigated in a forthcoming paper of the authors, in which we will also obtain results for non-compact spaces (cf. [5]).

References

[1] K. Borsuk, *Theory of retracts*, Warszawa (1967).

[2] J. Dugundji, *Locally equiconnected spaces and absolute*

neighborhood retracts, Fund. Math. 57 (1965), 187-193.

[3] R. H. Fox, *On fiber spaces*, II, Bull. Amer. Math. Soc. 49 (1943), 733-735.

[4] J. van Mill, *The superextension of the closed unit interval is homeomorphic to the Hilbert cube*, Fund. Math. 103 (1979), 151-175.

[5] ________ and M. van de Vel (in preparation).

[6] J. van Mill and E. Wattel, *An external characterization of spaces which admit binary normal subbases*, Am. J. Math. 100 (1978), 987-994.

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