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A NONSUPERCOMPACT CONTINUOUS IMAGE OF A SUPERCOMPACT SPACE

Charles F. Mills and Jan van Mill*

ABSTRACT. We give an example of a nonsupercompact continuous image of a supercompact space.

0. Introduction. This paper deals with supercompact spaces. A space is called *supercompact* (cf. de Groot [7]) provided it has a closed subbase such that any of its linked subsystems (a system of sets is called *linked* if any two of its members meet) has nonempty intersection. Much work has been done to show that certain spaces are supercompact and that certain spaces are not supercompact. We want to mention Strok & Szymański [11], who showed that all compact metric spaces are supercompact (easier proofs are available now, see van Douwen [4] and Mills [9]) and Bell [2] who gave the first examples of compact (Hausdorff) spaces which are not supercompact. Another big class of supercompact spaces has recently been discovered. Mills [10] has shown that every compact topological group is supercompact.

It has been open for some time whether every dyadic space (i.e. a space which is a continuous image of a family of two point discrete spaces) is supercompact. Notice that every compact topological group is dyadic (cf. Kuz'minov [8]) so that Mills' result stated above gives a partial answer to this question. Since every dyadic space is the continuous image of a supercompact space, the question arises whether the continuous image of a supercompact space is supercompact (cf. van Douwen & van Mill [6]). We will give an example of a nonsupercompact space which is a continuous image of a supercompact space (all our spaces are Hausdorff; a T_1 example was earlier given by Verbeek [12].). In addition we will prove a theorem which as an application allows us to give a surprisingly simple proof that a space like $\beta\omega$ is not supercompact.

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We finally want to note that the question whether a closed G_δ subset of a supercompact space is supercompact (cf. van Douwen & van Mill [6]) was answered in the negative recently by Bell [3].

1. **Supercompact spaces.** All topological spaces under discussion are assumed to be Tychonoff.

A closed subbase $\mathcal{S}$ for a space X with the property that any of its linked subsystems has nonempty intersection usually is called *binary*. It is an easy observation that a space X has a binary subbase iff X has a binary subbase closed under arbitrary intersections. We will assume, from now on, that every binary subbase is closed under arbitrary intersections.

Let $\mathcal{S}$ be a binary subbase for X. For $A \subset X$ define $I(A) \subset X$ by

$$I(A) = \cap\{S \in \mathcal{S} \mid A \subset S\}.$$

Notice that $I(A) \in \mathcal{S}$, hence that $cl_X(A) \subset I(A)$, that $I(I(A)) = I(A)$ and that $I(A) \subset I(B)$ if $A \subset B$, for all $A,B \subset X$. The following simple lemma is due to van Douwen & van Mill [6]. Although it is simple, it is quite useful.

1.1. LEMMA. *Let $\mathcal{S}$ be a binary subbase for X and let $p \in X$. If U is a neighborhood of p and if A is a subset of X with $p \in cl_X(A)$, then there is a subset B of A with $p \in cl_X(B)$ and $I(B) \subset U$.*

This lemma is the basic tool in proving the main result of this section. Recall that d(X) denotes the density of a space X. We give ordinals the order topology; a cardinal number is an initial ordinal number. A set $A \subset X$ is called a $G_{\delta,\kappa}$ set if A is an intersection of κ open sets in X.

1.2. THEOREM. *Let X be a space and suppose that X contains a closed $G_{\delta,d(X)}$ set which can be mapped onto $d(X)^+ + 1$. Then X is not a continuous image of a supercompact space.*

PROOF. To the contrary assume that Y is a supercompact space, with binary subbase $\mathcal{S}$, which can be mapped by f onto X. For convenience set $\kappa = d(X)$. Let A be a closed $G_{\delta,\kappa}$ set in X which admits a surjective mapping $g\colon A \to \kappa^+ + 1$. Write $X - A = \cup_{\alpha<\kappa} C_\alpha$, where the C_α's are closed. Let $D = \{d_\alpha \mid \alpha < \kappa\}$ be a dense subset for X and for each $\alpha < \kappa$ take a point $d'_\alpha \in Y$ such that $f(d'_\alpha) = d_\alpha$. Let D′ be the set of points thus obtained. Define

$$Z = \mathrm{cl}_Y(D') \cap f^{-1}(A).$$

Since D is dense and f is closed we see that $f[Z] = A$.

By induction we will construct for each $\alpha < \kappa$ a set $F_\alpha \in \mathcal{S}$ with the following properties:

(i) $F_\alpha \subset \cap_{\beta<\alpha} F_\beta$;

(ii) $g[f[F_\alpha \cap Z]] \cap \kappa^+$ is unbounded;

(iii) $F_\alpha \subset Y - f^{-1}(C_\alpha)$.

Suppose that the F_α's are constructed for all $\alpha < \beta$. Define $F = \cap_{\alpha<\beta} F_\alpha$ if $\beta > 0$ and $F = Y$ if $\beta = 0$. By (i) and by the compactness of Y and X we have that

$$g[f[F \cap Z]] = \cap_{\alpha<\beta} g[f[F_\alpha \cap Z]]$$

and hence, by (ii), $g[f[F \cap Z]] \cap \kappa^+$ is unbounded. Since $\mathcal{S}$ is a subbase there is a finite $\mathcal{F}_\beta \subset \mathcal{S}$ such that

$$f^{-1}(A) \subset \cup \mathcal{F}_\beta \subset Y - f^{-1}(C_\beta).$$

Define $\hat{\mathcal{F}}_\beta = \{H \cap (F \cap Z) | H \in \mathcal{F}_\beta\}$. Then $g[f[F \cap Z]] = \cup\{g[f[E]] | E \in \hat{\mathcal{F}}_\beta\}$ and consequently there is an $E \in \hat{\mathcal{F}}_\beta$ such that $g[f[E]] \cap \kappa^+$ is unbounded. Take $H \in \mathcal{F}_\beta$ such that $H \cap (F \cap Z) = E$ and define $F_\beta = H \cap F$. It is clear that F_β defined in this way satisfies our inductive assumptions.

By the compactness of X and Y we have, by (i), that

$$(*) \quad \cap_{\alpha<\kappa} g[f[F_\alpha \cap Z]] = g[f[\cap_{\alpha<\kappa} F_\alpha \cap Z]].$$

Define $F = \cap_{\alpha<\kappa} F_\alpha$. Notice that $F \subset f^{-1}(A)$. By (*) and by (ii) we also have that $g[f[F \cap Z]] \cap \kappa^+$ is unbounded.

For each $\alpha < \kappa$ take a point

$$e_\alpha \in \cap_{y \in F} I(\{d'_\alpha, y\}) \cap F.$$

Notice that it is possible to take such a point since $\mathcal{S}$ is binary. Define $E = \{e_\alpha | \alpha < \kappa\}$. Notice that $E \subset F$.

CLAIM. $f[F \cap Z] \subset \mathrm{cl}_X(f[E])$.

Indeed, take $x_0 \in f[F \cap Z]$ and let U be any open neighborhood of x_0. Take $y_0 \in F \cap Z$ such that $f(y_0) = x_0$. Since $y_0 \in Z \subset \mathrm{cl}_Y(D')$, by Lemma 1.1 there is a subset $B \subset D'$ such that $y_0 \in \mathrm{cl}_Y(B)$ and $I(B) \subset f^{-1}(U)$. Take $d'_\alpha \in B$ arbitrarily. Then

$$e_\alpha \in \cap_{y \in F} I(\{d'_\alpha, y\}) \cap F \subset I(\{d'_\alpha, y_0\}) \subset I(B) \subset f^{-1}(U).$$

Hence $f(e_\alpha) \in U$. This proves the claim.

We conclude that $g[f[F \cap Z]] \subset g[cl_X(f[E])] \subset cl_{\kappa^+ +1}(g[f[E]])$. Since $|E| = \kappa$ this contradicts the fact that $g[f[F \cap Z]] \cap \kappa^+$ is unbounded.

Notice that this theorem gives an easy proof that no compact preimage of $\beta\omega$ (i.e. for example $\beta\omega - \omega$) is supercompact since clearly $\beta\omega - \omega$ can be mapped onto $\omega_1 + 1$. The first proof that $\beta\omega$ is not supercompact was found by Bell [2].

2. **The example.** In this section we will present an example of a compact space D which is not embeddable as a G_δ subset of a supercompact space while it is a continuous image of a supercompact space under a two to one mapping. That D could be a candidate to be a nonsupercompact continuous image of a supercompact space was suggested to us by Eric van Douwen.

Recall that a cardinal κ is *regular* if κ is not the sum of fewer, smaller cardinals. In addition, a set C is a *cub* in κ if it is closed and unbounded in κ. Also, $S \subset \kappa$ is called *stationary* in κ if $S \cap C \neq \phi$ for every set C which is a cub in κ. We heavily rely on the following lemma which is well known; short proofs are to be found in van Douwen & Lutzer [5] and Baumgartner & Prikry [1].

2.1. LEMMA. (*Pressing Down Lemma*): *Let* S *be a stationary subset of a regular uncountable cardinal* κ *and suppose that* $f: S \to \kappa$ *is a function such that* $f(x) < x$ *for all* $x \in S - \{0\}$. *Then for some* $y \in \kappa$ *the set* $f^{-1}[\{y\}]$ *is stationary in* κ.

We will refer to Lemma 2.1 as PDL.

We will now describe D. Let $X = (\omega_1 + 1) \times (\omega_1 + 1)$ and $Z = \{\langle\alpha,\beta\rangle \in X | \beta \leqslant \alpha\}$. It is easily seen that Z is supercompact. Indeed, the collection

$$\{([\alpha,\beta] \times [\alpha^1,\beta^1]) \cap Z | \alpha,\beta,\alpha^1,\beta^1 \in \omega_1 + 1\}$$

is a binary subbase for Z. Now, D is the quotient space obtained from Z by collapsing for each $\alpha < \omega_1$ the set $\{\langle\alpha,\alpha\rangle,\langle\omega_1,\alpha\rangle\}$ to one point. We will show that D is not embeddable as a G_δ subset of a supercompact space. To the contrary, assume that Y is a supercompact space with binary subbase $\mathcal{S}$ which contains D as a G_δ. We will derive a contradiction. To simplify the notation let us make the following conventions:

(i) the point $\{\langle\alpha,\alpha\rangle,\langle\omega_1,\alpha\rangle\}$ of D will be denoted by p_α and $\langle\omega_1,\omega_1\rangle$ is p_{ω_1}.

(ii) $P = \{p_\alpha | \alpha \leqslant \omega_1\}$ and $Z^* = D - P$;

(iii) for each $\langle\gamma,\beta\rangle \in Z^*$ we set $T(\langle\gamma,\beta\rangle,p_\beta) = \{p_\beta\} \cup \{\langle\delta,\beta\rangle \in Z^* | \delta \geqslant \gamma\}$.

Now fix $\alpha < \omega_1$. Notice that for each γ with $\alpha < \gamma < \omega_1$ the set $T(\langle\gamma,\alpha\rangle,p_\alpha)$ is a G_δ in Y.

CLAIM. If $\alpha < \gamma < \omega_1$ we can find $S_\gamma \in \mathcal{S}$ such that $S_\gamma \subset T(\langle\gamma,\alpha\rangle,p_\alpha)$ and p_α is a limit point of S_γ.

We use a technique similar to that used in the proof of Theorem 1.2. Say $T(\langle\gamma,\alpha\rangle,p_\alpha) = \cap_{n<\omega} U_n$, where each U_n is open in Y. For each $n < \omega$ there is a finite subcollection $\boldsymbol{F}_n \subset \mathcal{S}$ such that

$$T(\langle\gamma,\alpha\rangle,p_\alpha) \subset \cup \boldsymbol{F}_n \subset U_n.$$

But then there must be an $F_n \in \boldsymbol{F}_n$ such that F_n meets $T(\langle\gamma,\alpha\rangle,p_\alpha)$ in an unbounded set; that is

$$\{\beta < \omega_1 | \langle\beta,\alpha\rangle \in F_n\}$$

is cub. But then $\cap_{n\in\omega} F_n$ is cub, and the claim is proved.

We claim further that if $T = \{\min(\cap \mathcal{A}) | \mathcal{A} \subset \{S_\gamma | \alpha < \gamma < \omega_1\}\}$ then T contains a cub. For assume not: then there is a stationary $S \subset \omega_1 \times \{\alpha\}$ disjoint from T. Then also $A = \cup\{S_\gamma \cap S | \alpha < \gamma < \omega_1\}$ is stationary. Define $h: A \to \omega_1 \times \{\alpha\}$ by $h(a) = \min \cap \{S_\gamma | \alpha < \gamma < \omega_1$ and $a \in S_\gamma\}$. Since $S \cap T = \phi$ we have $h(a) < a$ for all $a \in A$. By PDL there is a stationary $B \subset A$ and a $\gamma < \omega_1$ such that for all $\langle\beta,\alpha\rangle \in B$ we have that $h(\langle\beta,\alpha\rangle) = \langle\gamma,\alpha\rangle$. Since B is stationary it meets $S_{\gamma+1}$; say $\langle\delta,\alpha\rangle \in B \cap S_{\gamma+1}$. But since $S_{\gamma+1} \subset T(\langle\gamma+1,\alpha\rangle,p_\alpha)$ we have that $h(\langle\delta,\alpha\rangle) \neq \langle\gamma,\alpha\rangle$, which is a contradiction. We have proved:

FACT 1. For every $\alpha < \omega_1$ there is a cub $C_\alpha \subset \omega_1$ such that for every $\langle\beta,\alpha\rangle \in C_\alpha \times \{\alpha\}$ we have that

$$I(\{\langle\beta,\alpha\rangle,p_\alpha\}) \subset T(\langle\beta,\alpha\rangle,p_\alpha).$$

Now take $\beta \in C_\alpha$. We claim that there is $h(\beta) < \beta$ such that for all $\gamma \in (h(\beta),\beta] \cap C_\alpha$ the point $\langle\beta,\alpha\rangle$ is in $I(\{\langle\gamma,\alpha\rangle,p_\alpha\})$. If β is a successor, then there is nothing to prove, so suppose that β is a limit ordinal. Now suppose that our claim is not true. Then there are $\eta_n < \beta (n < \omega)$ belonging to C_α such that $\sup\{\eta_n | n < \omega\} = \beta$ while in addition $\langle\beta,\alpha\rangle \notin I(\{\langle\eta_n,\alpha\rangle,p_\alpha\})$ for every $n < \omega$. There is a clopen neighborhood U of $\langle\beta,\alpha\rangle$ in D such that U does not intersect $T(\langle\beta,\alpha\rangle,p_\alpha) - \{\langle\beta,\alpha\rangle\}$. Let V be any open set in Y such that $V \cap D = U$. By Lemma 1.1 there is a set $E \subset \{\langle\eta_n,\alpha\rangle | n < \omega\}$ such that $\sup E = \langle\beta,\alpha\rangle$ while moreover $I(E) \subset U$. Take $\langle\eta_n,\alpha\rangle \in E$ arbitrarily. Then

$I(\{\langle\eta_n,\alpha\rangle,\langle\beta,\alpha\rangle\}) \subset I(E) \subset U$. Consequently

$$I(\{\langle\eta_n,\alpha\rangle,\langle\beta,\alpha\rangle\}) \cap I(\{\langle\beta,\alpha\rangle,p_\alpha\}) \cap I(\{\langle\eta_n,\alpha\rangle,p_\alpha\})$$

$$\subset U \cap T(\langle\beta,\alpha\rangle,p_\alpha) \cap T(\langle\eta_n,\alpha\rangle,p_\alpha)$$

$$\subset \{\langle\beta,\alpha\rangle\} \cap T(\langle\eta_n,\alpha\rangle,p_\alpha) = \phi,$$

which contradicts the binarity of S.

Now, by PDL, there is a stationary $S \subset C_\alpha$ and there is a $\gamma < \omega_1$ such that for each $s \in S$, $h(s) = \gamma$. Then by the definition of h, for each $\beta > \gamma$ we have that $(S - \beta) \times \{\alpha\} \subset I(\{\langle\beta,\alpha\rangle,p_\alpha\})$. We have verified the following fact.

FACT 2. For every $\alpha < \omega_1$ there is a stationary $D_\alpha \subset C_\alpha$ and a $\gamma_\alpha < \omega_1$ such that for every $\beta \geqslant \gamma_\alpha$ we have that $(D_\alpha \times \{\alpha\}) \cap T(\langle\beta,\alpha\rangle,p_\alpha) \subset I(\{\langle\beta,\alpha\rangle,p_\alpha\})$.

We may assume that $\gamma_\alpha \in D_\alpha$ and also that $\gamma_\alpha > \alpha$. It is an easy consequence of PDL that we may find $\eta < \omega_1$ for which there is a sequence $\{\alpha_n | n < \omega\}$ such that $\langle\gamma_{\alpha_n},\alpha_n\rangle$ converges to $\langle\eta,\eta\rangle$. For suppose that such a sequence cannot be found. Then, for each $\beta < \omega_1$ set $h(\beta) = \sup\{\alpha < \omega_1 | \alpha < \beta \text{ and } \gamma_\alpha < \beta\}$. Then h presses down and is monotone. By PDL there is a stationary $S \subset \omega_1$ and a $\gamma < \omega_1$ such that $h(\beta) = \gamma$ for all $\beta \in S$. Take $\alpha \in S$ such that $\alpha > \gamma$. There is a $\xi \in S$ such that $\xi > \gamma_\alpha$. Then $h(\xi) > \gamma$, which is a contradiction.

Write $T_\eta = \{\langle\alpha,\beta\rangle \in Z^* | \max\{\alpha,\beta\} \leqslant \eta\} \cup \{p_\alpha | \alpha \leqslant \eta\}$. We claim that there are infinitely many n such that $I(\{p_{\alpha_n},p_\eta\}) \cap D \subset T_\eta$. It suffices to prove that every infinite subset of ω contains such n. Let $\{k_n | n < \omega\} \subset \omega$ be strictly increasing. Then $p_{\alpha_{k_n}}$ converges to p_η. For $\beta < \omega_1$, define

$$T_\eta^\beta = \{\langle\gamma,\xi\rangle \in Z^* | \gamma > \beta \text{ and } \xi \leqslant \eta\} \cup T_\eta.$$

Then T_η^β is an open neighborhood of p_η in D. By Lemma 1.1 there is an $n(\beta) < \omega$ such that $I(\{p_{\alpha_{k_{n(\beta)}}},p_\eta\}) \cap D \subset T_\eta^\beta$. There is $m < \omega$ such that $E = \{\beta < \omega_1 | n(\beta) = m\}$ is uncountable. Then

$$I(\{p_{\alpha_{k_n}},p_\eta\}) \cap D \subset \cap_{\beta\in E} T_\eta^\beta = T_\eta,$$

which proves the claim. Hence we may assume that $I(\{p_{\alpha_n},p_\eta\}) \cap D \subset T_\eta$ for all $n < \omega$. By fact (1) and fact (2) we have that

(*) $I(\{p_{\alpha_n},p_\eta\}) \cap I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_{\alpha_n}\}) = \{p_{\alpha_n}\}$

for every $n < \omega$. Set $E = \bigcap_{n<\omega} \overline{D}_{\alpha_n}$ (closure is taken in ω_1). Then E is closed and unbounded since D_{α_n} is stationary. Take $\beta \in E$ such that $\beta > \eta$. Define $A = (\{\beta\} \times \beta) \cup \{p_\beta\}$. Then A is closed in Y and does not contain p_η. By Lemma 1.1 there is an $n < \omega$ for which $I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_\eta\}) \subset Y - A$ while in addition $\gamma_{\alpha_n} < \beta$. We claim that $p_{\alpha_n} \notin I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_\eta\})$. For suppose that p_{α_n} belongs to $I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_\eta\})$. Then $\{p_{\alpha_n},\langle\gamma_{\alpha_n},\alpha_n\rangle\} \subset I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_\eta\})$ and consequently

$$I(\{p_{\alpha_n},\langle\gamma_{\alpha_n},\alpha_n\rangle\}) = I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_{\alpha_n}\}) \subset I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_\eta\}).$$

However, we will prove that $\langle\beta,\alpha_n\rangle \in I(\{p_{\alpha_n},\langle\gamma_{\alpha_n},\alpha_n\rangle\})$ which obviously is a contradiction. Indeed, since $\beta \in \overline{D}_{\alpha_n}$ and since $I(\{p_{\alpha_n},\langle\gamma_{\alpha_n},\alpha_n\rangle\}) \cap (\omega_1 \times \{\alpha_n\})$ is closed in $\omega_1 \times \{\alpha_n\}$ this is a direct consequence of Fact 2.

By (*) above we now have that

$$I(\{p_{\alpha_n},p_\eta\}) \cap I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_{\alpha_n}\}) \cap I(\{\langle\gamma_{\alpha_n},\alpha_n\rangle,p_\eta\}) = \phi,$$

which contradicts the binarity of $\mathcal{S}$.

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University of Wisconsin, Madison
Madison, Wisconsin 53706

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