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NOT EVERY K_1-EMBEDDED SUBSPACE IS K_0-EMBEDDED

JAN van MILL

0. Introduction. All topological spaces under discussion are assumed to be Tychonoff.

For any topological space X let $\tau(X)$ denote the topology of X. If $X \subset Y$ then a function $\kappa : \tau(X) \to \tau(Y)$ is called an *extender* provided that $\kappa(U) \cap X = U$ for all $U \in \tau(X)$. In addition, X is said to be *K_n-embedded* in Y (cf. [**3**]) provided there is an extender $\kappa : \tau(X) \to \tau(Y)$ such that

if $n = 0$ then $\kappa(\emptyset) = \emptyset$ and $\kappa(V) \cap \kappa(W) = \kappa(V \cap W)$ for all $V, W \in \tau(X)$;

if $n > 0$ then $\kappa(V_0) \cap \ldots \cap \kappa(V_n) = \emptyset$ whenever $V_i \cap V_j = \emptyset$ for $0 < i < j \leqq n$ and $V_0, \ldots, V_n \in \tau(X)$.

The extender κ is called a K_n-function (cf. [**3**]).

Eric van Douwen has asked whether there is a space X with a subspace Z which is K_1-embedded but not K_0-embedded. The aim of this note is to answer this question.

Example 0.1. There is a separable first countable compact space X which has a closed subspace Z which is K_1-embedded but not K_0-embedded.

Let n be a positive integer and let $X \subset Y$. An extender $\kappa : \tau(X) \to \tau(Y)$ is called an *M_n-function* (cf. [**2**]) if $\bigcap_{i=0}^n \kappa(U_i) = \emptyset$ for all $U_i \in \tau(X)$ $(i \leqq n)$ satisfying $\bigcap_{i=0}^n U_i = \emptyset$. The subspace X is said to be *M_n-embedded* in Y.

The following example answers another natural question.

Example 0.2. For every $n \geqq 1$ there is a compact space X_n which has a closed subspace Z_n which is M_n-embedded in X_n but which is not M_i-embedded in X_n for all $i > n$.

The spaces X_n in Example 0.2 unfortunately are not first countable.

1. Hyperspace-like extensions. If A is a set and κ is any cardinal, define (as usual)

$$[A]^\kappa := \{B \subset A \mid |B| = \kappa\}$$
$$[A]^{\leqq\kappa} := \{B \subset A \mid |B| \leqq \kappa\}$$
$$[A]^{<\kappa} := \{B \subset A \mid |B| < \kappa\}.$$

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Let X be a topological space and let $n \geqq 3$ be fixed. Define

$$M_n(X) := [X]^{\leqq n} - [X]^2.$$

In addition, for all $A \subset X$ define

$$\langle A \rangle_n := \{F \in M_n(X) \mid |F - A| \leqq 1\} - \{\{x\} \mid x \in X - A\}$$

and

$$(A)_n := \{F \in M_n(X) \mid |F \cap A| \geqq 2\} \cup \{\{x\} \mid x \in A\}$$

respectively.

LEMMA 1.1. *Let X be a topological space and let $n \geqq 3$ be fixed. Then*
(a) $\langle A \rangle_n \subset (A)_n$ *for all* $A \subset X$;
(b) *for any two* $A, B \subset X$, *if* $A \subset B$ *then* $\langle A \rangle_n \subset \langle B \rangle_n$ *and* $(A)_n \subset (B)_n$;
(c) *if* $A \cup B = X$ *then* $\langle A \rangle_n \cup (B)_n = M_n(X)$;
(d) *if* $A, B \subset X$ *and* $A \cap B = \emptyset$ *then* $\langle A \rangle_n \cap (B)_n = \emptyset$.

The simple proof of this lemma is left to the reader.

We now take the collection

$$\{\langle U \rangle_n \mid U \in \tau(X)\} \cup \{(U)_n \mid U \in \tau(X)\}$$

as an open subbase for a topology on $M_n(X)$. By Lemma 1.1 the collection

$$\{\langle Z \rangle_n \mid Z \text{ is a zero-set of } X\} \cup \{(Z)_n \mid Z \text{ is a zero-set of } X\}$$

is a closed subbase for $M_n(X)$ which satisfies the conditions of subbase normality and subbase regularity (in the sense of [**5**]). This implies that $M_n(X)$ is Tychonoff, cf. [**5**].

It is easily seen that the function $i : X \to M_n(X)$ defined by $i(x) := \{x\}$ is a topological embedding. We will identify X and $i[X]$.

LEMMA 1.2. *Let X be a topological space and let $n \geqq 3$ be fixed. Then*
(a) *X is closed in $M_n(X)$;*
(b) *X is first countable if and only if $M_n(X)$ is first countable;*
(c) *X is separable if and only if $M_n(X)$ is separable;*
(d) *X is compact if and only if $M_n(X)$ is compact.*

Proof. The easy proofs of (a), (b) and (c) are left to the reader. To prove (d) first notice that if $M_n(X)$ is compact then by (a) X is compact. Now assume that X is compact. Define $M_2(X) = X$. By induction on n ($n \geqq 2$) we will show that $M_n(X)$ is compact. Clearly $M_2(X)$ is compact. Now assume that $M_{n-1}(X)$ is compact. By the lemma of Alexander we need only show that a cover of type

$$(*) \quad \{\langle U_i \rangle_n \mid U_i \in \tau(X)\ (i \in I)\} \cup \{(V_j)_n \mid V_j \in \tau(X)\ (j \in J)\}$$

has a finite subcover. Since $M_{n-1}(X) \subset M_n(X)$ and since by induction hypothesis $M_{n-1}(X)$ is compact, we may choose a finite $F \subset I$ and a finite $G \subset J$ such that

$$M_{n-1}(X) \subset \bigcup_{i \in F} \langle U_i \rangle_n \cup \bigcup_{j \in G} (V_j)_n.$$

Define

$$Z = \{x = \langle x_1, \ldots, x_n \rangle \in X^n | \forall\, i \in F : | \{x_1, \ldots, x_n\} - U_i| > 1\} \\ \cap \{x \in X^n | \forall j \in G : | \{x_1, \ldots, x_n\} - V_j| > 1\}.$$

It is clear that Z is a closed subspace of the compact space X^n. Suppose that there is an $x = \langle x_1, \ldots, x_n \rangle \in Z$ such that $H = \{x_1, \ldots, x_n\}$ has cardinality less than or equal to 2. Then

$$H \cap (\bigcup_{i \in F} U_i \cup \bigcup_{j \in G} V_j) = \emptyset$$

and since

$$\bigcup_{i \in F} U_i \cup \bigcup_{j \in G} V_j = X$$

this is a contradiction. We conclude that the function $f : Z \rightarrow M_n(X)$ defined by

$$f(\langle x_1, \ldots, x_n \rangle) := \{x_1, \ldots, x_n\}$$

is well-defined. An easy check shows that f is continuous. Hence $f[Z]$ is compact. Obviously

$$M_n(X) - (\bigcup_{i \in F} \langle U_i \rangle_n \cup \bigcup_{j \in G} (V_j)_n) \subset f[Z].$$

We conclude that $(*)$ has a finite subcovering.

2. The examples. We first fix some notation. If A and B are sets, ${}^A B$ is the set of functions from A to B. We are interested in ${}^\alpha 2$, for ordinals $\alpha \leqq \omega$. An element of ${}^\alpha 2$ can be seen as an α-sequence of 0's and 1's. As usual we denote $\bigcup_{n<\omega} {}^n 2$ by ${}^{\underset{\smile}{\omega}}2$. For each $f \in {}^\omega 2$ let

$$I(f) = \{f \restriction n | n \in \omega\},$$

the set of initial sequences of f. It is clear that

(1) if f, $g \in {}^\omega 2$ are distinct, then $I(f) \cap I(g)$ is finite.

Hence, $\{I(f) | f \in {}^\omega 2\}$ is an almost disjoint collection of subsets of the countable set ${}^{\underset{\smile}{\omega}}2$.

The collection $\{I(f) | f \in {}^\omega 2\}$ has an important property:

$(*)$ *for every uncountable subset G of ${}^\omega 2$ there is a $g \in G$ and an infinite $H \subset G - \{g\}$ such that $I(h) \cap I(h') \subset I(g)$ for any two distinct, h, $h' \in H$.*

This was shown in [**4**].

The set $T = {}^{\underset{\smile}{\omega}}2 \cup {}^\omega 2$ is a tree, partially ordered by inclusion, the so-called Cantor tree, cf. [**6**]. The tree T is topologized in the following way: points of

$^{\underset{\smile}{\omega}}2$ are isolated, and a basic neighborhood of $f \in {}^{\omega}2$ contains f and all but finitely many points of $I(f)$.

We can now construct Example 0.1.

2.1. *Construction of Example* 0.1. Let γT be a first countable compactification of T. Such a compactification is described in [**4**]. Let $X = M_3(\gamma T)$ (cf. Section 1) and let $Z = \gamma T$. Then X is separable and first countable (cf. Lemma 1.2).

That Z is K_1-embedded in X is trivial; it is easily seen that $\kappa : \tau(Z) \to \tau(X)$ defined by $\kappa(U) = \langle U \rangle_3$ is a K_1-function.

Let us now show that Z is not K_0-embedded in X. The proof is an adaptation of a proof in [**4**].

To the contrary, assume that $\kappa : \tau(Z) \to \tau(X)$ is a K_0-function. For each $f \in {}^{\omega}2$ let $U(f) = \kappa(I(f) \cup \{f\})$. Then $U(f)$ is a neighborhood of f in X. Since

$$\{\langle V \rangle_3 | f \in V \in \tau(Z)\}$$

is a neighborhood base of f in X (the reader should verify this) we can take $V(f) \in \tau(Z)$ such that

$$f \in V(f) \subset \langle V(f) \rangle_3 \subset U(f) = \kappa(I(f) \cup \{f\}).$$

Since $\{V(f) \cap {}^{\underset{\smile}{\omega}}2 | f \in {}^{\omega}2\}$ has cardinality 2^{ω} there is an uncountable $G \subset {}^{\omega}2$ and a point $p \in {}^{\underset{\smile}{\omega}}2$ such that

$$p \in \textstyle\bigcap_{g \in G} V(g) \cap {}^{\underset{\smile}{\omega}}2.$$

By $(*)$ above there is a $g \in G$ and an infinite $H \subset G - \{g\}$ such that $I(h) \cap I(h') \subset I(g)$ for any two distinct $h,\ h' \in H$. Since $V(h) \cap {}^{\underset{\smile}{\omega}}2$ is infinite for all $h \in H$ we conclude that

$$\{V(h) - (I(g) \cup \{g\}) | \ h \in H\}$$

is a disjoint collection of nonempty subsets of Z.

Since $I(g) \cup \{g\}$ is clopen in Z so is $W = Z - (I(g) \cup \{g\})$. For every $w \in W$ let $O(w) \subset W$ be open such that

$$w \in O(w) \subset \langle O(w) \rangle_3 \subset \kappa(W).$$

By the compactness of W there is a finite $F \subset W$ such that

$$W \subset \textstyle\bigcup_{x \in F} O(x) \subset \bigcup_{x \in F} \langle O(x) \rangle_3 \subset \kappa(W).$$

Since F is finite there is an $x \in F$ and there are distinct $h,\ h' \in H$ such that $O(x)$ intersects both $V(h)$ and $V(h')$. Take $p(h) \in O(x) \cap V(h)$ and $p(h') \in O(x) \cap V(h')$. Notice that $p(h) \neq p(h')$. Define $B = \{p,\ p(h),\ p(h')\}$. Then

$$B \in \langle O(x) \rangle_3 \cap \langle V(h) \rangle_3 \cap \langle V(h') \rangle_3 \subset \kappa(W) \cap \kappa(I(h) \cup \{h\}) \\ \cap \kappa(I(h') \cup \{h'\}).$$

Now, since

$$\kappa(W) \cap \kappa(I(h) \cup \{h\}) \cap \kappa(I(h') \cup \{h'\}) \subset \kappa(W \cap (I(h) \cup \{h\}) \cap (I(h') \cup \{h'\})) = \kappa(\emptyset) = \emptyset,$$

this is a contradiction.

For the construction of Example 0.2 we need a theorem in [**1**]. Let N denote the set of natural numbers.

THEOREM 2.2. (cf. [**1**]). *Let $n \geqq 2$. Let $\mathcal{J} \subset \mathcal{P}(N)$ and let $g : \mathcal{P}(N) \to [\mathcal{J}]^{<\omega}$ such that for all $A \in \mathcal{P}(N)$ we have $A = \bigcup g(A)$. Then there is a collection $\mathcal{H} \in [\mathcal{P}(N)]^n$ and for each $H \in \mathcal{H}$ there is a $G_H \in g(H)$ such that*

(i) $\bigcap \mathcal{H} = \emptyset$;

(ii) *for all $\mathcal{B} \in [\{G_H \mid H \in \mathcal{H}\}]^{n-1}$ we have that $\bigcap \mathcal{B} \neq \emptyset$.*

This gives us Example 0.2.

2.3. *Construction of Example* 0.2. Let βN be the Čech–Stone compactification of N. Let $n \geqq 1$ be fixed. Let $Y = \beta N \cup [\beta N]^{n+2}$, regarded as a subspace of $M_{n+2}(\beta N)$. Let $X = \beta Y$ and $Z = \beta N$.

We first show that βN is M_n-embedded in X. Indeed, define

$$\kappa : \tau(\beta N) \to \tau(X)$$

by

$$\kappa(U) := X - \mathrm{cl}_X (Y - (\langle U \rangle_{n+2} \cap Y)).$$

We claim that κ defined in this way is an M_n-function. Indeed, take open sets $U_0, \ldots, U_n \in \tau(\beta N)$ such that $\bigcap_{i=0}^n U_i = \emptyset$. We claim that

$$\bigcap_{i=0}^n \langle U_i \rangle_{n+2} \cap Y = \emptyset.$$

Indeed, to the contrary, assume there is an $F \in \bigcap_{i=0}^n \langle U_i \rangle_{n+2} \cap Y$. For each $i \in \{0, 1, \ldots, n\}$ let $F_i := F \cap U_i$. Then $|F_i| \geqq n + 1$ and since $|F| = n + 2$ there is a point $x \in \bigcap_{i=0}^n F_i$. Then $x \in \bigcap_{i=0}^n U_i$ which is a contradiction. Hence

$$\bigcap_{i=0}^n \langle U_i \rangle_{n+2} \cap Y = \emptyset.$$

However, since Y is dense in X, this implies that $\bigcap_{i=0}^n \kappa(U_i) = \emptyset$.

We now show that βN is not M_{n+1}-embedded in X. It can easily be seen that this implies that βN is not M_i-embedded in X for all $i \geqq n + 1$. The proof is inspired by a construction in [**1**].

Let $\rho : \tau(\beta N) \to \tau(X)$ be any extender. For all $A \subset N$ we have that

$$A \subset \mathrm{cl}_{\beta N}(A) \subset \rho(\mathrm{cl}_{\beta N}(A)).$$

Since $\mathrm{cl}_{\beta N}(A)$ is compact, with the same technique as used in 2.1, there is a finite $\mathfrak{F}(A) \subset \tau(\beta N)$ such that

$$\mathrm{cl}_{\beta N}(A) \subset \bigcup_{F \in \mathcal{F}(A)} \langle F \rangle_{n+2} \subset \rho(\mathrm{cl}_{\beta N}(A)).$$

Define a function $g : \mathscr{P}(N) \to [\mathscr{P}(N)]^{<\omega}$ by

$$g(A) = \{F \cap N | \, F \in \mathfrak{F}(A)\}.$$

Notice that $A = \bigcup g(A)$ for all $A \subset N$. By Theorem 2.2 there are $A_0, \ldots, A_{n+1} \subset N$ and for each $0 \leqq i \leqq n + 1$ there is a $G_i \in g(A_i)$ such that

(a) $\bigcap_{i=0}^{n+1} A_i = \emptyset$;
(b) $\bigcap_{i=0}^{m-1} G_i \cap \bigcap_{i=m+1}^{n+1} G_i \neq \emptyset$ for all $0 \leqq m \leqq n + 1$.

For all $0 \leqq m \leqq n + 1$ take

$$x_m \in \bigcap_{i=0}^{m-1} G_i \cap \bigcap_{i=m+1}^{n+1} G_i.$$

Since $\bigcap_{i=0}^{n+1} A_i = \emptyset$ we have that $H = \{x_i | \, 0 \leqq i \leqq n + 1\}$ has cardinality $n + 2$ and hence is a point of Y. For all $0 \leqq i \leqq n + 1$ take $F_i \in \mathfrak{F}(A_i)$ such that $F_i \cap N = G_i$. Then

$$H \in \bigcap_{i=0}^{n+1} \langle F_i \rangle_{n+2} \subset \bigcap_{i=0}^{n+1} \rho(\mathrm{cl}_{\beta N}(A_i)).$$

Since $\bigcap_{i=0}^{n+1} \mathrm{cl}_{\beta N}(A_i) = \emptyset$ we find that ρ is not an M_{n+1}-function.

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University of Wisconsin,
Madison, Wisconsin