GENERALIZED ORDERED SPACES AT THE VRIJE UNIVERSITEIT

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1. Introduction

In 1968 Maarten Maurice was appointed full professor of pure mathematics with special emphasis on Topology at the Vrije Universiteit. As Maarten wrote his Ph. D. thesis on ordered spaces [8], it was only natural that a research group was founded working in that area. Maarten was a very inspiring teacher of mathematics at all levels. He attracted many students and as a result of that, under his supervision, 7 Doctoral Theses were prepared at the Vrije Universiteit:

J. van Dalen, *Finite products of locally compact ordered spaces* (1972)
H. Kok, *Connected orderable spaces* (1973)
M. J. Faber, *Metrizability in generalized ordered spaces* (1974)
E. K. van Douwen, *Simultaneous extension of continuous functions* (1975)
A. E. Brouwer, *Treelike spaces and related connected topological spaces* (1976)
J. M. van Wouwe, *GO-spaces and generalizations of metrizability* (1978)
K. P. Hart, *Coverings, trees and continua* (1984)

In six of these theses the central theme is normality, connectivity and metrizability in ordered spaces, in the seventh one ordered spaces are merely touched upon. In this paper we will briefly survey the main results in the theses mentioned.

Maarten was also 'copromotor' of Wattel [21] and 'referent' of Bruijning [2], Koetsier [6] and Van der Bijl [15].

2. The doctoral theses

To make the presentation unified we fix some terminology. A *Linearly Ordered Topological Space* (LOTS) is a linearly ordered set $(X, <)$ endowed with its order topology $\tau(<)$. A *Generalized Ordered space* (GO-space) is a triple $(X, <, \tau)$, where $<$ is a linear order on X and where τ is a topology, finer than the order topology but with a base of convex sets; it turns out that GO-spaces are precisely the subspaces of LOTS. Indeed a GO-space may be embedded into a LOTS in an order preserving fashion as a closed subset or as a dense subset. Finally a *weakly ordered space* is a triple $(X, <, \tau)$, where $<$ is a linear order and τ is a topology that is finer than the order topology.

Linearly ordered spaces enter General Topology at various levels. In a beginner's course of General Topology they come right after metric spaces as sources of structures that come with a 'natural' topology. At the research level the classes of LOTS and GO-spaces are important because they generally serve as a first place to test hypotheses. Indeed, one of the universal counterexamples from topology, the Sorgenfrey line, is a GO-space.

Generally, the extra structure that one gets from the linear order makes the proofs go smoother and questions tend to have the 'correct' answer. If this 'correct' answer is not forthcoming then this is almost invariably a signal that the original problem will be quite difficult. A hallmark is Suslin's problem from 1923 that asks whether the real line is characterized by being a linearly ordered set without gaps or jumps in which every pairwise disjoint family of intervals is countable — it was well-known that having a countable dense set suffices. It turned out that Suslin's problem was undecidable — much as Euclid's fifth postulate is undecidable: neither a positive nor a negative answer will give rise to contradictions.

Questions concerning orderability of topological spaces and of representing spaces as continuous images of ordered spaces continue to spur much research; at present a lot of effort is being spent in trying to determine whether compact monotonically normal spaces are the continuous images of (compact) ordered spaces.

Needless to say that there was a lot of contact between Maarten's group and the other researchers in the field of ordered spaces; H. Herrlich served as referee for Faber's thesis and D.J. Lutzer was the referee for the dissertations of Van Wouwe and Hart.

2.1. The thesis of J. van Dalen

Van Dalen's thesis [14] can be seen as an attempt to generalize well-known theorems about Euclidean spaces to products of locally compact ordered spaces, where 'ordered space' means LOTS. The main results are

1. A product of n connected ordered spaces has large inductive dimension n, provided the product itself is normal.
2. Invariance of Domain in the generalized plane (a product of two connected ordered spaces without end points).
3. A No-Retraction Theorem for finite products of compact connected ordered spaces.

There are also some examples that show that not all results about the plane or about n-space admit generalization. For example, there is no counterpart of the Schoenflies Theorem which says that a homeomorphism between Jordan curves in the plane can be extended to a homeomorphism between the respective bounded domains. In a non-metrizable order-homogeneous and reversible connected ordered space a square and a triangle have homeomorphic boundaries but the sets themselves are not homeomorphic.

This thesis also contains Question 3.1 which we will come back to in Section 3.

2.2. The thesis of H. Kok

Kok's thesis [7] deals with connectivity in ordered spaces. In this thesis 'ordered space' means weakly ordered space; LOTS are referred to as 'strictly ordered'. The emphasis is on *connected* ordered spaces, or rather conditions that ensure that a connected (T_1-) space is orderable. In the thesis one finds some 20 necessary conditions for a space to be orderable, each of which isolates some property of the family of intervals or of the points in a connected ordered space.

Save for a few questions Kok's thesis presents a complete picture of the relations between the properties; this picture is neatly summarized in two diagrams, and a chart with 50 examples that shows that no other relations exist.

2.3. The thesis of M. J. Faber

Faber's thesis [4] deals with characterizations of various topological properties of GO-spaces in terms of the order structure of the space under consideration.

Typical results of this nature are the following:

1. A GO-space $X = (X, <, \tau)$ is compact if and only if X has neither gaps nor pseudo-gaps, except for the two pseudo-end gaps.
2. A GO-space $X = (X, <, \tau)$ is paracompact if and only if for each gap and each pseudo-gap (A, B) in X, there exist discrete subsets $L \subseteq A$ and $R \subseteq B$ which are, respectively, cofinal in A and coinitial in B

As the title of Faber's thesis indicates, the central results are the ones related to metrizability in GO-spaces.

Theorem 2.1. *Let $X = (X, <, \tau)$ be a GO-space. Then the following statements are equivalent:*

1. *X is metrizable.*
2. *There exists a subset D in X such that*
   (a) *D is dense in X,*
   (b) *D contains all pseudo-gaps of X, and*
   (c) *D is σ-discrete (in X).*
3. *There exists a sequence of open covers $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of X such that*

(a) $\bigcap_{n=1}^{\infty} \mathrm{St}(p,\mathcal{U}_n) = \{p\}$ *for every* $p \in X$*, and*

(b) $\{\mathrm{St}(p,\mathcal{U}_n)\}_{n=1}^{\infty}$ *is a local base at all* $p \in X$ *but for a* σ*-discrete set in* X.

Faber also proved that each metrizable GO-space $X = (X, <, \tau)$ has a σ-discrete base consisting of convex open sets. One might think that this result can be generalized to the effect that the base consists of open intervals exclusively. But, interestingly, this is not the case. Faber presents an example of a LOTS having the property that no σ-disjoint collection of open intervals covers it.

2.4. The thesis of E. K. van Douwen

E. K. van Douwen started his mathematical career in Delft as a Ph. D. student of J. M. Aarts. He continued his studies under Maarten Maurice at the Vrije Universiteit, where he got his Ph. D.

The Van Douwen Thesis does not deal with ordered spaces, so in fact falls outside the scope of this article. The article would however not be complete without some lines devoted to the work of Van Douwen and so we leave the world of ordered spaces for (just) a moment.

Let X be a topological space. As usual $C^*(X)$ denotes the vector space of bounded real-valued continuous functions on X. If A is a subspace of X then a function $\Psi\colon C^*(A) \to C^*(X)$ is called an *extender* if for each $f \in C^*(A)$ the function $\Psi(f)$ extends f. A space X is said to have property D_c^*, where c is a real number greater than or equal to 1, if for every nonempty closed subspace A of X there is a linear extender $\Psi\colon C^*(A) \to C^*(X)$ with norm not exceeding c. The Dugundji Extension Theorem implies that every metrizable space has property D_1^*.

Again, let X be a topological space. It will be convenient to let τX denote the topology of X. The space X is said to be a K_n-space if for every subspace A of X there is a function $\kappa\colon \tau A \to \tau X$ such that

1. $A \cap \kappa(U) = U$ for every $U \in \tau A$ (i.e. κ extends open subsets of A to open subsets of X),
2. if $\{U_i\}_{i=0}^{n}$ is a family of $n+1$ pairwise disjoint open subsets of A then the intersection $\bigcap_{i=0}^{n} \kappa(U_i)$ is empty.

In his thesis [16], Van Douwen proved that a space with property D_c^* is a K_n-space, where n is the smallest integer larger than $\frac{1}{2}(c-1)$. In addition, for every n he gave an example of a first countable cosmic (= continuous image of a separable metrizable space) space H_n which is not a K_n-space. Hence H_∞, the topological sum of the spaces H_n, does not have property D_c^* for any c. This answered a question of E. A. Michael in the negative. For more information on Van Douwen's mathematical work, see [18].

2.5. The thesis of A. E. Brouwer

Brouwer's thesis [1] can be considered as a sequel to Kok's thesis in that it continues and completes the study begun there. It is shown among other things that the four question marks in the table of [7, p. 85] should all be minus signs. The thesis also contains a thorough study of treelike spaces (a space is called treelike if for any two distinct points in the space there is a third which separates them) and a complete structural classification of spaces that have the property that the complement of a connected set can have at most two components.

2.6. The thesis of J. M. van Wouwe

Just as Brouwer's thesis was a sequel to Kok's thesis, Van Wouwe's thesis can be thought of as continuing the line of research started in Faber's thesis.

Ever since the first metrization theorems appeared people have been inventing and studying properties that generalize parts of the necessary and sufficient conditions for metrizability. In his thesis [20] Van Wouwe studies several of these generalized metrizability properties in the class of GO-spaces.

Let $X = (X, <, \tau)$ be a GO-space. Van Wouwe constructs a natural equivalence relation $\mathcal{G}_\mathcal{X}$ on X such that $gX = X/\mathcal{G}_\mathcal{X}$ is a GO-space and moreover is metrizable if and only if X is a p-space (in the sense of Arhangel′skiĭ). He then proceeds to define an equivalence relation $\mathcal{C}_\mathcal{X}$ on X such that $cX = X/\mathcal{C}_\mathcal{X}$ is a GO-space and moreover is metrizable if and only if X is an M-space (in the sense of Morita). Since it is clear that the metrizability of gX implies the metrizability of cX, Van Wouwe arrives at the interesting conclusion that every GO-space that is a p-space is also an M-space. He also proves that every perfectly normal GO-spaces that is a Σ-space is also an M-space. He defines a third equivalence relation $\mathcal{L}_\mathcal{X}$ on X such that $X/\mathcal{L}_\mathcal{X}$ is a GO-space; in addition, X is a Σ-space if and only if $X/\mathcal{L}_\mathcal{X}$ is metrizable and each $L \in \mathcal{L}_\mathcal{X}$ has a Σ-network. His final results deal with images and pre-images of GO-spaces under various mappings. He proves, for example, that the familiar Sorgenfrey line is not the image of a LOTS under an open-and-closed map. Interestingly, Hušek and Kulpa [**?**] showed that each GO-space is the open image of a LOTS.

2.7. The thesis of K. P. Hart

Hart's thesis [5] consists of three parts. The first parts deals with covering properties that every LOTS and GO-space has and determines their relations outside of the class of ordered spaces. In the second part the topological structure of ω_1-trees is investigated, with emphasis on covering and separation properties. Finally, the third part contains a general method for constructing ordered continua with(out) various homogeneity properties.

3. Maarten's questions

Maarten posed several intriguing questions, some of which are still unsolved. We shall discuss a few of them. The first, already alluded to when we discussed Van Dalen's thesis, asks:

Question 3.1. *Is Colorado homeomorphic to Utah?*

This formulation is after the title of the paper [10] that contains its solution. To formulate it in a precise manner we consider a compact connected ordered space X, with minimum 0 and maximum 1. We choose two points, a and b, in X and we consider the following subset U (Utah) of $X \times X$:

$$U = \bigl\{(x, y) : x \leq a \text{ or } y \leq b\bigr\}.$$

The question asks whether U is homeomorphic to $C = X^2$ (Colorado). By Van Dalen's Invariance of Domain theorem the boundaries of U and C will have to be homeomorphic as well. This, plus the fact that one does not want the answer to depend on the particular points a and b, leads one to consider order-homogeneous spaces, spaces in which every closed interval is similar to X.

In [10] Mills gave a negative answer by showing that if X is a compact connected ordered space without separable intervals then the autohomeomorphisms of X^2 have a particularly simple structure: they are of the form $\phi \times \psi$, possibly composed with the reflection map $\langle x, y\rangle \mapsto \langle y, x\rangle$.

It is not too hard to show that if $h : C \to U$ were a homeomorphism then $h^{-1} \circ (\mathrm{Id} \times \phi) \circ h$ would not be of this form, where ϕ is a homeomorphism of X that acts nontrivially on $[a, 1]$.

A question that generated a lot of research is the following:

Question 3.2. *Can one decompose the real line into two mutually homeomorphic homogeneous subsets?*

An positive answer was claimed by J. Menu in a preprint which never appeared. A second answer was given by J. van Mill in [19].

The simplest example that we are aware of takes a Hamel base for $\mathbb{R}$ over $\mathbb{Q}$ that contains π. The decomposition is then obtained as follows:

$$A = \{r \in \mathbb{R} : \pi_r \in [2n, 2n + 1) \text{ for some } n \in \mathbb{Z}\},$$

where π_r denotes the πth coordinate of r. The translation $x \mapsto x + \pi$ is a homeomorphism between A and its complement. The only hard part is to show that A is homogeneous; here one employs the fact that $\mathbb{R}$ is a vector space over $\mathbb{Q}$ and an approximation technique.

A very natural question was:

Question 3.3. *Are there paracompact spaces X and Y such that $X \times Y$ is normal but not paracompact?*

This question was inspired by the fact that the first examples on nonpreservation of paracompactness in products all produced nonnormal products. It was finally settled by Przymusiński in a very strong way, first assuming $\mathsf{MA} + \neg\mathsf{CH}$ in [11] and later in ZFC alone in [12]: there is a Lindelöf space X with X^2 normal but not paracompact. A good survey on questions (and answers) on normality in products is Przymusiński's paper [13].

It is easy to prove that if a GO-space X has a σ-discrete dense subset, then X is perfectly normal. The converse of this result is not provable in ZFC since a Suslin line is perfectly normal but has no σ-discrete dense subset. So Maarten asked:

Question 3.4. *Is there a 'real' example of a perfectly normal LOTS which does not have a σ-discrete dense subset?*

This question is still unsolved, despite various attempts. It is closely related to an older problem posed by Heath: Is there a 'real' example of a perfectly normal LOTS which has a point-countable base and yet is not metrizable?

In his thesis [8] Maarten constructed a family of ω_1 pairwise nonhomeomorphic (infinite) (topologically) homogeneous compact LOTS's and asked on p. 9:

Question 3.5. *Are there are any other (infinite) homogeneous compact LOTS's?*

In [9] he solved this problem by constructing another family consisting of ω_1 pairwise nonhomeomorphic homogeneous compact LOTS's. The definitive answer to Maarten's question was given by Van Douwen in [17]. He constructed $2^{\mathfrak{c}}$ such spaces (this number is best possible).

An interesting feature of Maarten's examples is that most of them are not separable, in fact they have cellularity $\mathfrak{c}$. This result motivated Van Douwen to ask the following now famous problem:

Question 3.6 (Van Douwen's Problem). *Is there a compact homogeneous space with cellularity greater than $\mathfrak{c}$?*

For the first construction of compact rigid LOTS's, see the paper [3] by Maarten and his advisor De Groot.

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