

# Actions on the Hilbert cube

Jan van Mill

*To Cor Baayen, at the occasion of his retirement*

*Abstract:* We provide a negative answer to Problem 933 in the “Open Problems in Topology Book”.

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## 1. INTRODUCTION

Let  $Q$  denote the Hilbert cube  $\prod_{i=1}^{\infty} [-1, 1]_i$ . In the “Open Problems in Topology Book”, WEST [2] asks the following (Problem #933):

*Let the compact Lie group  $G$  act semifreely on  $Q$  in two ways such that their fixed point sets are identical. If the orbit spaces are ANR's, are the actions conjugate?*

The aim of this note is to present a counterexample to this problem. For all undefined notions we refer to [1].

## 2. THE EXAMPLE

Let  $G$  be a group and let  $\pi: G \times X \rightarrow X$  be an action from  $G$  on  $X$ . Define  $\text{Fix}(G) = \{x \in X : (\forall g \in G)(\pi(g, x) = x)\}$ . It is clear that  $\text{Fix}(G)$  is a closed subset of  $X$ : it is called the *fixed-point set* of  $G$ . The action  $\pi$  is called *semifree* if it is free off  $\text{Fix}(G)$ , i.e., if  $x \in X \setminus \text{Fix}(G)$  and  $\pi(g, x) = x$  for some  $g \in G$  then  $g$  is the identity element of  $G$ . The space of orbits of the action  $\pi$  will be denoted by  $X/G$ . Let  $\mathbb{I}$  denote the interval  $[0, 1]$ .

Let  $G$  denote the compact Lie group  $\mathbb{T} \times \mathbb{Z}_2$ , where  $\mathbb{T}$  denotes the circle group. We identify  $\mathbb{Z}_2$  and the subgroup  $\{-1, 1\}$  of  $\mathbb{T}$ . In addition,  $D$  denotes  $\{z \in \mathbb{C} : |z| \leq 1\}$ . We let  $G$  act on  $D \times D$  in the obvious way:

$$((g, \varepsilon), (x, y)) \mapsto (g \cdot x, \varepsilon \cdot y) \quad (g \in \mathbb{T}, \varepsilon \in \{-1, 1\}, x, y \in D),$$

where “ $\cdot$ ” means complex multiplication. Observe that this action is semifree, and that its fixed-point set contains the point  $(0, 0)$  only. Also, observe that  $(D \times D)/G \approx \mathbb{I} \times D$ .

**Lemma 2.1.** *Let  $H$  denote either  $G$  or  $\mathbb{T}$ . There is a semifree action of  $H$  on  $Q \times \mathbb{I}$  having  $Q \times \{0\}$  as its fixed-point set. Moreover,  $(Q \times \mathbb{I})/G$  and  $Q$  are homeomorphic.*

**PROOF.** We will only prove the lemma for  $G$  since the proof for  $\mathbb{T}$  is entirely similar. We first let  $G$  act on  $X = D \times D \times Q$  as follows:

$$((g, \varepsilon), (x, y, z)) \mapsto (g \cdot x, \varepsilon \cdot y, z) \quad (g \in \mathbb{T}, \varepsilon \in \{-1, 1\}, x, y \in D, z \in Q).$$

This action is semifree and its fixed-point set is equal to  $\{(0,0)\} \times Q$ . Also observe that  $X/G \approx \mathbb{I} \times D \times Q$ .

We now let  $G$  act coordinatewise on the infinite product  $X^\infty$ . This action is again semifree, having the diagonal  $\Delta$  of  $\{(0,0)\} \times Q$  in  $X^\infty$  as its fixed-point set. Also,  $X^\infty/G$  is homeomorphic to  $(I \times D \times Q)^\infty \approx Q$ . Since  $\Delta$  projects onto a proper subset of  $X$  in every coordinate direction of  $X^\infty$ , it is a  $Z$ -set. Since  $X^\infty \approx Q$  there consequently is a homeomorphism of pairs  $(X^\infty, \Delta) \rightarrow (Q \times \mathbb{I}, Q \times \{0\})$ . We are done.

We will now describe two actions of  $G$  on  $Q \times [-1, 1]$ . By Lemma 2.1 there is a semifree action  $\alpha_r: \mathbb{T} \times Q \times \mathbb{I} \rightarrow Q \times \mathbb{I}$  having  $Q \times \{0\}$  as its fixed point set, while moreover  $Q \times \mathbb{I}/G \approx Q$ . We let  $\mathbb{T}$  act on  $Q \times [-1, 0]$  as follows:

$$(z, (q, t)) \mapsto (\bar{q}, s) \quad \text{iff} \quad \alpha_r(z, (q, -t)) = (\bar{q}, -s).$$

We will denote this action by  $\alpha_l$ . So  $\alpha = \alpha_l \cup \alpha_r$  is an action of  $\mathbb{T}$  onto  $Q \times [-1, 1]$ , having  $Q \times \{0\}$  as its fixed-point set. Now define  $\bar{\alpha}: G \times (Q \times [-1, 1]) \rightarrow Q \times [-1, 1]$  as follows:

$$\bar{\alpha}((z, \varepsilon), (q, t)) = \begin{cases} \alpha(z, (q, t)), & (\varepsilon = 1), \\ \alpha(z, (q, -t)), & (\varepsilon = -1). \end{cases}$$

Then  $\bar{\alpha}$  is a semifree action of  $G$  onto  $Q \times [-1, 1]$  having  $Q \times \{0\}$  as its fixed-point set, while moreover  $(Q \times [-1, 1])/\bar{\alpha} \approx Q$ . Observe the following triviality.

**Lemma 2.2.** *If  $A \subseteq Q \times [-1, 1]$  is  $\bar{\alpha}$ -invariant such that  $A$  is not contained in  $Q \times \{0\}$ , then  $A$  intersects  $Q \times (0, 1]$  as well as  $Q \times [-1, 0)$ .*

We will now describe the second action on  $Q \times [-1, 1]$ . By Lemma 2.1 there is a semifree action  $\beta_r: G \times Q \times \mathbb{I} \rightarrow Q \times \mathbb{I}$  having  $Q \times \{0\}$  as its fixed point set, while moreover  $Q \times \mathbb{I}/G \approx Q$ . Construct  $\beta_l$  from  $\beta_r$  in the same way we constructed  $\alpha_l$  from  $\alpha_r$ . Then  $\beta = \beta_l \cup \beta_r$  is a semifree action from  $G$  onto  $Q \times [-1, 1]$  having  $Q \times \{0\}$  as its fixed-point set. Moreover,  $(Q \times \mathbb{I})/\beta$  is the union of two Hilbert cubes, meeting in a third Hilbert cube, hence is an **AR**. (It can be shown that  $(Q \times \mathbb{I})/\beta \approx Q$ .)

Now assume that the two actions  $\bar{\alpha}$  and  $\beta$  are conjugate. Let  $\tau: Q \times [-1, 1] \rightarrow Q \times [-1, 1]$  be a homeomorphism such that for every  $g \in G$ ,  $\beta(g) = \tau^{-1} \circ \bar{\alpha}(g) \circ \tau$ . Then  $\tau(Q \times (0, 1])$  is a connected  $\bar{\alpha}$ -invariant subset of  $Q \times [-1, 1]$  which misses  $Q \times \{0\}$ . This contradicts Lemma 2.2.

## REFERENCES

- [1] J. van Mill. *Infinite-Dimensional Topology: prerequisites and introduction*. North-Holland Publishing Company, Amsterdam, 1989.
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Faculteit Wiskunde en Informatica  
 Vrije Universiteit  
 De Boelelaan 1081a  
 1081 HV Amsterdam

The Netherlands  
(email: [vanmillcs.vu.nl](mailto:vanmillcs.vu.nl))