# Contents

# Student seminar theoretical physics 2001/02

# 1 Conventions

The following conventions will be used:

- $\hbar = c = k_\mathrm{B} = 1$. $1\ \mathrm{GeV}^{-1} = 1.9733 \times 10^{-14}$ cm $= 6.5822 \times 10^{-25}$ sec.
  $1\ \mathrm{GeV} = 1.1606 \times 10^{13}$ K.

- Minkowski metric

$$\begin{aligned}
x_\mu &= \eta_{\mu\nu} x^\nu, \quad \eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = 1, &(1)\\
x^0 &= -x_0, \quad x_k = x^k, \quad k = 1, 2, 3, &(2)\\
x^2 &= x_\mu x^\mu, &(3)\\
\partial_\mu &= \frac{\partial}{\partial x^\mu}. &(4)
\end{aligned}$$

- Electrodynamics

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad E_k = F^{0k}, \quad B_k = \frac{1}{2}\,\epsilon_{klm} F_{lm}. \qquad (5)$$

- Geometrodynamics (as in Misner, Thorne & Wheeler)

$$\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu, &(6)\\
g_{\mu\nu} g^{\nu\rho} &= \delta^\rho_\mu, &(7)\\
V_\mu &= g_{\mu\nu} V^\nu, \quad V^\mu = g^{\mu\nu} V_\nu, \quad g^\mu{}_\nu = g_\nu{}^\mu = \delta^\mu_\nu, &(8)\\
\Gamma^\alpha{}_{\beta\mu} &= \frac{1}{2}\, g^{\alpha\rho}(\partial_\mu g_{\rho\beta} + \partial_\beta g_{\rho\mu} - \partial_\rho g_{\beta\mu}), &(9)\\
D_\mu V^\alpha &= \partial_\mu V^\alpha + \Gamma^\alpha{}_{\beta\mu} V^\beta, \quad D_\mu W_\alpha = \partial_\mu W_\alpha - W_\beta \Gamma^\beta{}_{\alpha\mu}, &(10)\\
R^\alpha{}_{\beta\mu\nu} &= \partial_\mu \Gamma^\alpha{}_{\beta\nu} + \Gamma^\alpha{}_{\gamma\mu}\Gamma^\gamma{}_{\beta\nu} - (\mu \leftrightarrow \nu), &(11)\\
R_{\beta\nu} &= R^\alpha{}_{\beta\alpha\nu}, &(12)\\
R &= g^{\mu\nu} R_{\mu\nu}, &(13)\\
g &= \det \hat{g}, \quad \hat{g} \equiv \text{matrix } (g_{\mu\nu}). &(14)
\end{aligned}$$

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The definition of the covariant derivative $D_\mu$ in terms of the connection $\Gamma^\alpha{}_{\beta\mu}$ appears to be the same for everyone, hence also the relation between the connection and the metric $g_{\mu\nu}$. Weinberg has the same sign of metric, but opposite sign of Riemann tensor $R^\alpha{}_{\beta\mu\nu}$, hence also opposite sign of Ricci tensor $R_{\mu\nu}$ and scalar curvature $R$; he defines $g$ as minus the determinant of the metric. Kolb & Turner and also Peacock have the opposite sign of metric. We shall denote (with Garcia-Bellido) the scale factor by $a(t)$ (Weinberg, Kolb & Turner and Peacock use $R(t)$, Peacock defines $a(t) = R(t)/R(t_0)$).

- Our Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\gamma_0 = -\gamma^0$, $\gamma_0^\dagger = -\gamma_0$, $\gamma_k^\dagger = \gamma_k$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger$, $\beta = i\gamma^0$, $\alpha_k = -\gamma^0\gamma^k$. Furthermore, $\bar\psi \equiv \psi^\dagger\beta$ and the charge-conjugation matrix $C$ has the properties $C = -C^T$, $C^\dagger = C^{-1}$, $\gamma_\mu^T = -C^\dagger\gamma_\mu C$. The charge conjugates of $\psi$ and $\bar\psi$ are $\psi^{(c)} = (\bar\psi C)^T$ and $\bar\psi^{(c)} = -(C^\dagger\psi)^T$.

# 2 Special Relativity, Electrodynamics

The action for the electromagnetic field $A_\mu(x)$ coupled to a current $j^\mu(x)$ is given by

$$S_A = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right), \tag{15}$$

where the integration is over some compact region in spacetime. The action for a system of point particles with masses $m_\alpha$, $\alpha = 1, 2, \cdots$, following trajectories $z_\alpha^\mu(t)$, is given by

$$S_m = -\sum_\alpha m_\alpha \int dt \sqrt{-\dot{z}_\alpha^\mu \dot{z}_{\alpha\mu}}, \quad \dot{z}_\alpha^\mu = \frac{d}{dt} z_\alpha^\mu, \quad z_\alpha^0 = t. \tag{16}$$

If the particles have electric charges $q_\alpha$, their electromagnetic current is given by

$$j^\mu(x) = \sum_\alpha q_\alpha \int dt\, \dot{z}_\alpha^\mu(t)\, \delta^4(x - z_\alpha(t)). \tag{17}$$

N.B. here $t$ is a dummy integration variable, $t \neq x^0$. The electrodynamic action of the coupled particle-field system is given by

$$S = S_A + S_m, \tag{18}$$

with the above expression for the current (17).

Consider variations $\delta A_\mu$ and $\delta z_\alpha^\mu$, which vanish at the boundary of the integration region in the expression for the action. The variation of the action, $\delta S = S[A + \delta A, z + \delta z] - S[A, z]$ to first order in $\delta A_\mu$ and $\delta z_\alpha^\mu$, can be written in the form

$$\delta S = \int d^4x\, C^\mu(x) \delta A_\mu(x) + \sum_\alpha \int dt\, C_{\alpha\mu}(t) \delta z_\alpha^\mu(t) + O(\delta A^2, \delta z^2), \tag{19}$$

where partial integration has been used to remove differentiations of $\delta A$ and $\delta z$. By definition, the coefficients $C$ are the functional derivatives of $S$, usually denoted as $\delta S/\delta A_\mu(x) = C^\mu(x)$ and $\delta S/\delta z_\alpha^\mu(t) = C_{\alpha\mu}(t)$.[1]

a. Verify

$$\frac{\delta S}{\delta A_\nu} = \partial_\mu F^{\mu\nu} + j^\nu, \tag{20}$$

$$\frac{\delta S}{\delta z_\alpha^\mu} = -m_\alpha \frac{d}{dt} \left( \frac{\dot{z}_{\alpha\mu}}{\sqrt{-\dot{z}_\alpha^2}} \right) + q_\alpha \dot{z}_\alpha^\nu F_{\mu\nu}. \tag{21}$$

Setting these to zero gives the electrodynamic equations of motion.

[1] The variation of a function $f(z)$ of one variable $z$ is simply $\delta f(z) = (df(z)/dz)\delta z$. The variation of a function of many variables $z_k$ is $\delta f(z) = \sum_k (\partial f(z)/\partial z_k) \delta z_k$. In case of continuous labels, e.g. $k = 1, 2, \cdots \to t \in (-\infty, \infty)$, we get a functional $f[z]$, and the variational derivative is the generalization of the partial derivative, $\delta f[z] = \int dt\, (\delta f[z]/\delta z(t)) \delta z(t)$.

b. Express the particle equations of motion (*after* calculating $\delta S/\delta z^\mu_\alpha(t)$) in terms of the propertimes defined by

$$d\tau_\alpha = \sqrt{-\dot{z}^\mu_\alpha \dot{z}_{\alpha\mu}}\, dt, \tag{22}$$

and the four-velocities

$$u^\mu_\alpha = \frac{\dot{z}^\mu_\alpha}{\sqrt{-\dot{z}^\mu_\alpha \dot{z}_{\alpha\mu}}} = \frac{dz^\mu_\alpha}{d\tau_\alpha}. \tag{23}$$

Note that $u^2 = -1$ and that the four-momentum of a particle is defined as

$$p^\mu = m\frac{dz^\mu}{d\tau}, \tag{24}$$

with the property $p^2 = -m^2$.

c. Express $S$ in terms of the propertimes and verify that it is Lorentz invariant:

$$S[A', z'] = S[A, z], \tag{25}$$

where $A'_\mu$ and $z'^\mu_\alpha$ are the Lorentz transforms of $A_\mu$ and $z^\mu_\alpha$,

$$\begin{aligned} x'^\mu &= \ell^\mu{}_\nu x^\nu, \quad x'^2 = x^2, \quad \det \hat{\ell} = 1, && (26)\\ A'_\mu(x') &= \ell_\mu{}^\nu A_\nu(x) \quad (\text{or } A'_\mu(x) = \ell_\mu{}^\nu A_\nu(\ell^{-1}x)), && (27)\\ z'^\mu_\alpha(t') &= \ell^\mu{}_\nu z^\nu_\alpha(t). && (28) \end{aligned}$$

# 3 General Relativity, Geometrodynamics

The action for the gravitational field coupled to a set of point particles with masses $m_\alpha$, $\alpha = 1, 2, \cdots$, following trajectories $z^\mu_\alpha(t)$, is given by

$$\begin{aligned} S &= S_g + S_m, && (29)\\ S_g &= \int d^4x \sqrt{-g}\, \frac{1}{16\pi G}(R - 2\Lambda), && (30)\\ S_m &= -\sum_\alpha m_\alpha \int dt \sqrt{-g_{\mu\nu}(z_\alpha)\dot{z}^\mu_\alpha \dot{z}^\nu_\alpha}, \quad \dot{z}^\mu_\alpha = \frac{d}{dt} z^\mu_\alpha, \quad z^0_\alpha(t) = t. && (31) \end{aligned}$$

Here $G$ is Newton's constant and $\Lambda$ is the cosmological constant.

a. Consider variations $\delta g_{\mu\nu}$, which are zero on the boundary of the integration region in the expression for the action. The energy-momentum tensor is defined by

$$T^{\mu\nu}(x) \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}(x)}. \tag{32}$$

Verify that

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{-g(x)}} \sum_\alpha m_\alpha \int dt\, \delta^4(x - z_\alpha(t)) \frac{\dot{z}^\mu_\alpha(t) \dot{z}^\nu_\alpha(t)}{\sqrt{-\dot{z}^\lambda_\alpha(t) \dot{z}_{\alpha\lambda}(t)}}, \tag{33}$$

b. Express $T^{\mu\nu}$ in terms of the propertimes $\tau_\alpha$, related to $t$ by

$$d\tau_\alpha = dt \sqrt{-g_{\mu\nu}(z(t)) \frac{dz^\mu_\alpha}{dt} \frac{dz^\nu_\alpha}{dt}}. \tag{34}$$

Recall that the fourmomentum of a particle is given by

$$p^\mu_\alpha = m_\alpha u^\mu_\alpha. \tag{35}$$

Verify that

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{-g(x)}} \sum_\alpha \int d\tau\, \delta^4(x - z_\alpha(\tau))\, \frac{p^\mu_\alpha(\tau) p^\nu_\alpha(\tau)}{m_\alpha}. \tag{36}$$

Now verify the following result

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)} = -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu} + 8\pi G T^{\mu\nu}, \tag{37}$$

in steps (cf. Weinberg):

c1.

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g}\, g^{\mu\nu} \delta g_{\mu\nu} \tag{38}$$

(hint: recall Cramer's formula for the inverse of a matrix, i.e. $g^{\alpha\kappa} = \frac{1}{g} \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\kappa\lambda\mu\nu} g_{\beta\lambda} g_{\gamma\mu} g_{\delta\nu}$, and $\epsilon_{\alpha\beta\gamma\delta}\, g = \epsilon^{\kappa\lambda\mu\nu} g_{\alpha\kappa} g_{\beta\lambda} g_{\gamma\mu} g_{\delta\nu}$);

c2.

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\beta\nu} \delta g_{\alpha\beta} \tag{39}$$

(hint: use $\hat{g}\hat{g}^{-1} = 1$);

c3.

$$\delta \Gamma^\alpha{}_{\beta\mu} = \frac{1}{2} g^{\alpha\rho} (D_\mu \delta g_{\rho\beta} + D_\beta \delta g_{\rho\mu} - D_\rho \delta g_{\beta\mu}), \tag{40}$$

which implies that $\delta\Gamma^\alpha{}_{\beta\mu}$ is a tensor;

c4.

$$\delta R_{\beta\nu} = D_\alpha \delta \Gamma^\alpha{}_{\beta\nu} - D_\nu \delta \Gamma^\alpha{}_{\alpha\beta}, \tag{41}$$

Palatini's identity;

c5.
$$D_\mu g_{\alpha\beta} = 0, \tag{42}$$
the metric tensor is covariantly constant;

c6.
$$\Gamma^\alpha{}_{\alpha\mu} = \frac{1}{2}\, g^{\alpha\beta}\partial_\mu g_{\alpha\beta} \quad (= \frac{1}{2}\,\mathrm{Tr}\,\hat{g}^{-1}\partial_\mu \hat{g} = \frac{1}{2}\partial_\mu \mathrm{Tr}\,\ln \hat{g}); \tag{43}$$

c7. the covariant divergence: for any vector field $V^\mu$,
$$D_\mu V^\mu = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}V^\mu); \tag{44}$$

c8. putting things together,
$$\sqrt{-g}\, g^{\beta\nu}\delta R_{\beta\nu} = \partial_\alpha(\sqrt{-g}g^{\beta\nu}\delta\Gamma^\alpha{}_{\beta\nu}) - \partial_\nu(\sqrt{-g}g^{\beta\nu}\delta\Gamma^\alpha{}_{\alpha\beta}); \tag{45}$$

c9. finally,
$$\begin{aligned}\delta S_g &= \frac{1}{16\pi G}\int d^4x\,[(R-2\Lambda)\delta\sqrt{-g} + \sqrt{-g}\,R_{\mu\nu}\delta g^{\mu\nu} \\ &\quad + \text{surface terms}],\end{aligned} \tag{46}$$
where the surface terms are zero, gives (37).

Setting the left hand side of (37) to zero we get the Einstein equations.

d. By varying the action as a functional of $z^\mu_\alpha$, find the equation of motion for particle $\alpha$.

e. Express this equation in terms of the propertime $\tau_\alpha$. The result is the geodesic equation (see e.g. Weinberg Sect. 3.3):
$$\frac{d^2 z^\mu_\alpha}{d\tau_\alpha} + \Gamma^\mu{}_{\rho\sigma}(z_\alpha)\frac{dz^\rho_\alpha}{d\tau_\alpha}\,\frac{dz^\sigma_\alpha}{d\tau_\alpha} = 0. \tag{47}$$

# 4 General coordinate invariance

The gravitational action is invariant (a scalar) under general coordinate transformations which reduce to the identity at the boundary of the spacetime integration. For the action (29) this means $S[g', z'] = S[g, z]$, where $g'_{\mu\nu}$ and $z'^\mu_\alpha$ are the transformed metric and particle trajectories. Under a general coordinate transformation $x'^\mu = f^\mu(x)$ the metric transforms as
$$g'_{\mu\nu}(x') = \frac{\partial x^\kappa}{\partial x'^\mu}\,\frac{\partial x^\lambda}{\partial x'^\nu}\, g_{\kappa\lambda}(x). \tag{48}$$

Consider an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x), \tag{49}$$

with infinitesimal $\epsilon^{\mu}(x)$. Its effect on the metric may be interpreted as a variation,

$$\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x), \tag{50}$$

i.e.

$$\delta g_{\mu\nu} = -\partial_{\mu}\epsilon^{\alpha} g_{\alpha\nu} - \partial_{\nu}\epsilon^{\alpha} g_{\mu\alpha} - \epsilon^{\alpha}\partial_{\alpha} g_{\mu\nu}. \tag{51}$$

The action $S_g$ is invariant under such variations of $g_{\mu\nu}$, $\delta S_g/\delta\epsilon^{\mu} = 0$.

a. Use the invariance of $S_g$ for arbitrary $\epsilon^{\mu}(x)$ to show that

$$D_{\mu}(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R + \Lambda g^{\mu\nu}) = 0. \tag{52}$$

For $\Lambda = 0$ these equations are known as the contracted Bianchi identities. From the Einstein equations now also follows the covariant energy-momentum conservation law, $D_{\mu} T^{\mu\nu} = 0$.

# 5 Friedmann-Lemaitre-Robertson-Walker Metric

The Robertson-Walker metric is given by $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$,

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{53}$$

where $a(t)$ is the scale factor and $k = -1$, 0 or 1, for a universe that has negative, zero or positive spatial curvature, respectively.

The spatial part of the metric, $g_{mn}$, describes a homogeneous and isotropic space of constant scalar curvature ${}^3R$, the sign of ${}^3R$ is equal to $k$. We shall get a feeling for this in the following exercises.

a. For $k = 0$ we have flat space $R^3$.

b. The three-sphere $S^3$ can be defined as the collection of points in $R^4$ satisfying $X_1^2 + X_2^2 + X_3^2 + X_4^2 = a^2$; $a$ is called its radius. This space can be described by intrinsic coordinates $(\chi, \theta, \phi)$ such that

$$\begin{aligned} X_1 &= a\sin\chi\sin\theta\cos\phi, \quad X_2 = a\sin\chi\sin\theta\sin\phi, \\ X_3 &= a\sin\chi\cos\theta, \quad X_4 = a\cos\chi. \end{aligned} \tag{54}$$

Obtain the metric of this three-sphere in terms of $(\chi, \theta, \phi)$ and specify the domain of the coordinates $(\chi, \theta, \phi)$.

Another coordinate system is $(r, \theta, \phi)$ such that $r = \sin\chi$. Verify that this corresponds to the spatial part of the RW metric for $k = 1$ and give the domain of $r$.

c. The hyperbolic space $H^3$ can be defined by $X_1^2 + X_2^2 + X_3^2 - X_4^2 = -a^2$. We get the corresponding metric from the $S^3$ metric by the substitution $\chi \to i\chi$, $a \to -ia$.

   Give the metric, the domain of coordinates and verify that it corresponds to RW with $k = -1$.

d. Let $V(l)$ be the volume within geodesic distance $l$ from the origin. There is a general formula for $V(l)$ in $d$-dimensional space with euclidean signature, for small $l$:

$$V(l) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}\, l^d \left[1 - \frac{{}^dR}{6(d+2)}\, l^2 + O(l^4)\right], \tag{55}$$

   where ${}^dR$ is the scalar curvature at the origin and $\Gamma$ is the usual factorial function.

   Use this formula to calculate ${}^3R$ for the spatial RW metric.

   It can be shown (see e.g. Weinberg) that the spatial Riemann tensor is given by

$${}^3R_{bcmn} = \frac{k}{a^2}(g_{bm}g_{cn} - g_{bn}g_{cm}). \tag{56}$$

   This can be verified by (cumbersome) direct calculation. The tensor structure of this relation is a consequence of the isotropy and homogeneity of the RW spaces. Given (56), ${}^3R_{mn}$ and ${}^3R$ follow by contraction with $g^{mn}$.

   Verify that this gives the same ${}^3R$ as found from (55).

e. Derive the red shift relation $1 + z \equiv \lambda_0/\lambda_1 = a_0/a_1$ ($\lambda_0 \leftrightarrow$ detection, $\lambda_1 \leftrightarrow$ emmision, cf. Kolb and Turner sect. 2.3, Weinberg sect. 14.3).

   Derive Hubble's law

$$H_0 d_L = z + \frac{1}{2}(1 - q_0)z^2 + \cdots, \tag{57}$$

   where $d_L = a_0 r_1(1+z)$ is the luminosity distance, $H = \dot{a}/a$ ($\dot{a} \equiv da/dt$) is the Hubble rate and $q = -(\ddot{a}/\dot{a})(a/\dot{a})$ the deceleration (cf. Kolb and Turner sect. 2.3, Weinberg sect. 14.4). The subscript 0 indicates the current epoch, i.e. $a_0$ is the scale factor 'now'.

The Einstein equations for the RW metric can be obtained by going through the following steps.

f. Verify that the non-zero components of the connection ${\Gamma^\kappa}_{\lambda\mu}$ involving at least one time-like index are given by

$$\begin{aligned} {\Gamma^0}_{lm} &= \frac{\dot{a}}{a}\, g_{lm}, && (58) \\ {\Gamma^k}_{l0} &= \frac{\dot{a}}{a}\, \delta^k_l. && (59) \end{aligned}$$

g. Verify that the non-zero components of the Ricci tensor $R_{\mu\nu}$ and the scalar curvature $R$ are given by

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a}, && (60) \\ R_{mn} &= \left(\frac{2\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} + \frac{2k}{a^2}\right) g_{mn}, && (61) \\ R &= 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). && (62) \end{aligned}$$

h. Write down the Einstein equations for the RW metric, separately for $(\mu,\nu) = (0,0)$, $(0,n)$ and $(m,n)$.

i. In section 3 we derived the geodesic equation of motion (47) for massive particles,

$$\frac{du^\mu}{d\tau} + {\Gamma^\mu}_{\rho\sigma}(z) u^\rho u^\sigma = 0, \qquad u^\mu = \frac{dz^\mu}{d\tau}. \tag{63}$$

Specialize to the RW form of the metric and show that $|\mathbf{u}|$ defined by $|\mathbf{u}| \equiv \sqrt{g_{ij} u^i u^j}$ is proportional to $1/a$. (Hint: use the $\mu = 0$ equation, $g_{\mu\nu} u^\mu u^\nu = -1$ and $u^0 = dt/d\tau$.) It follows that the magnitude of the three momentum of a freely propagating particle 'red shifts' as $a^{-1}$ (see Kolb and Turner sect. 2.2).

# 6 Friedmann and Einstein equations

The energy-momentum tensor of an ideal fluid is characterized by a local four-velocity field $u^\mu(x)$, energy density $\rho(x)$ and pressure $p(x)$. The general form of $T^{\mu\nu}$ is then a linear combination of $u^\mu u^\nu$ and $g^{\mu\nu}$,

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}. \tag{64}$$

Examining this expression in a comoving ($u^\mu(x) = \delta^\mu_0$), local Lorentz ($g^{\mu\nu} = \eta^{\mu\nu}$) frame shows that $\rho$ is indeed the energy density and $p$ the pressure: $T^{00} = \rho$, $T^{mn} = p\delta_{mn}$.

a. Show that for the RW metric $D_\mu T^{\mu\nu} = 0$ is equivalent to

$$\partial_0(\rho a^3) = -p\partial_0(a^3) \tag{65}$$

which is analogous to the first law of equilibrium thermodynamics. As an intermediate step one may derive eq. (5.4.3) in Weinberg:

$$D_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}(\rho+p)u^\mu u^\nu] + g^{\mu\nu}\partial_\mu p + \Gamma^\nu{}_{\mu\lambda}(\rho+p)u^\mu u^\lambda. \tag{66}$$

b. Show that the Einstein equations reduce to

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} - \Lambda = 8\pi G\rho, \tag{67}$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \Lambda = 8\pi G p. \tag{68}$$

The first is called the Friedmann equation. The cosmological constant can be absorbed in $\rho$ and $p$ via $\rho \to \rho + \Lambda/8\pi G$, $p \to p - \Lambda/8\pi G$.

Verify that these equations are consistent with (65).

c. Find an expression for the expansion age of a matter dominated universe in terms of $\Omega_{\mathrm{M}0} < 1$ and $H_0$. Do the same for a flat universe $\Omega_0 = \Omega_{\mathrm{M}0} + \Omega_{\Lambda 0} = 1$ (cf. eq. (3.63) in Peacock, and/or eqs. (57) and (58) in Garcia-Bellido).

# 7 Equilibrium Thermodynamics and Particle Distribution Functions

The cosmological energy-momentum tensor can be modeled as an ideal fluid corresponding to particles in local equilibrium. In a first approximation the equilibrium properties are evaluated within special relativity. In **Minkowsky space** the particles of a given species are described by a distribution function $f(x, \mathbf{p})$, such that

$$g f(x, \mathbf{p}) \frac{d^3p}{(2\pi)^3} \tag{69}$$

is the average density (number of particles per unit volume) of particles with momentum in $d^3p$ around $\mathbf{p}$ at $\mathbf{x}$ and time $x^0$; here $g$ represents the internal degrees of freedom of the particle. For example, $g = 2$ for the photon, which has two independent spin states.
The density of the particles is given by

$$n(x) = g \int \frac{d^3p}{(2\pi)^3} f(x, \mathbf{p}). \tag{70}$$

The energy density is

$$\rho(x) = g \int \frac{d^3p}{(2\pi)^3}\, f(x,\mathbf{p}) E(\mathbf{p}), \quad E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2} = p^0, \tag{71}$$

with $m$ the mass of the particles.

To find the expression for the full energy-momentum tensor, we consider its expression for a system of point particles found earlier in sect. 3,

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{-g(x)}} \sum_\alpha \int dt\, \sqrt{-\dot{z}_\alpha^\lambda(t)\dot{z}_{\alpha\lambda}(t)}\, \delta^4(x - z_\alpha(t)) \frac{p_\alpha^\mu(t) p_\alpha^\nu(t)}{m_\alpha}, \tag{72}$$

where $p_\alpha^\mu = m_\alpha \dot{z}_\alpha^\mu / \sqrt{-\dot{z}_\alpha^\lambda \dot{z}_{\lambda\alpha}} = m_\alpha dz_\alpha^\mu / d\tau_\alpha$, $z_\alpha^0 = t$. Note that the dummy $t$ is not equal to $x^0$. Specializing to the Minkowsky spacetime and particles of one species, this can be written in the form

$$\begin{aligned} T^{\mu\nu}(x) &= g \int \frac{d^3p}{(2\pi)^3}\, f(x,\mathbf{p})\, \frac{p^\mu p^\nu}{E(\mathbf{p})}, & (73)\\ g f(x,\mathbf{p}) &= \sum_\alpha \delta^3(\mathbf{x} - \mathbf{z}_\alpha(x^0))\, (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}_\alpha(x^0)). & (74) \end{aligned}$$

a. Show that (73,74) is compatible with (72). From this example we conclude that in the Minkowsky case the correct expression for $T^{\mu\nu}(x)$ in terms of a general $f(x,\mathbf{p})$ is given by

$$T^{\mu\nu}(x) = g \int \frac{d^3p}{(2\pi)^3}\, f(x,\mathbf{p})\, \frac{p^\mu p^\nu}{E(\mathbf{p})}. \tag{75}$$

c. Show that the pressure can be written as

$$p(x) = g \int \frac{d^3p}{(2\pi)^3}\, f(x,\mathbf{p}) \frac{\mathbf{p}^2}{3E(\mathbf{p})}. \tag{76}$$

Let the equilibrium distribution function be given by

$$f(\mathbf{p}) = [\exp(E(\mathbf{p}) - \mu)/T) \mp 1]^{-1}, \tag{77}$$

where $\mu$ is the chemical potential and the upper sign is for bosons, which obey Bose-Einstein (BE) statistics, the lower sign for fermions, which obey Fermi-Dirac (FD) statistics.

d. Show that for $T \ll m$ (nonrelativistic limit),

$$\begin{aligned} n &= g\left(\frac{mT}{2\pi}\right)^{3/2} \exp[-(m-\mu)/T], & (78)\\ \rho &= nm, & (79)\\ p &= nT \ll \rho. & (80) \end{aligned}$$

e. Show that for $T \gg m$ (relativistic limit),

$$n = \frac{g}{2\pi^2} T^3 \int_0^\infty dx \, \frac{x^2}{e^{x-\tilde{\mu}} \mp 1}, \quad \tilde{\mu} \equiv \frac{\mu}{T}, \tag{81}$$

$$\rho = \frac{g}{2\pi^2} T^4 \int_0^\infty dx \, \frac{x^3}{e^{x-\tilde{\mu}} \mp 1}, \tag{82}$$

$$p = \frac{1}{3} \rho. \tag{83}$$

f. By expanding the denominators in $\exp(-x)$, show that

$$\int_0^\infty dx \, \frac{x^n}{e^x - 1} = n! \, \zeta(n+1), \tag{84}$$

$$\int_0^\infty dx \, \frac{x^n}{e^x + 1} = n! \, (1 - 2^{-n}) \zeta(n+1), \tag{85}$$

where $\zeta(n)$ is the Riemann zeta function,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}. \tag{86}$$

We have $\zeta(2) = \pi^2/6$, $\zeta(3) = 1.202\ldots$, $\zeta(4) = \pi^4/90$.

g. Show that for $T \gg m, \mu$,

$$n = \frac{\zeta(3)}{\pi^2} \, gT^3, \quad \text{BE}, \tag{87}$$

$$= \frac{3}{4} \frac{\zeta(3)}{\pi^2} \, gT^3, \quad \text{FD}, \tag{88}$$

$$\rho = \frac{\pi^2}{30} \, gT^4, \quad \text{BE}, \tag{89}$$

$$= \frac{7}{8} \frac{\pi^2}{30} \, gT^4, \quad \text{FD}. \tag{90}$$

h. Show that for degenerate relativistic fermions ($T \gg m$, $\mu \gg T$),

$$n = \frac{1}{6\pi^2} \, g\mu^3, \tag{91}$$

$$\rho = \frac{1}{8\pi^2} \, g\mu^4. \tag{92}$$

i. For fermions, let $+$ denote particles and $-$ antiparticles. Assume $\mu_+ = -\mu_-$. Show that

$$n_+ - n_- = \frac{1}{6\pi^2} \, gT^3 \left( \pi^2 \frac{\mu}{T} + \frac{\mu^3}{T^3} \right), \quad T \gg m, \tag{93}$$

$$= 2g \left( \frac{mT}{2\pi} \right)^{3/2} \sinh\left( \frac{\mu}{T} \right) \exp\left( -\frac{m}{T} \right), \quad T \ll m, \tag{94}$$

where $\mu = \mu_+$.

j. Assume a radiation dominated universe, writing

$$\rho = 3p = \frac{\pi^2}{30}\, g_* T^4, \tag{95}$$

where $g_*$ is an effective number of degrees of freedom,

$$g_* = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 . \tag{96}$$

Verify that the Einstein equations determine the Hubble rate and cosmic time as

$$H = 0.331\, \sqrt{g_*}\, \frac{T^2}{m_{\rm P}}, \tag{97}$$

$$t = \frac{1.510}{\sqrt{g_*}}\, \frac{m_{\rm P}}{T^2}, \tag{98}$$

where $m_{\rm P} = (8\pi G)^{-1/2} = 2.436 \times 10^{18}$ GeV is the Planck mass. (N.B. Another definition commonly used is $m_{\rm P} = G^{-1/2} = 1.221 \times 10^{19}$ GeV). Note that $a \propto t^{1/2}$.

# 8 Quantum fields

This section contains an introduction to quantum field theory. We start with the scalar field.

## 8.1 Quantized scalar field

We start in a general geometry described by a metric $g_{\mu\nu}$. A typical action for a scalar field $\phi$ is given by

$$S = -\int d^4x\,\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \tag{99}$$

$$V(\phi) = \frac{1}{2}\kappa_0\phi^2 + \frac{1}{4}\lambda_0\phi^4 + \epsilon_0. \tag{100}$$

The role of the parameters $\kappa_0$, $\lambda_0$ and $\epsilon_0$ will become clear in the following. The action is dimensionless (in $\hbar = c = 1$ units), so the dimension of $\varphi$, $\kappa_0$, $\lambda_0$ and $\epsilon_0$ is -1, -2, 0 and -4, in length units, respectively, or equivalently in mass units: 1, 2, 0 and 4.

a. Verify this. Note that $\epsilon_0$ (more precisely $8\pi G\epsilon_0$) is a contribution to the cosmological constant.

The energy-momentum tensor for the field is given by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left[\frac{1}{2}\,\partial_\kappa\phi\partial^\kappa\phi + V(\phi)\right]. \tag{101}$$

b. Verify this.

We now specialize to Minkowski space, with metric $\eta_{\mu\nu}$. The action can be written as ($x^0 = t$)

$$S = \int dt\, L, \quad L = \int d^3x\,\left[\frac{1}{2}\,\dot{\phi}^2 - V(\phi) - \frac{1}{2}\,\partial_k\phi\partial_k\phi\right], \tag{102}$$

where the dot denotes $\partial/\partial t$. This looks like a sum of systems, one for each $\mathbf{x}$, which are coupled by the spatial gradient term. The canonical momentum conjugate to $\phi$ is given by

$$\pi(x,t) = \frac{\delta L}{\delta\dot{\phi}(x)} = \dot{\phi}(x,t), \tag{103}$$

and the hamiltonian

$$H = \int d^3x\,\pi\dot{\phi} - L, \tag{104}$$

$$= \int d^3x\,\left[\frac{1}{2}\,\pi^2 + V(\phi) + \frac{1}{2}\,\partial_k\phi\partial_k\phi\right]. \tag{105}$$

c. Verify that the total energy and total momentum in the field is given by

$$P^0 \equiv \int d^3x\, T^{00} = H, \tag{106}$$

$$P^k \equiv \int d^3x\, T^{0k} = -\int d^3x\, \pi \partial_k \phi. \tag{107}$$

We now quantize the theory by replacing $\varphi$ and $\pi$ by operators $\varphi$ and $\pi$ in Hilbert space such that the basic Poisson brackets correspond to commutators, say at $t = 0$,

$$[\varphi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}), \quad [\varphi(\mathbf{x}), \varphi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0. \tag{108}$$

These are called the canonical commutation relations. In the Heisenberg picture (where the operators are time dependent) they are supposed to hold at equal times. The above relations are a straightforward generalization of the case of discretely many variables. One realization of the commutation relations is the coordinate representation:

$$\varphi(\mathbf{x}) \to \text{multiplication by } \varphi(\mathbf{x}), \quad \pi(\mathbf{x}) \to \frac{\delta}{i\delta\varphi(\mathbf{x})}, \tag{109}$$

acting on Schrödinger wave *functionals* $\psi[\varphi]$. This realization is basic to the path integral approach. We shall follow another approach which is geared to the particle interpretation of the quantized field.

It turns out, after quantization, that the parameters in the action are not the parameters used for parametrizing physical quantities, such as scattering cross sections or even the classical field equations in a classical approximation to the quantum theory. This is the reason why the starting parameters in the quantum theory have the subscript 0: $\epsilon_0$, $\kappa_0$, $\lambda_0$. They are called the bare parameters. The physically more relevant quantities are then denoted by $\epsilon$, $\kappa$ and $\lambda$, and are called the renormalized (or dressed) parameters. In perturbation theory (expansion in $\lambda$) one finds

$$\lambda_0 = \lambda + O(\lambda^2), \quad \kappa_0 = \kappa + O(\lambda), \quad \epsilon_0 = \epsilon + O(1). \tag{110}$$

A related aspect has to do with the fact that fields represent an infinite number of degrees of freedom, which easily leads to divergent integrals in perturbation theory. It turns out that such divergencies can be absorbed in the bare parameters, such that the renormalized ones come out finite. We shall see in the following how this works in specific examples.

## 8.2 Free field

For $\lambda_0 = 0$ the hamiltonian is quadratic in the canonical variables. This is called the free theory, because it is equivalent to a collection of harmonic oscillators, as will now be shown by going over to 'momentum space'. To simplify the presentation we first assume only one spatial dimension. Afterwards, we can easily generalize back to three spatial dimensions. We furthermore assume space to be a circle with circumference $L$, i.e. $0 \leq x \leq L$ with periodic boundary conditions at $0, L$ and $\int dx = \int_0^L dx$. We expand the fields at time $t = 0$ in Fourier modes,

$$\begin{aligned} \varphi(x) &= \frac{1}{\sqrt{L}} \sum_p e^{ipx}\, \tilde{\varphi}_p, \quad \pi(x) = \frac{1}{\sqrt{L}} \sum_p e^{ipx}\, \tilde{\pi}_p, && (111)\\ \tilde{\varphi}_p &= \frac{1}{\sqrt{L}} \int_0^L dx\, e^{-ipx}\, \varphi(x), \quad \tilde{\pi}_p = \frac{1}{\sqrt{L}} \int_0^L dx\, e^{-ipx}\, \pi(x), && (112) \end{aligned}$$

where $p = 2\pi n/L$, $n = 0, \pm 1, \pm 2, \cdots$. The modes are eigenfunctions of the gradient operator $\partial/\partial x$ with periodic boundary conditions. Since the fields are hermitian, $\varphi^\dagger(x) = \varphi(x)$, the Fourier components satisfy the relations

$$\tilde{\varphi}_p^\dagger = \tilde{\varphi}_{-p}, \quad \tilde{\pi}_p^\dagger = \tilde{\pi}_{-p}. \tag{113}$$

The hamiltonian and the momentum operator are diagonal in this representation:

$$\begin{aligned} H &= \sum_p \frac{1}{2}[\tilde{\pi}_p^\dagger \tilde{\pi}_p + (p^2 + \kappa)\tilde{\varphi}_p^\dagger \tilde{\varphi}_p] + \epsilon_0 L, && (114)\\ P &= -\sum_p \tilde{\pi}_p^\dagger \tilde{\varphi}_p ip. && (115) \end{aligned}$$

Notice that we have replaced $\epsilon$ by $\epsilon_0$, in accordance with (110). Furthermore, for free fields $\kappa_0 = \kappa$. The hamiltonian looks like that of a sum of harmonic oscillators with frequencies

$$\omega_p = \sqrt{p^2 + m^2}, \quad m^2 = \kappa, \tag{116}$$

where we have chosen $\kappa > 0$. As in the case of the harmonic oscillator, it is very useful to introduce creation and annihilation operators, $a_p^\dagger$ and $a_p$, one for each mode:

$$\begin{aligned} a_p &= \frac{1}{\sqrt{2\omega_p}}(\omega_p \tilde{\varphi}_p + i\tilde{\pi}_p), \quad a_p^\dagger = \frac{1}{\sqrt{2\omega_p}}(\omega_p \tilde{\varphi}_{-p} - i\tilde{\pi}_{-p}), && (117)\\ \tilde{\varphi}_p &= \frac{1}{\sqrt{2\omega_p}}(a_p + a_{-p}^\dagger), \quad \tilde{\pi}_p = \frac{1}{\sqrt{2\omega_p}}(-i\omega_p a_p + i\omega_p a_{-p}^\dagger), && (118) \end{aligned}$$

where we used (113). The creation and annihilation operators satisfy the commutation relations

$$[a_p, a_q^\dagger] = \delta_{pq}, \quad [a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0. \tag{119}$$

The hamiltonian and the momentum operator can now be written in the form

$$\begin{aligned} H &= \sum_p \frac{1}{2}(a_p^\dagger a_p + a_p a_p^\dagger)\omega_p + \epsilon_0 L = \sum_p (a_p^\dagger a_p + \frac{1}{2})\omega_p + \epsilon_0 L, && (120)\\ P &= \sum_p a_p^\dagger a_p\, p. && (121) \end{aligned}$$

We see that the hamiltonian is just that of a sum of independent harmonic oscillators. The simultaneous eigenstates of $H$ and $P$ are obtained from the ground state $|0\rangle$ satisfying

$$a_p|0\rangle = 0, \quad \langle 0|0\rangle = 1, \tag{122}$$

by application of the creation operators,

$$|\{n_p\}\rangle = \prod_p \frac{(a_p^\dagger)^{n_p}}{\sqrt{n_p!}}|0\rangle, \tag{123}$$

with the occupation numbers $n_p = 0, 1, \cdots$. All eigenstates are normalized to 1. The eigenvalues are given by

$$\begin{aligned} H|\{n_p\}\rangle &= (E_0 + \sum_p n_p\omega_p)|\{n_p\}\rangle, \quad E_0 = \epsilon_0 L + \sum_p \frac{1}{2}\omega_p, &(124)\\ P|\{n_p\}\rangle &= (\sum_p n_p\, p)|\{n_p\}\rangle. &(125)\end{aligned}$$

Consider now the ground state energy density:

$$\begin{aligned} \epsilon &\equiv \frac{E_0}{L} = \epsilon_0 + \frac{1}{L}\sum_p \frac{1}{2}\omega_p &(126)\\ &\to \epsilon_0 + \int_{-\infty}^{\infty} \frac{dp}{2\pi}\,\frac{1}{2}\sqrt{p^2+m^2}, \quad L\to\infty. &(127)\end{aligned}$$

The integral in the last line is the limit of a Riemann sum:

$$\frac{1}{L}\sum_p F(p) = \frac{1}{2\pi}\sum_p \Delta p\, F(p) \to \int_{-\infty}^{\infty}\frac{dp}{2\pi}\,F(p), \quad \Delta p = \frac{2\pi}{L}. \tag{128}$$

The ground state energy as written is infinite, because the integral diverges at large $p$. The reason is that we are dealing with an infinite number of degrees of freedom. However, we can absorb this infinity in $\epsilon_0$, such that $\epsilon$ is finite. We come back to this shortly.

We now generalize to three spatial dimensions. Let us choose $\epsilon_0$ such that $\epsilon = 0$. Then we can summarize as follows:

$$\begin{aligned} \varphi(\mathbf{x}) &= \sum_{\mathbf{p}} \left[ a_{\mathbf{p}} \frac{e^{i\mathbf{p}\mathbf{x}}}{\sqrt{2\omega_{\mathbf{p}}L^3}} + a_{\mathbf{p}}^\dagger \frac{e^{-i\mathbf{p}\mathbf{x}}}{\sqrt{2\omega_{\mathbf{p}}L^3}} \right], &(129)\\ \pi(\mathbf{x}) &= \sum_{\mathbf{p}} \left[ -i\omega_{\mathbf{p}} a_{\mathbf{p}} \frac{e^{i\mathbf{p}\mathbf{x}}}{\sqrt{2\omega_{\mathbf{p}}L^3}} + i\omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger \frac{e^{-i\mathbf{p}\mathbf{x}}}{\sqrt{2\omega_{\mathbf{p}}L^3}} \right], &(130)\\ [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= \delta_{\mathbf{p},\mathbf{q}}, \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0, &(131)\\ P^\mu &= \sum_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\, p^\mu, \quad P^0 = H, \quad p^0 = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}, &(132)\\ P^\mu|0\rangle &= 0, \quad P^\mu|\mathbf{p}\rangle = p^\mu|\mathbf{p}\rangle, \quad |\mathbf{p}\rangle \equiv a_{\mathbf{p}}^\dagger|0\rangle = |1_{\mathbf{p}}\rangle, &(133)\\ P^\mu|\mathbf{p}_1\mathbf{p}_2\rangle &= (p_1^\mu + p_2^\mu)|\mathbf{p}_1\mathbf{p}_2\rangle, \quad |\mathbf{p}_1\mathbf{p}_2\rangle \equiv a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger|0\rangle, &(134)\end{aligned}$$

etc. In (133) we used the convention that only non-zero occupation numbers are shown in the ket.

The interpretation of the scalar field model in terms of a collection of free particles is very suggestive. The ground state $|0\rangle$ is interpreted as representing the vacuum. The one particle state $|\mathbf{p}\rangle$ is the state with $n_{\mathbf{p}} = 1$ and all other $n_{\mathbf{q}} = 0$, $\mathbf{q} \neq \mathbf{p}$. The *mass* of the particles is $m = \sqrt{\kappa}$. Their *spin* is zero since their is no further index besides $\mathbf{p}$ to indicate a spin degree of freedom. More formally, it can be shown that a particle state at rest ($\mathbf{p} = 0$) is invariant under rotations, so its total angular momentum is identically zero and the particles are spinless.

The two particle state[2] $|\mathbf{p}_1\mathbf{p}_2\rangle$ is symmetric in the interchange of the labels $\mathbf{p}_1$ and $\mathbf{p}_2$: the particles are *bosons*.

## 8.3 Renormalization of the cosmological constant

We now return to the energy density in the groundstate, $\epsilon$. It is the vacuum expectation value of $T^{00}$. Calculating the expectation value of the full energy-momentum tensor gives in the infinite volume limit

$$\langle 0|T^{\mu\nu}(x)|0\rangle = -\epsilon_0\eta^{\mu\nu} + \int d\omega_p\, p^\mu p^\nu, \tag{135}$$

where we introduced the notation

$$d\omega_p \equiv \frac{d^3p}{(2\pi)^3 2p^0}. \tag{136}$$

Apart from conveniently absorbing numerical factors, this volume element of integration has the important property that it is Lorentz invariant (cf. Problem 1.1):

$$d\omega_{\ell p} = d\omega_p. \tag{137}$$

It follows that the integral should be an invariant tensor under Lorentz transformations, hence proportional to $\eta^{\mu\nu}$:

$$\langle 0|T^{\mu\nu}(x)|0\rangle = -\epsilon\eta^{\mu\nu}. \tag{138}$$

We can now interpret $\epsilon$ as the true contribution to the cosmological constant, while $\epsilon_0$ is just a parameter in the action. In standard jargon, $8\pi G\epsilon$ is the renormalized (or dressed) cosmological constant, and $8\pi G\epsilon_0$ the bare cosmological constant.

However, the integral (135) is badly divergent at large momenta. To make sense of it we should regularize it. Even better, we can start with a regularized formulation of the theory such that at every stage we have well defined expressions. This can be done, e.g. by replacing the spacetime continuum by a lattice,

[2] This state can also be written as $|1_{\mathbf{p}_1}1_{\mathbf{p}_2}\rangle$, or $\sqrt{2}|2_{\mathbf{p}}\rangle$ if $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$.

but it is cumbersome and we have learned that in many cases it is sufficient to deal with the problem 'on the fly', by regulating divergent integrals in a consistent manner. We could simply cut off the momentum integration at $|\mathbf{p}| = \Lambda$,

$$\begin{aligned} \langle 0|T^{00}|0\rangle &= \epsilon_0 + \frac{4\pi}{2(2\pi)^3}\int_0^\Lambda dp\, p^2\sqrt{p^2+m^2}, & (139)\\ \langle 0|T^{kl}|0\rangle &= -\epsilon_0\delta_{kl} + \frac{4\pi}{2(2\pi)^3}\int_0^\Lambda dp\, p^2\frac{p^k p^l}{\sqrt{p^2+m^2}}. & (140)\end{aligned}$$

The problem with this is that it is not consistent with Lorentz invariance: we are treating space and time differently and $\langle 0|T^{\mu\nu}|0\rangle$ will not be proportional to $\eta^{\mu\nu}$ this way. There are Lorentz covariant regularizations, for example dimensional regularization or Pauli-Villars regularization. The latter is simplest here to present and is as follows. Define $\langle 0|T^{\mu\nu}|0\rangle$ as

$$\langle 0|T^{\mu\nu}|0\rangle = -\epsilon_0\eta^{\mu\nu} + \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\sum_i c_i\,\frac{p^\mu(m_i)p^\nu(m_i)}{p^0(m_i)}, \qquad (141)$$

where the coefficients $c_i$ and the masses $m_i$ are chosen such that the integral converges, whith $c_1 \equiv 1$ and $m_1 \equiv m$. This regularization is Lorentz invariant because the $c_i$ and $m_i$ are invariant. The masses $m_i$, $i > 1$ are sent to infinity after calculating the integral. Then the result diverges again but we cancel this by a suitable choice of $\epsilon_0$. We shall not go further into details here.

Having set the vacuum energy density equal to zero we can now ask meaningful questions about the energy of the ground state in a finite volume. A famous example is the Casimir effect. This was originally discovered in QED but it applies also to our scalar field *mutatus mutandis* (two free massless scalar fields to represent the two spin states of the photon, Dirichlet boundary conditions). However, let us use the language of QED anyway as it is more intuitive. Consider two parallel plates of a conductor a distance $a$ apart, with $a$ much smaller than the linear size $L$ of the plates. The presence of the plates is taken into account by imposing boundary conditions corresponding to a perfect conductor. This shifts the ground state energy inside and outside the plates relative to the vacuum, and the result is (see e.g. Itzykson and Zuber sect. 3-2-4, Van Baal sect. 2)

$$\Delta E = \frac{-\hbar\pi^2 L^2}{720 a^3}. \qquad (142)$$

It corresponds to a tiny attractive force which has been verified by experiment.

## 8.4 Simple scattering

When the action is of higher than second order in the fields the theory is said to be interacting, because there is then no Fourier or other representation in which the harmonic oscillators are uncoupled. In our scalar field model the higher order term is the anharmonic $\varphi^4$ term in the action, the strength of which is monitored by $\lambda$, the coupling constant. Its presence changes the eigenvalues and eigenvectors of $P^\mu$, and we have to recalculate the ground state and the single and multiparticle states. A useful tool is perturbation theory, making an expansion in powers of $\lambda$. One of the most interesting new possible effects is scattering. Fortunately, to study scattering to lowest non-trivial order we only need to know the particle states in zeroth order, i.e. the free states, and we shall not need to renormalize $\kappa$ and $\lambda$.

Consider the scattering $1 + 2 \to 3 + 4$. We start with a free two-particle state $|\mathbf{p}_1\mathbf{p}_2\rangle$ at time $t = 0$ and wish to calculate the probability amplitude for the transition to another such state $|\mathbf{p}_3\mathbf{p}_4\rangle$ at a later time $t$,

$$\langle\mathbf{p}_3\mathbf{p}_4|U(t,0)|\mathbf{p}_1\mathbf{p}_2\rangle, \quad U(t,0) = e^{-iHt}, \tag{143}$$

where $U(t,0)$ is the evolution operator. The hamiltonian $H$ has the form

$$H = H_0 + H_1, \quad H_1 = \int d^3x\, \frac{1}{4}\lambda\varphi^4. \tag{144}$$

with $H_0$ the free hamiltonian of the previous sections:

$$H_0|\mathbf{p}_1\mathbf{p}_2\rangle = (E_1 + E_2)|\mathbf{p}_1\mathbf{p}_2\rangle, \quad E_1 = E(\mathbf{p}_1) = \sqrt{\mathbf{p}_1^2 + m^2}, \tag{145}$$

etc. For non-trivial scattering the final state is different from the initial state and the result would then be zero if $H_1$ were zero. Hence the scattering amplitude is at least of order $H_1$ (order $\lambda$). We want to expand the evolution operator in powers of $H_1$. It is wrong to simply expand the exponential because $H_1$ and $H_0$ do not commute. This is a standard problem in time dependent perturbation theory, which we will solve here by introducing

$$V(t) = e^{iH_0t}\, e^{-iH_0t - iH_1t}, \tag{146}$$

differentiating this with respect to $t$,

$$i\partial_t V(t) = e^{iH_0t}\, H_1\, e^{-iH_0t - iH_1t} = e^{iH_0t}\, H_1\, e^{-iH_0t} + O(H_1^2), \tag{147}$$

and integrating this, which after multuplication by $\exp(-iH_0t)$ gives the result

$$e^{-i(H_0+H_1)t} = e^{-iH_0t} - ie^{-iH_0t}\int_0^t dt'\, e^{iH_0t'}\, H_1\, e^{-iH_0t'} + O(H_1^2). \tag{148}$$

It follows that

$$\begin{aligned} \langle\mathbf{p}_3\mathbf{p}_4|U(t,0)|\mathbf{p}_1\mathbf{p}_2\rangle &= e^{-i(E_3+E_4)t}\, \frac{1 - e^{i\Delta Et}}{\Delta E}\langle\mathbf{p}_3\mathbf{p}_4|H_1|\mathbf{p}_1\mathbf{p}_2\rangle, && (149)\\ \Delta E &= E_3 + E_4 - E_1 - E_2, && (150) \end{aligned}$$

and

$$|\langle \mathbf{p}_3\mathbf{p}_4|U(t,0)|\mathbf{p}_1\mathbf{p}_2\rangle|^2 = \frac{2-2\cos(\Delta E t)}{(\Delta E)^2}\,|\langle \mathbf{p}_3\mathbf{p}_4|H_1|\mathbf{p}_1\mathbf{p}_2\rangle|^2. \tag{151}$$

We now turn to the matrix element of $H_1$. Using

$$\varphi(\mathbf{x}) = \sum_{\mathbf{q}} \left[ \frac{e^{i\mathbf{q}\mathbf{x}}}{\sqrt{2E(\mathbf{q})L^3}}\, a_{\mathbf{q}} + \frac{e^{-i\mathbf{q}\mathbf{x}}}{\sqrt{2E(\mathbf{q})L^3}}\, a^\dagger_{\mathbf{q}} \right], \tag{152}$$

and using the fact that only terms contribute which do not change the number of particles (i.e. same number of annihilation and creation operators), we get terms of the form

$$\begin{aligned}
\langle \mathbf{p}_3\mathbf{p}_4|a^\dagger_{\mathbf{q}_3}a^\dagger_{\mathbf{q}_4}a_{\mathbf{q}_1}a_{\mathbf{q}_2}|\mathbf{p}_1\mathbf{p}_2\rangle &= (\delta_{\mathbf{q}_1,\mathbf{p}_1}\delta_{\mathbf{q}_2,\mathbf{p}_2} + \delta_{\mathbf{q}_1,\mathbf{p}_2}\delta_{\mathbf{q}_2,\mathbf{p}_1})\langle \mathbf{p}_3\mathbf{p}_4|\mathbf{q}_3\mathbf{q}_4\rangle &(153)\\
&\to 2\delta_{\mathbf{q}_1,\mathbf{p}_1}\delta_{\mathbf{q}_2,\mathbf{p}_2}\langle \mathbf{p}_3\mathbf{p}_4|\mathbf{q}_3\mathbf{q}_4\rangle\\
&= 2\delta_{\mathbf{q}_1,\mathbf{p}_1}\delta_{\mathbf{q}_2,\mathbf{p}_2}(\delta_{\mathbf{q}_3,\mathbf{p}_3}\delta_{\mathbf{q}_4,\mathbf{p}_4} + \delta_{\mathbf{q}_3,\mathbf{p}_4}\delta_{\mathbf{q}_4,\mathbf{p}_3})\\
&\to 4\delta_{\mathbf{q}_1,\mathbf{p}_1}\delta_{\mathbf{q}_2,\mathbf{p}_2}\delta_{\mathbf{q}_3,\mathbf{p}_3}\delta_{\mathbf{q}_4,\mathbf{p}_4}, &(154)
\end{aligned}$$

where the arrows indicate equivalence under relabeling of the dummy **q**s which are to be summed over. (In (153) we worked the $a_{\mathbf{q}}$s to the right using the commutation relations until we got $a_{\mathbf{q}}|0\rangle = 0$.) There are five more such contributions, differering in the order of the operators ($aa^\dagger a^\dagger a$, …, $aaa^\dagger a^\dagger$), which each give equivalent results (terms like $\delta_{\mathbf{q}_i,\mathbf{q}_j}$ do not contribute because the initial and final states differ). The result is then

$$\langle \mathbf{p}_3\mathbf{p}_4|H_1|\mathbf{p}_1\mathbf{p}_2\rangle = \frac{6\lambda}{\prod_i \sqrt{2E_iL^3}} \int d^3x\, e^{i(-\mathbf{p}_3-\mathbf{p}_4+\mathbf{p}_1+\mathbf{p}_2)\mathbf{x}} = \frac{6\lambda L^3}{\prod_i \sqrt{2E_iL^3}}\,\delta_{\mathbf{p}_3+\mathbf{p}_4,\mathbf{p}_1+\mathbf{p}_2}. \tag{155}$$

This gives for the probability

$$|\langle \mathbf{p}_3\mathbf{p}_4|U(t,0)|\mathbf{p}_1\mathbf{p}_2\rangle|^2 = \frac{(6\lambda)^2L^6}{L^{12}\prod_i 2E_i}\,\frac{2-2\cos(\Delta E t)}{(\Delta E)^2}\,\delta_{\mathbf{p}_3+\mathbf{p}_4,\mathbf{p}_1+\mathbf{p}_2}. \tag{156}$$

We are interested in scattering into a domain $\Delta$ of final momenta,

$$\begin{aligned}
&\sum_{(\mathbf{p}_3,\mathbf{p}_4)\in\Delta} |\langle \mathbf{p}_3\mathbf{p}_4|U(t,0)|\mathbf{p}_1\mathbf{p}_2\rangle|^2 &(157)\\
\to\;& \frac{L^{-3}(6\lambda)^2}{4E_1E_2}\int_\Delta d\omega_3\, d\omega_4\, \frac{2-2\cos(\Delta E t)}{(\Delta E)^2}\,(2\pi)^3\delta^3(\mathbf{p}_3+\mathbf{p}_4-\mathbf{p}_1-\mathbf{p}_2), &(158)\\
&d\omega_i = \frac{d^3p_i}{(2\pi)^3 2E_i}, &(159)
\end{aligned}$$

where the arrow indicates the infinite volume limit (128), which also implies

$$L^3\delta_{\mathbf{p},\mathbf{q}} \to (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}). \tag{160}$$

For large times $t$ (on the scale of the typical inverse energies $E^{-1}$) we have the identity

$$\frac{2-2\cos(\Delta E t)}{(\Delta E)^2} = t\,2\pi\delta(\Delta E) + O(1/t). \tag{161}$$

This can be shown by integration with a test function $F(E)$:

$$\begin{aligned}
\int_{-\infty}^{\infty} dE\, F(E)\, \frac{2-2\cos Et}{E^2} &= t\int_{-\infty}^{\infty} du\, F\left(\frac{u}{t}\right) \frac{2-2\cos u}{u^2} && (162)\\
&= t\left[F(0)\int_{-\infty}^{\infty} du\, \frac{2-2\cos u}{u^2} + O(t^{-2})\right] && (163)\\
&= tF(0)\,2\pi + O(t^{-1}), && (164)
\end{aligned}$$

where we used $F(u/t) = F(0) + F'(0)u/t + O(t^{-2})$; the $F'(0)$ term drops out by symmetry.

Summarizing, we have the following result for the probability *rate*:

$$\begin{aligned}
\Gamma_\Delta &= \frac{\partial}{\partial t} \sum_{(\mathbf{p}_3,\mathbf{p}_4)\in\Delta} |\langle \mathbf{p}_3\mathbf{p}_4|U(t,0)|\mathbf{p}_1\mathbf{p}_2\rangle|^2 && (165)\\
&= \frac{L^{-3}}{4E_1E_2}\int_\Delta d\omega_3\, d\omega_4\, (2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\,(6\lambda)^2. && (166)
\end{aligned}$$

The probablity rate implies an *event rate*, which is expected to be proportional to the overlap of particle densities $\int d^3x\, n_1n_2$. The results of scattering experiments are expressed in terms of the *cross section* $\sigma_\Delta$. In a reference frame where the initial particle momenta are aligned it is defined by

$$\Gamma_\Delta^{\text{event}} = \sigma_\Delta v_{12}\int d^3x\, n_1n_2, \tag{167}$$

with

$$v_{12} = |\mathbf{p}_1/E_1 - \mathbf{p}_2/E_2| \tag{168}$$

the relative velocity. Realizing that $\Gamma_\Delta^{\text{event}} = \Gamma_\Delta$ if we normalize to unit initial particle number, $\int d^3x\, n_{1,2} = 1$, and that the density of our initial particles is $n_{1,2} = 1/L^3$, we have the result

$$\sigma_\Delta = \frac{1}{4E_1E_2v_{12}}\int_\Delta d\omega_3\, d\omega_4\, (2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\,|\mathcal{M}|^2. \tag{169}$$

where $\mathcal{M}$ (called the invariant scattering amplitude) is in this case given by

$$|\mathcal{M}|^2 = (6\lambda)^2. \tag{170}$$

The prefactor can be expressed as a Lorentz scalar,

$$E_1E_2v_{12} = \sqrt{(p_1p_2)^2 - m_1^2m_2^2}, \tag{171}$$

and we see that if the integration domain $\Delta$ is invariantly specified, the cross section is a Lorentz scalar. For example, integrating over all momenta gives the total cross section (cf. Problem 3)

$$\sigma = \frac{1}{32\pi s}\,(6\lambda)^2, \quad s \equiv -(p_1+p_2)^2, \tag{172}$$

where the Lorentz invariant $s$ is equal to the total energy squared in the center of mass frame. In a more detailed specification of $\Delta$ we can fix the invariant momentum transfer $t$. The corresponding cross section is conventially written $d\sigma/dt$ (cf. Problem 3):

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s(s-4m^2)}\,(6\lambda)^2, \quad t \equiv -(p_1-p_3)^2. \tag{173}$$

In the center of mass frame defined by $\mathbf{p}_1+\mathbf{p}_2=0$, we have $t=-2|\mathbf{p}_1|^2(1-\cos\theta)$, with $\theta$ the scattering angle, the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$, and $|\mathbf{p}_1|^2=(s-4m^2)/4$. So we see that the differential cross section

$$\left[\frac{d\sigma}{d\Omega}\right]_{\rm cm} = \frac{1}{64\pi^2 s}(6\lambda)^2 \tag{174}$$

is isotropic. This is special to the $\varphi^4$ theory, later we will encounter more interesting differential cross sections.

The above derivation of the scattering amplitude has the benefit that it is short. In higher orders it gets complicated because it lacks manifest Lorentz covariance. Only the end results are covariant or invariant. Later we will develop more sophisticated calculational techniques which are manifestly covariant. Conceptually the above derivation can be improved by considering wave packet states which are localized in space (unlike the plane wave states used here which correspond to uniform density).

## 8.5 Decay

Apart from leading to scattering, interactions may cause particles to be unstable, transforming them into two or more particles of a different species. For example, neutral pions are unstable and decay predominantly into two photons, $\pi^0 \to \gamma+\gamma$, with a mean life time $\tau = 8.8\times 10^{-17}$ sec. The mean life time is the inverse of the decay rate $\Gamma$.

The possibility of decay can be illustrated by the following simple model involving only spinless particles. The model is specified by the action

$$S[\chi,\varphi] = -\int d^4x\,\left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}M^2\chi^2 + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4}\varphi^4 + \frac{g}{2}\varphi^2\chi\right), \tag{175}$$

which describes two types of particles "$\chi$" and "$\varphi$", with masses $M$ and $m$, respectively. There are two interaction terms,

$$H_{\rm int} = \int d^3x \, \left(\frac{1}{2} g\varphi^2\chi + \frac{1}{4}\lambda\varphi^4\right), \tag{176}$$

with strengths parametrized by the coupling constants $g$ and $\lambda$ ($g$ has dimension of mass). Apart from new types of scattering, the $g\varphi^2\chi$ term also allows for transitions $\chi \leftrightarrow \varphi + \varphi$.

Suppose at time zero the initial state contains only one $\chi$-particle with four-momentum $p$. The probability at a later time $t$ for the decay into two $\varphi$-particles with momenta $q_1$ and $q_2$ is then $|\langle \mathbf{q}_1(\varphi)\mathbf{q}_2(\varphi)|U(t,0)|\mathbf{p}(\chi)\rangle|^2$ (we use the same notation as in the scattering case). Going through similar steps as in the derivation of the rate for scattering, gives for the decay rate

$$\begin{aligned} \Gamma = \frac{1}{\tau} &= \frac{\partial}{\partial t}\frac{1}{2}\sum_{\mathbf{q}_1\mathbf{q}_2} |\langle \mathbf{q}_1(\varphi)\mathbf{q}_2(\varphi)|U(t,0)|\mathbf{p}(\chi)\rangle|^2 && (177)\\ &= \frac{1}{2p^0}\frac{1}{2}\int d\omega_1 d\omega_2 (2\pi)^4\delta(q_1+q_2-p)\, g^2, && (178) \end{aligned}$$

where we used

$$\begin{aligned} \langle \mathbf{q}_1(\varphi)\mathbf{q}_2(\varphi)|H_{\rm int}|\mathbf{p}(\chi)\rangle &= \frac{g}{2}\int d^3x \sum_{\mathbf{p}'\mathbf{q}_1'\mathbf{q}_2'} \frac{1}{\sqrt{8q_1'^0 q_2'^0 p'^0 L^9}} e^{i(\mathbf{p}'-\mathbf{q}_1'-\mathbf{q}_2')\cdot\mathbf{x}} \\ &\quad \langle \mathbf{q}_1(\varphi)\mathbf{q}_2(\varphi)|a^\dagger_{\mathbf{q}_1'}(\varphi)a^\dagger_{\mathbf{q}_1'}(\varphi)a_{\mathbf{p}'}(\chi)|\mathbf{p}(\chi)\rangle && (179)\\ &= g\frac{1}{\sqrt{8q_1^0 q_2^0 p^0 L^3}}\,\delta_{\mathbf{q}_1+\mathbf{q}_2,\mathbf{p}}. && (180) \end{aligned}$$

The explicit factor 1/2 in (177) avoids double counting the two identical particles in the final state.

This example illustrates that the transition at relatively large times on the scale of $m^{-1}$, $M^{-1}$ (i.e. 'the decay'), is only possible if energy-momentum is conserved: $q_1 + q_2 = p$. Examining this for the case of a $\chi$-particle at rest one finds that this leads to the condition

$$M \geq 2m. \tag{181}$$

The integral in (178) is Lorentz invariant. It depends only on $g^2$, $M$ and $m$ (cf. Problem 3),

$$\Gamma = \frac{q}{16\pi M p^0}\, g^2, \quad q = \frac{1}{2}\sqrt{M^2 - 4m^2}. \tag{182}$$

For a moving $\chi$-particle the factor $1/p^0$ in (178) expresses the expected time dilatation.

The unstable particles can be produced in scattering, e.g. $\varphi(\mathbf{q}_1) + \varphi(\mathbf{q}_2) \to \chi(\mathbf{p})$, which is just the inverse of the decay process.

## 8.6 Symmetry, charge and antiparticles

Consider the theory described by two scalar fields $\phi_1$ and $\phi_2$ with action

$$S = -\int d^4x\, \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k + V(\sqrt{\phi_k\phi_k})\right]. \tag{183}$$

We use the summation convention also for indices like $k$: $\phi_k\phi_k \equiv \sum_{k=1}^{2}\phi_k\phi_k$. The above action is invariant under rotations in 'internal space'

$$\phi_1' = \cos\alpha\,\phi_1 - \sin\alpha\,\phi_2, \quad \phi_2' = \sin\alpha\,\phi_1 + \cos\alpha\,\phi_2, \quad S[\phi'] = S[\phi]. \tag{184}$$

Infinitesimal rotations can be written in the form

$$\delta\phi_k = -\epsilon_{kl}\phi_l\delta\alpha, \quad \epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0, \tag{185}$$

with infinitesimal rotation angle $\delta\alpha$. To a continuous symmetry corresponds a conserved quantity, usually called 'charge' (Noether's theorem). This can be seen as follows. Consider infinitesimal rotation angles depending on space-time: $\delta\alpha(x)$. The action is now in general not invariant anymore because $\alpha$ depends on $x$; for an infinitesimal rotation,

$$\delta S = \int d^4x\, j^\mu\partial_\mu\delta\alpha = -\int d^4x\,\partial_\mu j^\mu\delta\alpha. \tag{186}$$

However, if the $\phi_k$ satisfy the field equations (equations of motion), then $\delta S = 0$ and we have a local balance equation (a 'conserved current'),

$$\partial_\mu j^\mu = 0, \tag{187}$$

with a corresponding conserved charge

$$Q = \int d^3x\, j^0. \tag{188}$$

a. Show that the current is given by

$$j^\mu = \epsilon_{kl}\partial^\mu\phi_k\,\phi_l. \tag{189}$$

b. Derive the field equations from the stationary action principle.

c. Verify using the field equations that $\partial_\mu j^\mu = 0$.

d. In the quantum theory $Q$ is an operator, which can be expressed in the creation and annihilation operators at time zero. Show that

$$Q = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}k}(-i\epsilon_{kl})a_{\mathbf{p}l}. \tag{190}$$

e. Consider the free theory with $\lambda = 0$. Choosing the vacuum energy to be zero, the energy-momentum operator is given by

$$P^\mu = \sum_{\mathbf{p}} a^\dagger_{\mathbf{p}k} a_{\mathbf{p}k}\, p^\mu. \tag{191}$$

Since $Q$ is time independent it should be possible to diagonalize $Q$ and $P^0$ simultaneously. This can be done as follows. Define

$$a_{\mathbf{p}\pm} = \frac{1}{\sqrt{2}}(a_{\mathbf{p}1} \mp i a_{\mathbf{p}2}). \tag{192}$$

Show that in terms of the new creation and annihilation operators

$$Q = \sum_p (a^\dagger_{\mathbf{p}+} a_{\mathbf{p}+} - a^\dagger_{\mathbf{p}-} a_{\mathbf{p}-}), \tag{193}$$

$$P^\mu = \sum_p (a^\dagger_{\mathbf{p}+} a_{\mathbf{p}+} + a^\dagger_{\mathbf{p}-} a_{\mathbf{p}-})\, p^\mu. \tag{194}$$

The interpretation is as follows: $a^\dagger_{\mathbf{p}+}$ is the creation operator for particles, $a^\dagger_{\mathbf{p}-}$ is the creation operator for antiparticles. The particles have charge $+1$, the antiparticles have charge $-1$ and $Q$ counts the number of particles minus the number of antiparticles. Particles and antiparticles have the same mass.

## 8.7 Partition function, distribution function

The density matrix of a closed system in equilibrium is a function of the conserved quantities. In field theory these are the total energy, momentum and charge(s). For the system (183) there is only one charge and the canonical density operator is given by

$$\rho = \frac{1}{Z}\, e^{-\beta_\mu P^\mu + \alpha Q}, \quad Z = \operatorname{Tr} e^{-\beta_\mu P^\mu + \alpha Q}, \quad \operatorname{Tr}\rho = 1. \tag{195}$$

with $Z$ the grand canonical partition function. (The trace of an operator is defined as the trace of its matrix representation in an orthonormal basis:

$$\operatorname{Tr} O = \sum_i \langle i|O|i\rangle, \quad \langle i|j\rangle = \delta_{ij}, \quad \sum_i |i\rangle\langle i| = 1 = \text{unit operator.}) \tag{196}$$

The parameters $\beta_\mu$ and $\alpha$ are determined by the average energy-momentum and charge, which are given by

$$\langle P^\mu\rangle = \operatorname{Tr}\rho P^\mu = -\frac{\partial}{\partial\beta_\mu}\ln Z, \quad \langle Q\rangle = \operatorname{Tr}\rho Q = \frac{\partial}{\partial\alpha}\ln Z. \tag{197}$$

For a system at rest $\langle P^k\rangle = 0$, $\beta_k = 0$ and $\beta \equiv \beta_0 = 1/T$ is the inverse temperature.

For the free theory ($\lambda = 0$) the partition function is just a product of the independent mode contributions. The eigenmodes are characterized by $\mathbf{p}$ and $\pm$, where $+$ denotes the particles and $-$ the antiparticles. Let us lump these into a collective label $i$: $H = \epsilon_0 L^3 + \sum_i (a_i^\dagger a_i + 1/2)\omega_i$ and $Q = \sum_i a_i^\dagger a_i q_i$. The eigenvalues of $a_i^\dagger a_i$ are the occupation numbers $n_i = 0, 1, 2, \cdots$. Then

$$
\begin{aligned}
Z &= \mathrm{Tr}\, e^{-\beta H + \alpha Q} = e^{-\beta\epsilon_0 L^3} \mathrm{Tr}\, e^{\sum_i [-\beta\omega_i/2 + a_i^\dagger a_i(-\beta\omega_i + \alpha q_i)]} && (198)\\
&= e^{-\beta(\epsilon_0 L^3 + \sum_i \omega_i/2)} \sum_{\{n_i\}} e^{\sum_i n_i(-\beta\omega_i + \alpha q_i)]} && (199)\\
&= e^{-\beta(\epsilon_0 L^3 + \sum_i \omega_i/2)} \prod_i \sum_{n_i=0}^{\infty} e^{(-\beta\omega_i + \alpha q_i)n_i} && (200)\\
&= e^{-\beta(\epsilon_0 L^3 + \sum_i \omega_i/2)} \prod_i \frac{1}{1 - e^{-\beta\omega_i + \alpha q_i}}, && (201)\\
\ln Z &= -\beta\left(\epsilon_0 L^3 + \sum_i \frac{1}{2}\omega_i\right) - \sum_i \ln(1 - e^{-\beta\omega_i + \alpha q_i}), && (202)
\end{aligned}
$$

or more explicitly

$$
\begin{aligned}
\ln Z &= -\beta\left(\epsilon_0 L^3 + 2\sum_{\mathbf{p}} \frac{1}{2}\omega_i\right) \\
&\quad - \sum_{\mathbf{p}} \left[\ln(1 - e^{-\beta\omega_{\mathbf{p}} + \alpha}) + \ln(1 - e^{-\beta\omega_{\mathbf{p}} - \alpha})\right], && (203)\\
\frac{-\ln Z}{\beta L^3} &\to \epsilon_0 + 2\int \frac{d^3p}{(2\pi)^3}\, \frac{\omega_p}{2} \\
&\quad + \frac{1}{\beta}\int \frac{d^3p}{(2\pi)^3} \left[\ln(1 - e^{-\beta\omega_{\mathbf{p}} + \alpha}) + \ln(1 - e^{-\beta\omega_{\mathbf{p}} - \alpha})\right], && (204)
\end{aligned}
$$

for large $L$.

We recognize the temperature independent vacuum energy in the first two terms (really, the vacuum pressure – see below), which we have set to zero. Evaluating the average energy and charge from (197) we recognize the distribution functions:

$$
\begin{aligned}
U &\equiv \langle H\rangle = V\int \frac{d^3p}{(2\pi)^3}\, [f_+(\mathbf{p}) + f_-(\mathbf{p})]\omega_{\mathbf{p}}, \quad V = L^3, && (205)\\
N_Q &\equiv \langle Q\rangle = V\int \frac{d^3p}{(2\pi)^3}\, [f_+(\mathbf{p}) - f_-(\mathbf{p})], && (206)\\
f_\pm(\mathbf{p}) &= \frac{1}{e^{\beta(\omega_p \mp \mu)} - 1}, \quad \beta\mu = \alpha. && (207)
\end{aligned}
$$

(Note the difference in sign convention for $\alpha$ compared to (301).) We note in passing that the distribution function is just the average occupation number, as can be seen from (199)

$$\langle n_i \rangle = \frac{\partial}{\partial(\alpha q_i)} \ln Z = f_i. \tag{208}$$

In realistic theories there are usually several conserved charges $Q_A$, and the density matrix depends on several independent $\alpha_A$: $\rho = Z^{-1} \exp(-\beta H + \sum_A \alpha_A Q_A)$. Furthermore, more than one particle species contributes to a particular $Q_A$, and denoting the charge of species $k$ by $q_{Ak}$, we get the chemical potentials in the form

$$\mu_k = T \sum_A q_{Ak}, \quad T = 1/\beta, \tag{209}$$

as (up to a sign convention) in (301).

We can now also give a heuristic derivation of the final state enhancement factors $1+f_k(\mathbf{p})$ in the collision term of the Boltzmann equation. It is a consequence of the relations

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{210}$$

which hold for any mode in the occupation number representation. Consider again (153) generalized to arbitrary occupation numbers:

$$\begin{aligned} &\langle \{n_{\mathbf{p}_f}\}| a^\dagger_{\mathbf{q}_3} a^\dagger_{\mathbf{q}_4} a_{\mathbf{q}_1} a_{\mathbf{q}_2} |\{n_{\mathbf{p}_i}\}\rangle = \\ &\sqrt{n_{\mathbf{q}_1} n_{\mathbf{q}_2} (n_{\mathbf{q}_3}+1)(n_{\mathbf{q}_4}+1)}\, \langle \{n_{\mathbf{p}_f}\}|\{n_{\mathbf{p}_i}\}', n_{\mathbf{q}_1}-1, n_{\mathbf{q}_2}-1, n_{\mathbf{q}_3}+1, n_{\mathbf{q}_4}+1\rangle, \end{aligned} \tag{211}$$

where $i, f$ means 'initial', 'final' and the prime on $\{n\}'$ indicates that the elsewhere listed $n$s in the ket are to be omitted. We assumed all $\mathbf{q}$s to be different. Squaring, summing over all final states and replacing the occupation numbers by their averages according to some initial density matrix produces the factor $f(\mathbf{q}_1)f(\mathbf{q}_2)[1+f(\mathbf{q}_3)][1+f(\mathbf{q}_4)]$, with a summation over the $\mathbf{q}$s coming from the interaction hamiltonian, giving a total rate

$$\Gamma \propto \int d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4\, f(\mathbf{q}_1)f(\mathbf{q}_2)[1+f(\mathbf{q}_3)][1+f(\mathbf{q}_4)](2\pi)^4\delta^4(q_3+q_4-q_1-q_2)\,(6\lambda)^2. \tag{212}$$

To get the collision term in the Boltzman equation we can restrict a momentum integration in the initial or final state 'by hand'.

Finally, we recall the meaning of $\ln Z$. Calculating the entropy we find

$$S \equiv -\mathrm{Tr}\,\rho \ln \rho = \beta(U - \mu N_Q) + \ln Z, \tag{213}$$

a. Verify this.

or using standard thermodynamic relations

$$T \ln Z = -(U - TS - \mu N_Q) = -\Omega V = pV, \tag{214}$$

where $p$ is the pressure and $\Omega$ is the thermodynamic potential. Note that these are usually taken to depend explicitly on the temperature $T$ and chemical potential $\mu$ (not $\alpha$): $p = p(T, \mu)$, $\Omega = \Omega(T, \mu)$.

b. In terms of densities $\rho = U/V$, $s = S/V$, $n_Q = N_Q/V$, verify that

$$s = (\rho + p - \mu n_Q)/T. \tag{215}$$

c. By making a partial integration in (204), express it in the form (76).

## 8.8 Fermions

The relativistic field theoretic description of particles with spin is a lot more complicated. Here we give a plausible summary for free fermions, which we assume to have spin 1/2.

The analogue of the bosonic creation and annihilation operators are now $a^\dagger_{\mathbf{p}\lambda}$ and $a_{\mathbf{p}\lambda}$, where $\lambda = \pm$ indicates the spin polarization (e.g. the value of the helicity, the projection of the angular momentum along $\mathbf{p}$). There may also be a species label $k$. For simplicity let us lump these labels into the collective label $i = (\mathbf{p}, \lambda, k)$. For fermions the $a_i$ and $a_i^\dagger$ satisfy *anti*-commutation relations ($\{A, B\} \equiv AB + BA$):

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0. \tag{216}$$

In addition the fermionic operators commute with the bosonic ones. The vacuum is annihilated by the $a_i$,

$$a_i|0\rangle = 0, \tag{217}$$

and the $a_i^\dagger$ create the particles. These rules assure antisymmetric basis vectors

$$|ij\rangle \equiv a_i^\dagger a_j^\dagger|0\rangle = -|ji\rangle, \tag{218}$$

and the occupation numbers are limited to 0 or 1, since

$$(a_i^\dagger)^2 = \frac{1}{2}\{a_i^\dagger, a_i^\dagger\} = 0. \tag{219}$$

Without a bare cosmological constant the ground state energy would be negative,

$$H = \epsilon_0 L^3 + \sum_i \frac{1}{2}(a_i^\dagger a_i + a_i a_i^\dagger)E_i = \epsilon_0 L^3 + \sum_i (a_i^\dagger a_i - \frac{1}{2})E_i, \tag{220}$$

$$\frac{E_0}{L^3} = \epsilon_0 - \frac{1}{2}\sum_{\lambda k}\int \frac{d^3p}{(2\pi)^3}\, E_k(\mathbf{p}), \quad E_k(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_k^2}. \tag{221}$$

Charges are usually associated (in the Standard Model) with fermions rather than bosons and they have a similar form

$$Q_A = \sum_i a_i^\dagger a_i q_{Ai}, \tag{222}$$

with $q_{Ai}$ opposite for particles and antiparticles. The partition function follows easily by restricting the summations in (200) to $n_i = 0, 1$,

$$\frac{-\ln Z}{\beta L^3} = \epsilon_0 - \sum_{\lambda k} \int \frac{d^3p}{(2\pi)^3} \frac{E_k(\mathbf{p})}{2} - \frac{1}{\beta} \sum_{\lambda k} \int \frac{d^3p}{(2\pi)^3} \ln \left\{1 + e^{-\beta[E_k(\mathbf{p})-\mu_k]}\right\}. \quad (223)$$

Evaluating $\langle H \rangle$ and $\langle Q_A \rangle$ we encounter the fermion distribution function

$$f_k(\mathbf{p}) = \frac{1}{e^{\beta[E_k(\mathbf{p})-\mu_k]} + 1}, \quad (224)$$

as expected.

Finally, the Pauli blocking factors in the collision term in the Boltzmann equation follow from the analogue of (210),

$$a|n\rangle = n|n-1\rangle, \quad a^\dagger|n\rangle = (1-n)|n+1\rangle, \quad (225)$$

and the subsequent effective replacement $n_i \to f_i$.

## 8.9 Covariant normalization

We shall also use a convenient covariant normalization of particle states in the infinite volume limit

$$\langle p'|p\rangle = 2p^0(2\pi)^3\delta(\mathbf{p}' - \mathbf{p}), \quad p^0 = \sqrt{\mathbf{p}^2 + m^2}. \quad (226)$$

This has the property (cf. (137))

$$\int d\omega_p \, f(p) \, \langle p'|p\rangle = f(p'). \quad (227)$$

For the argument of ket and bra we use the four-momentum $p$, but note that here $p^0$ is not an independent variable. Comparing with our finite volume normalization we have

$$|p\rangle = \sqrt{2p^0L^3}|\mathbf{p}\rangle \quad (228)$$

(recall (160)). In infinite volume we expand the free scalar field in terms of covariant $a(p)$ and $a^\dagger(p)$,

$$\varphi(x) = \int d\omega_p \left[a(p)e^{ipx} + a^\dagger(p)e^{-ipx}\right]. \quad (229)$$

Comparison with the previous finite volume expansion at time zero

$$\varphi(\mathbf{x}) = \sum_{\mathbf{p}} \left[\frac{e^{i\mathbf{p}\mathbf{x}}}{\sqrt{2p^0L^3}} \, a_{\mathbf{p}} + \frac{e^{-i\mathbf{p}\mathbf{x}}}{\sqrt{2p^0L^3}} \, a^\dagger_{\mathbf{p}}\right] \quad (230)$$

shows that[3]

$$
\begin{aligned}
a(p) &= \sqrt{2p^0L^3}a_{\mathbf{p}}, && (231)\\
[a(p), a^\dagger(p')] &= 2p^0(2\pi)^3\delta(\mathbf{p}'-\mathbf{p}), && (232)\\
|p\rangle &= a^\dagger(p)|0\rangle, \quad \text{etc.}, && (233)\\
P^\mu &= \int d\omega_p\, a^\dagger(p)a(p)\, p^\mu. && (234)
\end{aligned}
$$

## 8.10 Problems

1. The integration volume element

$$d\omega_p \equiv \frac{d^3p}{(2\pi)^2 2p^0} \tag{235}$$

is Lorentz invariant i.e.

$$d\omega_{\ell p} = d\omega_p. \tag{236}$$

Verify this for a Lorentz tranformation along the 3-axis with velocity $v < 1$:

$$p'^0 = \gamma p^0 + \gamma v p^3, \quad p'^3 = \gamma p^3 + \gamma v p^0, \quad p'^1 = p^1, \quad p'^2 = p^2, \tag{237}$$

where $\gamma = 1/\sqrt{1-v^2}$ is the relativistic dilatation factor.

2. The Hilbert space for a system of arbitrarily many particles such as the free scalar field is called Fock space. A basis is given by $|0\rangle$, $|p\rangle$, $|p_1p_2\rangle$, etc. The states are normalized as

$$
\begin{aligned}
\langle p|q\rangle &= 2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{q}), \quad p^0 = \sqrt{m^2+\mathbf{p}^2},\\
\langle p_1p_2|q_1q_2\rangle &= \langle p_1|q_1\rangle\langle p_2|q_2\rangle + \langle p_1|q_2\rangle\langle p_2|q_1\rangle,
\end{aligned}
$$

etc. In general we get a sum over all permutations $\pi$ of $1, \cdots, n$ (the value of $n$ will be clear from the context),

$$\langle p_1\cdots p_m|q_1\cdots q_n\rangle = \delta_{mn}\sum_\pi \langle p_1|q_{\pi 1}\rangle\cdots\langle p_n|q_{\pi n}\rangle. \tag{238}$$

(For fermions the right hand side is completely antisymmetric in exchange of indices, which is represented by $\sum_\pi \to \sum_\pi (-1)^\pi$, with $(-1)^\pi \equiv 1\,(-1)$ for an even (odd) permutation.) The completeness relation can be written as

$$\hat{1} = \hat{1}_0 + \hat{1}_1 + \hat{1}_2 + \cdots, \tag{239}$$

where $\hat{1}_0 = |0\rangle\langle 0|$ and $\hat{1}_n$ is the unit operator in the $n$-particle subspace,

$$\hat{1}_n = \frac{1}{n!}\int d\omega_{p_1}\cdots d\omega_{p_n}\, |p_1\cdots p_n\rangle\langle p_1\cdots p_n|. \tag{240}$$

Verify this by taking matrix elements with $|q_1\cdots q_m\rangle$.

[3]For a free field $a(p)$ is time-independent, it is the value of the Heisenberg operator at time zero: $a(p,t) = a(p)\exp(-ip^0t)$.

3. In this problem we evaluate the remaining integrals encountered in two particle scattering and decay.

   The integral (called a phase space integral)

$$
\begin{aligned}
I(p) &= \int d\omega_{q_1}\, d\omega_{q_2}\, (2\pi)^4\delta^4(q_1+q_2-p), && (241)\\
d\omega_{q_i} &= \frac{d^3q_i}{(2\pi)^3 2\sqrt{\mathbf{q}_i^2+m_i^2}}, \quad i=1,2, && (242)
\end{aligned}
$$

   is Lorentz invariant, $I(p) = I(\ell p)$. It is convenient to evaluate it in the center of mass frame defined by $\boldsymbol{p} = 0$, in the following steps:

   a) integrate over $\mathbf{q}_2$ using the momentum conserving delta functions,
   b) choose spherical coordinates $\mathbf{q}_1 \to (q,\theta,\phi)$, $d^3q_1 = q^2dq\,d\Omega$, $d\Omega = d(\cos\theta)\,d\phi$,
   c) for the $q$ integral use the energy conserving delta function and the general formula

$$
\int_a^b dx\,\delta(f(x))g(x) = \sum_j \frac{1}{|f'(x_j)|}\,g(x_j), \tag{243}
$$

   where the summation is over the zero(s) $x_j$ of $f(x)$ in the interval $(a,b)$. In the present case the argument of the delta function, $\sqrt{m_1^2+q^2}+\sqrt{m_2^2+q^2}-p^0$, has only one zero. We use $s\equiv -p^2$, which is $p_0^2$ in the center of mass.

   Verify that the result is given by

$$
I = \frac{q}{16\pi^2\sqrt{s}}\int d\Omega\; = \frac{q}{4\pi\sqrt{s}}, \tag{244}
$$

   with

$$
q^2 = \frac{s^2+(m_1^2-m_2^2)^2}{4s} - \frac{m_1^2+m_2^2}{2}. \tag{245}
$$

   The application of $I(p)$ to two-particle decay is straightforward.

   In the application to scattering $1+2\to 3+4$, $q_1\to p_3$, $q_2\to p_4$, $p$ is the total incoming momentum, $P = p_1+p_2$, and $\theta$ may be the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$. The invariant amplitude $\mathcal{M}$ is a Lorentz invariant function of the momenta, so a function of the two independent invariants $s = -(p_1+p_2)^2 = -(p_3+p_4)^2$, and $t = -(p_1-p_3)^2 = -(p_2-p_4)^2$. The other invariant $u = -(p_1-p_4)^2 = -(p_2-p_3)^2$ is not independent, because $s+t+u = m_1^2+m_2^2+m_3^2+m_4^2$.

   The 'flux factor' is given by

$$
4E_1E_2v_{12} = 4\sqrt{(p_1p_2)^2-m_1^2m_2^2} = 4|\mathbf{p}_1|\sqrt{s}, \tag{246}
$$

   with $|\mathbf{p}_1|$ given by (245) with $q\to|\mathbf{p}_1|$.

For the total cross section we need to multiply by 1/2 if the two particles in the final state are identical, to avoid double counting. This is the same factor 1/2! as in (240), $n = 2$.

For the differential cross section we do not integrate over $\theta$ and $\phi$. Verify that in the center of mass frame

$$d\sigma = d\Omega \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} |\mathcal{M}|^2. \tag{247}$$

Alternatively, we can specify the invariant momentum transfer $t = -(p_3 - p_1)^2$. It is linearly related to $\cos\theta$, $dt = 2|\mathbf{p}_1|\,|\mathbf{p}_3|\,d\cos\theta$, so $d\sigma/dt$ can be simply be read off from $d\sigma/d\cos\theta = 2\pi d\sigma/d\Omega$. We can also insert a constraining delta function $\delta(t + (p_1 - p_3)^2)$ in the integral $I(p_1 + p_2)$ which gives the same result:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s\,\mathbf{p}_1^2} |\mathcal{M}|^2 \tag{248}$$

(this holds also in the unequal mass case).

## 8.11 Gauge invariance

The two scalar field model (183) is invariant under continuous rotations of the vector $(\phi_1, \phi_2)$. Such a symmetry is called a global symmetry, because the symmetry transformation does not depend on spacetime. By modifying the action we can extend this symmetry into a local symmetry, i.e. one in which the rotation angle $\alpha$ depends on spacetime, $\alpha = \alpha(x)$. Such local symmetries are called local gauge symmetries, or gauge symmetries for short.

We shall now describe the construction of gauge invariant actions. It is convenient to allow the scalar fields to be complex. A complex field is equivalent to two real fields, its real and imaginary parts. Consider a model with $n$ scalar fields $\phi_a$, $a = 1, \cdots, n$, which may be complex. The action

$$S = -\int d^4x\,[\partial_\mu\phi^\dagger\partial^\mu\phi + V(\phi)], \quad V(\phi) = \epsilon + \kappa\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{249}$$

($\phi^\dagger\phi = \phi_a^*\phi_a$), is invariant under global unitary transformations

$$\phi \to \phi' = U\phi, \quad U^\dagger = U^{-1} \tag{250}$$

(i.e. $\phi_a \to \phi'_a = U_{ab}\phi_b$, $U^*_{ba} = U^{-1}_{ab}$). If $\phi$ is real, $U$ is real and orthogonal. The transformations form a group $\mathcal{G}$ and the $U$'s are a representation of $\mathcal{G}$. Simple examples of $\mathcal{G}$ are the abelian (i.e. commutative) group U(1)= SO(2), the group of rotations in a plane, SO(3), the non-abelian group of rotations in three dimensions, and SU(2), the group of unitary $2 \times 2$ matrices with determinant one, which is the 'rotation group for spin' in quantum mechanics. We shall use $\mathcal{G} =$

SU(2) for illustration, reduction to U(1) or SO(3) is easy. Let $\omega_p$ be coordinates in $\mathcal{G}$, the angles parametrizing the rotations. For SU(2) there are three rotation angles $\omega_1$, $\omega_2$ and $\omega_3$. In the exponential parametrization

$$U = \exp(i\omega_p T_p) = 1 + i\omega_p T_p + O(\omega^2), \tag{251}$$

where the $T_p$ are hermitian $n \times n$ matrices representing the generators of the group. In the defining (fundamental) representation of SU(2), $n = 2$ and the $T$'s are half the Pauli matrices,

$$T_p = \frac{\sigma_p}{2}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{252}$$

In the vector representation (which is the defining representation of SO(3)), which is otherwise known as the adjoint representation, $n = 3$ and the $T$'s may be taken as

$$(T_p)_{ab} = -i\epsilon_{abp}. \tag{253}$$

Note that the corresponding $U$'s are real, and in this case $\phi$ is naturally real. The $T_p$ satisfy the commutation relations of angular momentum,

$$[T_p, T_q] = i\epsilon_{pqr} T_r, \tag{254}$$

and the orthonormality relations

$$\mathrm{Tr}\, T_p T_q = c\delta_{pq}, \tag{255}$$

with $c = 1/2$ for the defining representation and $c = 2$ for the vector representation. They form a vector like angular momentum in the sense that

$$U^{-1} T_p U = R_{pq} T_q, \tag{256}$$

with $R$ the vector representation (i.e. $R = \exp(i\omega_p T_p)$ with the $T$'s given by (253)). In general, the generators of SO(3) have the same commutation relations as those of SU(2).

The action (249) is not invariant under local transformations because the derivatives do not commute with $U$ and $U^\dagger$ when these depend on $x$. Therefore one introduces a covariant derivative $D_\mu$ with the property

$$D'_\mu \phi' = U\, D_\mu \phi. \tag{257}$$

Using this covariant derivative in place of the ordinary derivative yields a gauge invariant action: $(D'_\mu \phi')^\dagger D'^\mu \phi' = (D_\mu \phi)^\dagger U^\dagger U D^\mu \phi = (D_\mu \phi)^\dagger D^\mu \phi$. The covariant derivative depends on a matrix field $A_\mu$,

$$D_\mu \phi = \partial_\mu \phi - iA_\mu \phi, \tag{258}$$

(i.e. $D_\mu\phi_a = \partial_\mu\phi_a - iA_{ab\mu}\phi_b$). The field $A_\mu$ is called a gauge field. It should transform such that (257) holds:[4]

$$
\begin{aligned}
D'_\mu\phi' &= \partial_\mu\phi' - iA'_\mu\phi' = \partial_\mu U\phi + U\partial_\mu\phi - iA'_\mu U\phi && (259)\\
= U\,D_\mu\phi &= U\partial_\mu\phi - iUA_\mu\phi. && (260)
\end{aligned}
$$

Since $\phi$ is arbitrary we may compare the matrices multiplying it, which gives the transformation law for $A_\mu$:

$$A'_\mu = UA_\mu U^{-1} - i\partial_\mu UU^{-1}. \quad (261)$$

The dynamical variables are now assumed to be $\phi(x)$ and $A_\mu(x)$. Not all matrix elements $A_{ab\mu}(x)$ need to be independent. To find the minimal set of $A_\mu$'s we look more closely at the inhomogeneous term in their transformation law. Let $x' = x + v$ be close to $x$; then $U(x+v)U^{-1}(x)$ is close to the unit matrix and

$$U(x+v)U^{-1}(x) = 1 + v^\mu\partial_\mu U(x)U^{-1}(x) + O(v^2) = 1 + i\omega_p(x,v)T_p + O(v^2), \quad (262)$$

where $\omega_p$ is linear in $v$. It follows that $i\partial_\mu UU^{-1}$ can be written as a linear superposition of the generators $T_p$ with real coefficients. If we assume $A_\mu$ itself to have this form, then the first term in the transformation law (261) leaves this form unchanged because of (256). So we may write

$$A_\mu = A^p_\mu T_p, \quad (263)$$

with real $A^p_\mu$.

To describe the dynamics of the independent $A^p_\mu$ we need to add terms to the action involving time derivatives (and of course spatial derivatives because of Lorentz invariance). A form like $A^p_\mu A^{p\mu}$ is not gauge invariant because of the inhomogeneous term in (261). The commutator of two covariant derivatives contains derivatives of $A_\mu$,

$$F_{\mu\nu} \equiv i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \quad (264)$$

and transforms homogeneously,

$$
\begin{aligned}
F'_{\mu\nu}\phi' &= iD'_\mu D'_\nu\phi' - (\mu\leftrightarrow\nu) = U[iD_\mu D_\nu - (\mu\leftrightarrow\nu)]\phi = UF_{\mu\nu}U^{-1}\,\phi' && (265)\\
\rightarrow\quad & F'_{\mu\nu} = UF_{\mu\nu}U^{-1}. && (266)
\end{aligned}
$$

It is called the field-strength tensor. Using (263), (254) and (256) we have

$$
\begin{aligned}
F_{\mu\nu} &= F^p_{\mu\nu}T_p, \quad F^p_{\mu\nu} = \partial_\mu A^p_\nu - \partial_\nu A^p_\mu + \epsilon_{pqr}A^q_\mu A^r_\nu, && (267)\\
F'^p_{\mu\nu} &= R_{pq}F^q_{\mu\nu}. && (268)
\end{aligned}
$$

[4]The derivative $\partial_\mu$ acts only on the object immedeately following it.

The following is now a suitable gauge invariant action for the combined $(\phi, A_\mu)$ system:

$$S = -\int d^4x\, \left[(D_\mu\phi)^\dagger D^\mu\phi + V(\phi) + \frac{1}{4g^2} F^p_{\mu\nu} F^{p\mu\nu}\right]. \tag{269}$$

Here $g^2$ is called the gauge coupling constant. By rescaling $A_\mu \to gA_\mu$, it disappears from the $F^2$ term and reappears in the covariant derivative:

$$S = -\int d^4x\, \left[(\partial_\mu\phi - igA^p_\mu T_p\phi)^\dagger(\partial^\mu\phi - igA^{p\mu} T_p\phi) + V(\phi) + \frac{1}{4} F^p_{\mu\nu} F^{p\mu\nu}\right]. \tag{270}$$

In the U(1) case we have only one generator $T$ and the minimal scalar field content is just one complex $\phi$. The action for scalar electrodynamics is

$$S = -\int d^4x\, \left[(\partial_\mu\phi - ieA_\mu\phi)^*(\partial^\mu\phi - ieA^\mu\phi) + V(\phi) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right]. \tag{271}$$

where we have written conventionally $e$ in stead of $g$ and chosen unit charged particles, $T = 1$.[5] We recognize the action (15) for the Maxwell field. The electric current $j^\mu$ is to be identified from the equations of motion for the electromagnetic field $F_{\mu\nu}$:

$$-\partial_\mu F^{\mu\nu} = j^\mu = ie(D^\mu\phi)^*\phi - ie\phi^* D^\mu\phi. \tag{272}$$

It is not simply the analogue of (189) written in complex notation $\phi = (\phi_1 - i\phi_2)/\sqrt{2}$, since it also contains the vector potential $A_\mu$ through the covariant derivative.

We end this section by noting the similarities with General Relativity: $-iA_{ab\mu}$ is analogous to the connection $\Gamma^\alpha{}_{\beta\mu}$ and they play a similar role in the respective covariant derivatives (258) and (10), the definition of field strength tensor $-iF_{ab\mu\nu}$ in (264) is analogous to that of the Riemann tensor in (11). This is especially so in the vector representation in which $-iA_{ab\mu}$ and $-iF_{ab\mu\nu}$ are real.

[5] The minus sign in the definition of the covariant derivative (258) is chosen such that in the QED case a) the vector potential $A_\mu$ has the conventional sign ($\mathbf{B} = +\nabla\times\mathbf{A}$, $\mathbf{E} = -\partial_0\mathbf{A} - \nabla A^0$) and b) for $T = 1$ and $e > 0$ the particles (antiparticles) described by $\phi$ have positive (negative) unit charge.

# 9 Boltzmann equation

The Boltzmann equation is an important tool for the description of processes out of equilibrium, e.g. in the early universe. In this section we shall introduce this equation and derive its form in the expanding universe. First we summarize some formulas for scatering and decay, as they occur in field theory.

## 9.1 Summary: cross section and decay rate

We start with spinless particles. Consider the scattering $i + j \to 1 + \cdots + n$. The initial particles $i$ and $j$ have momenta $\mathbf{p}_i$ and $\mathbf{p}_j$. We assume a density of such particles, $n_i$ and $n_j$. Their relative velocity is $v_{ij}$. For example, $i$ means 'beam' and $j$ means 'target'. Then $v_{ij} = |\mathbf{p}_i/p_i^0|$. The event rate is given in terms of the cross section,

$$\Gamma_\Delta(t) = \sigma_\Delta\, v_{ij} \int d^3x\, n_i(\mathbf{x},t) n_j(\mathbf{x},t), \tag{273}$$

$$\sigma_\Delta = \int_\Delta d\sigma. \tag{274}$$

Here $\Delta$ specifies an integration region for the final momenta, $\Gamma_\Delta$ is the corresponding number of events per unit time and $\sigma_\Delta$ is an equivalent surface area, the cross section. If there is only one target particle, say at rest, we may write $n_j = \delta^3(\mathbf{x} - \mathbf{x}_j)$, which gives

$$\Gamma_\Delta(t) = \sigma_\Delta\, n_i(\mathbf{x}_j, t) v_{ij}, \quad \text{per target particle.} \tag{275}$$

In relativistic quantum (field) theory the cross section can be written as

$$d\sigma = d\omega_1 \cdots d\omega_K\, (2\pi)^4 \delta^4(p_1 + \cdots + p_K - p_i - p_j) |\mathcal{M}|^2 \frac{1}{4p_i^0 p_j^0 v_{ij}}, \tag{276}$$

with the Lorentz invariant volume element

$$d\omega_k \equiv \frac{d^3p_k}{(2\pi)^3 2p_k^0}, \quad p_k^0 = \sqrt{m_k^2 + \mathbf{p}_k^2}. \tag{277}$$

The quantity $\mathcal{M}$ is called the invariant amplitude. It is a Lorentz invariant function of the four-momenta (for spinless particles). (Another frequently used notation for $\mathcal{M}$ is $T$.) The combination $4p_i^0 p_j^0 v_{ij}$ can also be interpreted as a Lorentz invariant, the so-called flux factor $F \equiv 4\sqrt{(p_i^\mu p_{j\mu})^2 - m_i^2 m_j^2} = 4p_i^0 p_j^0 v_{ij}$, in case of parallel or antiparallel $\mathbf{p}_i$ and $\mathbf{p}_j$. Rewriting eq. (276) in terms of $F$, the differential cross section $d\sigma$ is Lorentz invariant by definition.

For identical particles we have to supply appropriate statistical factors, e.g. a factor $1/r!$ in case of $r$ identical particles in the final state.

If the particles have spin these formulas remain valid if we replace $|\mathcal{M}|^2$ by its average over spins $\overline{|\mathcal{M}|^2}$, and sum over the spins in the final state as appropriate according to the case at hand.

For a particle $i$ decaying into particles 1,2,... $K$ the formula for the decay rate is given by

$$\Gamma_\Delta = S\, g_1 \cdots g_K \int_\Delta d\Gamma, \tag{278}$$

$$d\Gamma = \frac{1}{2p_i^0} d\omega_1 \cdots d\omega_K\, (2\pi)^4 \delta^4(p_1 + \cdots p_K - p_i)\, \overline{|\mathcal{M}|^2}, \tag{279}$$

where $S$ is the statistical factor taking care of identical particles in the final state and $g_k$ is the number of spin states of particle $k$. The total decay rate refers usually to the particle rest frame, $p_i^0 \to m_i$. This rate then differs by the time dilatation factor $m_i/p_i^0$ in a general reference frame.

## 9.2 Introducing the Boltzmann equation

In the following the stage is special relativity.

Consider the distribution functions $f_k(x, \mathbf{p})$ for particles of type $k$, $k = e^+$, $e^-$, $\nu_e$, $\bar{\nu}_e$, $p$, $n$, etc. The particles carry conserved quantum numbers $q_{Ak}$, $A = Q$ (electric charge), $A = L$ (lepton number), $A = B$ (baryon number), etc. For example,

$$q_{Le^-} = q_{L\nu_e} = -q_{Le^+} = -q_{L\bar{\nu}_e} = 1, \quad q_{Lp} = q_{Ln} = q_{L\gamma} = 0, \tag{280}$$

$$q_{Bp} = q_{Bn} = q_{BH} = 1, \quad q_{Be^\pm} = q_{B\gamma} = 0, \tag{281}$$

$$q_{Qe^-} = -q_{Qp} = -1, \quad q_{Qn} = q_{Q\nu_e} = q_{QH} = q_{Q\gamma} = 0 \tag{282}$$

($H$ denotes the hydrogen atom). In a scattering process $i + j \to k + l$ we have

$$q_{Ai} + q_{Aj} = q_{Ak} + q_{Al}, \tag{283}$$

in addition to energy-momentum conservation,

$$p_i^\mu + p_j^\mu = p_k^\mu + p_l^\mu. \tag{284}$$

To the conserved quantities correspond current densities $\mathbf{j}_A(x)$ and 'charge' densities $j_A^0(x)$ which satisfy

$$\partial_\mu j_A^\mu(x) = 0. \tag{285}$$

Usually the four-currents $j^\mu$ are called 'currents' and $\partial_\mu j^\mu = 0$ is called 'current conservation'. Similarly, energy-momentum conservation is expressed by

$$\partial_\mu T^{\mu\nu}(x) = 0. \tag{286}$$

The currents and energy-momentum tensor are determined by the distribution functions as

$$j_A^\mu(x) = \sum_k q_{Ak}\, g_k \int_{\mathbf{p}} \frac{p_k^\mu}{p_k^0}\, f_k(x,\mathbf{p}), \tag{287}$$

$$T^{\mu\nu}(x) = \sum_k g_k \int_{\mathbf{p}} \frac{p_k^\mu p_k^\nu}{p_k^0}\, f_k(x,\mathbf{p}), \tag{288}$$

$$\int_{\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3}, \tag{289}$$

with $g_k$ the spin-weight factor. Note that the index $k$ is irrelevant here for $\mathbf{p}$ because it just a dummy variable, but not for $p_k^0$:

$$p_k^0 = \sqrt{\mathbf{p}^2 + m_k^2}, \tag{290}$$

where $m_k$ is the mass of particle species $k$.

The change in time of the distribution functions is described by the Boltzmann equation

$$L_k(f)(x,\mathbf{p}) = C_k(f)(x,\mathbf{p}). \tag{291}$$

Here $L_k$ describes the 'Liouville flow' due to the free motion of the particles (no external forces),

$$L_k(f) = \partial_0 f_k + v_k^i \frac{\partial}{\partial x^i} f_k = \frac{p_k^\mu}{p_k^0}\, \partial_\mu f_k \tag{292}$$

(the relativistic velocity is $v^i = \partial p^0/\partial p^i = p^i/p^0$), and $C_k(f)$ describes the effect of collisions. The Boltzmann equation has to be compatible with the local conservation laws corresponding to (285) and (286)

$$\sum_k q_{Ak}\, g_k \int_{\mathbf{p}} \frac{p_k^\mu}{p_k^0}\, \partial_\mu f_k(x,\mathbf{p}) = 0, \tag{293}$$

$$\sum_k g_k \int_{\mathbf{p}} \frac{p_k^\mu p_k^\nu}{p_k^0}\, \partial_\mu f_k(x,\mathbf{p}) = 0. \tag{294}$$

Using the Boltzmann equation (291) with (292) we see that the collision term has to satisfy

$$\sum_k q_{Ak} g_k \int_{\mathbf{p}} C_k(f)(x,\mathbf{p}) = 0, \tag{295}$$

$$\sum_k g_k \int_{\mathbf{p}} p_k^\mu C_k(f)(x,\mathbf{p}) = 0. \tag{296}$$

In many cases it is sufficient to take into account only binary collisions. Let $W_{ij|kl} = W_{ij|kl}(\mathbf{p}_i, \mathbf{p}_j; \mathbf{p}_k, \mathbf{p}_l)$ represent the probability per unit volume and per unit time that particles $i$ and $j$ collide into particles $k$ and $l$. In terms of $W$ the collision term can be written as the sum of a gain term and a loss term:

$$\begin{aligned} C_k(f) &= \frac{1}{2}\sum_{ijl} g_i g_j g_l \int_{\mathbf{p}_i\mathbf{p}_j\mathbf{p}_l} [f_i f_j W_{ij|kl}(1+\eta_k f_k)(1+\eta_l f_l) \\ &\quad - f_k f_l W_{kl|ij}(1+\eta_i f_i)(1+\eta_j f_j)]. \end{aligned} \tag{297}$$

We assume $W_{ij|kl} = W_{ji|kl} = W_{ij|lk}$ and the factor 1/2 avoids double counting over $i$ and $j$. The final state enhancement or suppression factors $(1+\eta f)$ take into account quantum statistics: $\eta = +1$ for bosons and $\eta = -1$ for fermions. We shall see that this leads to the expected equilibrium form for $f_k$. The transition probabilities $W_{ij|kl}$ are non-zero only if the conservation laws (283,284) are satisfied.

a. Verify that the relations (295,296) are satisfied by (297).

In equilibrium $f_k$ will be independent of $x$ and $C_k(f)$ should vanish. Assuming $W_{ij|kl} = W_{kl|ij}$, a solution is given by

$$\frac{f_k}{1+\eta_k f_k} = \exp(-\psi_k), \tag{298}$$

with $\psi_k$ satisfying $\psi_i + \psi_j = \psi_k + \psi_l$, hence a linear superposition of the conserved quantities:

$$\psi_k = \beta_\mu p_k^\mu + \sum_A \alpha_A q_{Ak} + \text{const.} \tag{299}$$

The $\alpha_A$ are the independent chemical potentials which determine the charge densities $j_A^0$ in the system. For a system in equilibrium at rest there is no prefered direction and $\boldsymbol{\beta} = 0$. Then the distribution function takes the usual Bose-Einstein or Fermi-Dirac equilibrium form

$$f_k(\mathbf{p}) = \frac{1}{e^{(p_k^0-\mu_k)/T} - \eta_k} \tag{300}$$

with

$$T = 1/\beta_0, \quad \mu_k = -T\sum_A \alpha_A q_{Ak}. \tag{301}$$

The const. in (299) is apparently zero.

b. By comparing with section 9.1, deduce that the $W$'s can be written as

$$W_{ij|kl} = \frac{1}{16 p_i^0 p_j^0 p_k^0 p_l^0}\,(2\pi)^4\delta^4(p_i + p_j - p_k - p_l)\,\overline{|\mathcal{M}_{ij|kl}|^2}. \tag{302}$$

A closer look shows that the assumed symmetry $W_{ij|kl} = W_{ji|kl} = W_{ij|lk}$ is standard in quantum field theory. The assumption of 'micro reversability', $W_{ij|kl} = W_{kl|ij}$ is usually satisfied to a good approximation. In case it is not, there should be a weaker property which still ensures the solution (298).

c. Consider the process $e^- + p \leftrightarrow H + \gamma$. Verify using (301) that in equilibrium the chemical potentials $\mu_k$ satisfy: $\mu_{e^-} + \mu_p = \mu_H$, $\mu_\gamma = 0$.

d. Consider the processes $e^- + p \leftrightarrow \nu_e + n$, $\bar{\nu}_e + p \leftrightarrow n + e^+$, $e^- + e^+ \leftrightarrow \gamma + \gamma$, at temperatures $T$ of order 10 MeV. Assume equilibrium, electric charge neutrality ($n_{e^-} - n_{e^+} = n_p$) and $n_p/n_\gamma = 10^{-10}$. Calculate $\mu_e/T$ and $\mu_p/T$ for $T/m_p = 0.01$.

In general the chemical potentials are determined by the conserved 'charges' ($Q$, $B$, $L$,...), or rather their densities, and in the expanding universe (in local equilibrium) they will be time and temperature dependent.

## 9.3 Boltzmann equation in the expanding universe

In the expanding universe there are new terms in the Boltzmann equation. We shall restrict ourselves to a flat RW spacetime and assume that the collisions occur on such a small scale in space and time that the collision term is not modified. Kolb and Turner give a modified Liouville term $L(f)$, eq. (5.4), which apparently assumes that the distribution functions depend explicitly also on $p^0$, in addition to $\mathbf{p}$. This appears to be in conflict with $p^0 = \sqrt{\mathbf{p}^2 + m^2}$; the steps leading to (5.7) are unclear. Here we shall assume that $f$ is s function of $\mathbf{x}$, $\mathbf{p}$ and time, as before, and derive the form of $L(f)$ by studying a realization of $f$ similar to eq. (74) in section 7.

Consider again (72), repeated here for convenience,

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{-g(x)}} \sum_\alpha \int dt\, \sqrt{-\dot{z}^\lambda_\alpha(t)\dot{z}_{\alpha\lambda}(t)}\, \delta^4(x - z_\alpha(t)) \frac{p^\mu_\alpha(t) p^\nu_\alpha(t)}{m_\alpha}, \tag{303}$$

where

$$p^\mu_\alpha(t) = m_\alpha \frac{\dot{z}^\mu_\alpha(t)}{\sqrt{-g_{\kappa\lambda}(z_\alpha(t))\dot{z}^\kappa_\alpha(t)\dot{z}^\lambda_\alpha(t)}}. \tag{304}$$

(Recall that $z^0_\alpha = t$ and that $t$ is just a dummy integration variable $\neq x^0$.) Specializing to one species this can be written as

$$\begin{aligned}
T^{\mu\nu}(x) &= \int \frac{d^3p}{(2\pi)^3}\, h(x, \mathbf{p})\, \frac{p^\mu(x) p^\nu(x)}{p^0(x)}, && (305)\\
p^0(x) &= \text{positive root of eq.: } g_{\kappa\lambda}(x) p^\kappa(x) p^\lambda(x) = -m^2, && (306)\\
h(x, \mathbf{p}) &= \frac{1}{\sqrt{-g(x)}} \sum_\alpha \delta^3(\mathbf{x} - \mathbf{z}_\alpha(x^0))\, (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}_\alpha(x^0)), && (307)
\end{aligned}$$

where we have assumed a spin-weight factor $g = 1$ to avoid confusion with the determinant of the metric, $g$. For simplicity we have written $p^\mu(x)$, but note that

$\mathbf{p}$ does not depend on $x$, only $p^0$ depends on $x$. On the other hand $p^\mu_\alpha$ depends on time via its dependence on $z_\alpha$, as in (304).

The distribution $h(x, \mathbf{p})$ corresponds to a realization of $f(x, \mathbf{p})$ in terms of particles labeled by $\alpha$. We shall now take the time derivative of $h(x, \mathbf{p})$, use the equation of motion for the particles and re-express the result as an equation for $h$. Then we shall assume this equation for $h$ to apply also to more general distribution functions $f(x, \mathbf{p})$. We first assume no scattering among the particles.

a. The equations of motion for the particles are the geodesic equations derived in Problems 3d,3e. Verify that they can be written as

$$\partial_0 p^\mu_\alpha = -\Gamma^\mu{}_{\rho\sigma}(z_\alpha)\frac{p^\rho_\alpha p^\sigma_\alpha}{p^0_\alpha}. \tag{308}$$

b. Verify (cf. sect. 3)

$$\partial_\mu \frac{1}{\sqrt{-g(x)}} = -\frac{1}{\sqrt{-g(x)}}\,\Gamma^\rho{}_{\rho\mu}(x) \tag{309}$$

c. Verify the following identity among delta functions (by integration with an arbitrary test function)

$$F(\mathbf{p}_\alpha)\frac{\partial}{\partial p^k}\,\delta^3(\mathbf{p}-\mathbf{p}_\alpha) = F(\mathbf{p})\frac{\partial}{\partial p^k}\,\delta^3(\mathbf{p}-\mathbf{p}_\alpha) + \delta^3(\mathbf{p}-\mathbf{p}_\alpha)\,\frac{\partial}{\partial p^k}F(\mathbf{p}). \tag{310}$$

d. Now take the time derivative of $h$ and show that the result can be written in the form

$$p^\mu\partial_\mu h - \Gamma^k{}_{\rho\sigma}p^\rho p^\sigma\frac{\partial}{\partial p^k}h + \left[p^\mu\Gamma^\rho{}_{\rho\mu} - \Gamma^k{}_{\rho\sigma}p^0\frac{\partial}{\partial p^k}\left(\frac{p^\rho p^\sigma}{p^0}\right)\right]h = 0, \tag{311}$$

where $h = h(x, \mathbf{p})$, $\Gamma^k{}_{\rho\sigma} = \Gamma^k{}_{\rho\sigma}(x)$, $p^0 = p^0(x)$.

e. Specializing to a flat Robertson-Walker metric, $g_{mn} = a^2(x^0)\delta_{mn}$, show that the above equation reduces to

$$\partial_0 h - 3\frac{\dot a}{a}\,h - 2\frac{\dot a}{a}\,p^k\frac{\partial}{\partial p^k}\,h = 0, \tag{312}$$

where we assumed homogeneous circumstances, $h = h(x^0, \mathbf{p})$.

We conclude that the Liouville part of the Boltzmann equation for $f = f(x^0, \mathbf{p})$ is

$$L(f) = \partial_0 f - 3\frac{\dot a}{a}\,f - 2\frac{\dot a}{a}\,p^i\frac{\partial}{\partial p^i}\,f \tag{313}$$

Note that the first term in (311) corresponds to (292).

The solution of (306) is now

$$p^0 = \sqrt{m^2 + a^2\mathbf{p}^2}. \tag{314}$$

It depends only on time via the scale factor $a(x^0)$. Eq. (314) suggests using the variable

$$\bar{\mathbf{p}} \equiv a\mathbf{p} \tag{315}$$

as the argument of the distribution function, since then $p^0 = \sqrt{m^2 + \bar{\mathbf{p}}^2}$, as usual. Let us write

$$f(t, \mathbf{p}) = a^3 \bar{f}(t, a\mathbf{p}) = a^3 \bar{f}(t, \bar{\mathbf{p}}). \tag{316}$$

f. Assuming $\bar{f}$ to depend only on $|\bar{\mathbf{p}}| \equiv \sqrt{\sum_{k=1}^{3} \bar{p}^k \bar{p}^k}$, verify

$$\begin{aligned} T^{kl} &= g^{kl} P, && (317) \\ P &= \int \frac{d^3\bar{p}}{(2\pi)^3}\, \bar{f}(t, \bar{\mathbf{p}}) \frac{|\bar{\mathbf{p}}|^2}{3p^0}, && (318) \end{aligned}$$

We see that the above equation has the familiar form of the pressure. The expression for $T^{00}$ looks similarly familiar when written in terms of $\bar{\mathbf{p}}$.

g. Verify

$$L(f) = a^3 \bar{L}(\bar{f}), \quad \bar{L}(\bar{f}) = \partial_0 \bar{f} - \frac{\dot{a}}{a} \bar{p}^k \frac{\partial}{\partial \bar{p}^k}\, \bar{f}. \tag{319}$$

h. The distribution function for a decoupled species satisfies $\bar{L}(\bar{f}) = 0$. Show that a general solution is given by

$$\bar{f}(t, \bar{\mathbf{p}}) = \tilde{f}(a(t)\bar{\mathbf{p}}). \tag{320}$$

Evidently, $\bar{\mathbf{p}}$ is to be identified with the usual momentum in a local Lorentz frame and $\bar{f}$ with the distribution function introduced earlier in the context of special relativity. The scattering terms in the Boltzmann equation will therefore have the usual form in terms of $\bar{\mathbf{p}}$. These arguments lead to the Bolzmann equation including scattering:

$$\bar{L}_k(\bar{f}) = \partial_0 \bar{f}_k - \frac{\dot{a}}{a} \bar{p}^i \frac{\partial}{\partial \bar{p}^i}\, \bar{f}_k = C_k(\bar{f}), \tag{321}$$

where we reintroduced the species label $k$.

**From now on we drop the bar** on $\bar{\mathbf{p}}$, $\bar{f}$ and $\bar{L}$.

i. Show that the density $n_k = g_k \int_{\mathbf{p}} f_k$ satisfies

$$\partial_0 n_k + 3\frac{\dot{a}}{a}\, n_k = g_k \int \frac{d^3p}{(2\pi)^3}\, C_k(f), \tag{322}$$

where we reintroduced the spin-weight $g_k$ since there is no more confusion with the metric anymore.

Using (302) and assuming $\overline{|\mathcal{M}_{ij|kl}|^2} = \overline{|\mathcal{M}_{kl|ij}|^2}$ we get

$$\begin{aligned} \partial_0 n_k + 3\frac{\dot{a}}{a}\, n_k &= \frac{1}{2}\, g_k \sum_{ijl} g_i g_j g_l \int d\omega_i\, d\omega_j\, d\omega_k\, d\omega_l\, (2\pi)^4 \delta^4(p_i + p_j - p_k - p_l) \\ & \overline{|\mathcal{M}_{ij|kl}|^2}\, [f_i f_j (1+\eta_k f_k)(1+\eta_l f_l) - f_k f_l (1+\eta_i f_i)(1+\eta_j f_j)]\,. \end{aligned} \tag{323}$$

This equation may be compared with eqs. (5.7)-(5.10) in Kolb and Turner, and is used in their discussion of the 'origin of species'.

If the collision terms $C_k$ dominates over the expansion term $\propto \dot{a}/a$, then we expect the distribution $f_k$ to acquire the equilibrium form

$$f_k(\mathbf{p}) = \frac{1}{e^{(\sqrt{m_k^2+\mathbf{p}^2}-\mu_k)/T} - \eta}, \tag{324}$$

with temperature and chemical potential that vary slowy on the collision time scale. As the universe cools down, the $f_k$ become smaller and after some time, depending on the strength of interactions of species $k$, the collision term $C_k$ becomes subdominant to the expansion term. The above form for $f_k$ is then then no longer a good approximation and the species $k$ is said to have decoupled from the thermal plasma and to have fallen out of equilibrium.

Consider the equation

$$\dot{f}_k(t, \mathbf{p}) - \frac{\dot{a}(t)}{a(t)} p^i \frac{\partial}{\partial p^i} f_k(t, \mathbf{p}) = 0. \tag{325}$$

According to (320), solutions may be found in the form

$$f_k(t, \mathbf{p}) = f_k\left(t_1, \frac{a(t)}{a(t_1)}\mathbf{p}\right). \tag{326}$$

which may be interpreted in the light of the result of problem 5.i, in which it was found that the magnitude of the momentum of a particle redshifts $\propto 1/a$. (Note that in problem 5.i the magnitude of the momentum $p^i$, which is related to the spatial component of the four-velocity by $p^i = mu^i$, was defined naturally in terms of the metric as $\sqrt{g_{ij}p^i p^j} = a|\mathbf{p}|$, and $a\mathbf{p} = \bar{\mathbf{p}}$ is denoted here by $\mathbf{p}$, since we omitted the 'bar' in our notation.)

In the aproximation that the decoupling takes place instantaneously at time $t_1$, there are two cases in which the distribution function can still be characterized by a time-dependent temperature $T(t)$:

- massless particles

$$f(t, \mathbf{p}) = \frac{1}{e^{|\mathbf{p}|/T(t)} - \eta}, \quad T(t) = \frac{a(t_1)}{a(t)} T(t_1); \tag{327}$$

- non-relativistic particles with $\mu = m$,

$$f(t, \mathbf{p}) = \frac{1}{e^{\mathbf{p}^2/2mT(t)} - \eta}, \quad T(t) = \frac{a(t_1)^2}{a(t)^2} T(t_1), \tag{328}$$

# 10 Freeze out

## 10.1 Simplified equation

As the temperature lowers, the density of massive particles decreases and would be ridiculously small ($\propto e^{-m/T}$) now, at 3 K or $3 \times 10^{-4}$ eV, if thermal equilibrium would be maintained. However, below a certain temperature, the collision term in the Boltzmann equation for some particle species will become negligible compared to the expansion term, upon which its distribution function freezes into a function of $a(t)p$, as in (326). The actual particle density then drops at a much slower rate than the equilibrium density and the number of particles in a comoving volume is conserved. In this way the expansion of the universe is vital for the 'origin of species'.

In this section we shall simplify coupled Boltzmann equations for several species into a much more pleasant equation, following section 5.2 in Kolb & Turner. Let $\psi$ and $\bar{\psi}$ be the particles that freeze out and suppose they annihilate into $X$ and $\bar{X}$ according to

$$\psi + \bar{\psi} \leftrightarrow X + \bar{X}, \tag{329}$$

apart from other processes such as $\psi + X \leftrightarrow \psi + X$, $\bar{\psi} + X \leftrightarrow \bar{\psi} + X$, etc. We suppose furthermore that there are other particles and processes such that $X$ and $\bar{X}$ are in thermal (i.e. kinetic and chemical) equilibrium, and $\psi$ and $\bar{\psi}$ in kinetic (but not necessarily chemical) equilibrium, at the time of freeze out. For example, the $\psi$s could be neutrinos, $\psi$, $\bar{\psi} = \nu$, $\bar{\nu}$, the $X$s electrons, $X$, $\bar{X} = e^-$, $e^+$, and the other particles photons. Another example is $\psi = e^-$, $\bar{\psi} = p$ (proton), $X = H$ (hydrogen atom) and $\bar{X} = \gamma$ (photon). Using the Boltzmann approximation for the distribution functions,

$$f_X(t, \mathbf{p}) = e^{(\mu_X(t) - E_X)/T(t)}, \quad f_{\bar{X}}(t, \mathbf{p}) = e^{(\mu_{\bar{X}}(t) - E_{\bar{X}})/T(t)}, \quad E_X = E_{\bar{X}} = \sqrt{m_X^2 + \mathbf{p}^2}, \tag{330}$$

thermal equilibrium means that the temperature $T$ and the chemical potentials $\mu_{X,\bar{X}}$ are fixed by the equilibrium conditions. Kinetic but not chemical equilibrium for the $\psi$s means that, in

$$f_\psi = e^{(\mu_\psi - E_\psi)/T}. \quad f_{\bar{\psi}} = e^{(\mu_{\bar{\psi}} - E_{\bar{\psi}})/T}, \tag{331}$$

only the temperature is fixed to the equilibrium temperature, whereas the chemical potentials $\mu_\psi$ and $\mu_{\bar\psi}$ may be out of equilibrium. Assuming also detailed balance, $\overline{|\mathcal{M}(\psi+\bar\psi\to X+\bar X)|^2} = \overline{|\mathcal{M}(X+\bar X\to \psi+\bar\psi)|^2}$ at given momenta, the Boltzmann equation for the number density of the $\psi$ takes the form[6]

$$\dot n_\psi + 3\frac{\dot a}{a} n_\psi = g_X^2 g_\psi^2 \int d\omega_\psi\, d\omega_{\bar\psi}\, d\omega_X\, d\omega_{\bar X}\, (2\pi)^4 \delta(p_\psi+p_{\bar\psi}-p_X-p_{\bar X})\, \overline{|\mathcal{M}|^2}\, (f_X f_{\bar X} - f_\psi f_{\bar\psi}). \tag{332}$$

Energy conservation, $E_X + E_{\bar X} = E_\psi + E_{\bar\psi}$, leads to

$$f_X f_{\bar X} = f_\psi^{\rm eq} f_{\bar\psi}^{\rm eq}, \tag{333}$$

where $\mu_X + \mu_{\bar X} = \mu_\psi^{\rm eq} + \mu_{\bar\psi}^{\rm eq}$. We also have

$$f_\psi f_{\bar\psi} = f_\psi^{\rm eq} f_{\bar\psi}^{\rm eq}\, e^{(\mu_\psi+\mu_{\bar\psi}-\mu_\psi^{\rm eq}-\mu_{\bar\psi}^{\rm eq})/T}. \tag{334}$$

So, in kinetic equilibrium, the momentum dependence in the product $f_\psi f_{\bar\psi}$ is equivalent to that of $f_\psi^{\rm eq} f_{\bar\psi}^{\rm eq}$. One now rewrites the Boltzmann equation in terms of the annihilation cross section of the $\psi$s ($v$ is the relative velocity cf. (276))

$$\sigma_{\psi\bar\psi\to X\bar X} v = g_X^2 \int d\omega_X\, d\omega_{\bar X}\, (2\pi)^4 \delta(p_\psi+p_{\bar\psi}-p_X-p_{\bar X})\, \overline{|\mathcal{M}_{\psi\bar\psi\to X\bar X}|^2}, \tag{335}$$

and its average in thermal equilibrium

$$\langle\sigma_{\rm ann} v\rangle = \frac{g_\psi^2 \int d\omega_\psi\, d\omega_{\bar\psi}\, f_\psi^{\rm eq} f_{\bar\psi}^{\rm eq} \sigma_{\psi\bar\psi\to X\bar X} v}{g_\psi^2 \int d\omega_\psi\, d\omega_{\bar\psi}\, f_\psi^{\rm eq} f_{\bar\psi}^{\rm eq}} = \frac{g_\psi^2 \int d\omega_\psi\, d\omega_{\bar\psi}\, f_\psi^{\rm eq} f_{\bar\psi}^{\rm eq}\, \sigma_{\psi\bar\psi\to X\bar X} v}{(n_\psi^{\rm eq})^2}, \tag{336}$$

resulting in the remarkably simple equation

$$\dot n_\psi + 3\frac{\dot a}{a} n_\psi = \langle\sigma_{\rm ann} v\rangle \left((n_\psi^{\rm eq})^2 - n_\psi^2\right). \tag{337}$$

It is useful to consider the ratio

$$Y = n_\psi / s, \tag{338}$$

where $s$ is the entropy density. Entropy conservation $sa^3 =$ constant (cf. problem 1) leads to

$$\frac{dY}{dt} = s\langle\sigma_{\rm ann} v\rangle \left(Y_{\rm eq}^2 - Y^2\right). \tag{339}$$

We get the Hubble rate $H = \dot a/a$ back into the picture by using

$$x = \frac{m}{T}, \tag{340}$$

[6] There is no factor 1/2 here compared to (323) because we do not sum here separately over $\psi+\bar\psi \leftrightarrow X+\bar X$ and $\psi+\bar\psi \leftrightarrow \bar X + X$.

as a variable representing time, where $m$ is some mass, e.g. $m = m_\psi$ if $\psi$ is massive. In the radiation dominated era (recall (97))

$$H = \frac{1}{2t} = \text{constant} \times T^2, \tag{341}$$

which leads to

$$\frac{x}{Y_{\rm eq}} \frac{dY}{dx} = \frac{\Gamma_{\rm ann}}{H} \left(1 - \frac{Y^2}{Y_{\rm eq}^2}\right), \tag{342}$$

where $\Gamma_{\rm ann}$ is the annihilation rate in thermal equilibrium,

$$\Gamma_{\rm ann} = \langle \sigma_{\rm ann} v \rangle n_\psi^{\rm eq}. \tag{343}$$

Equation (342) suggests indeed that $Y \approx Y_{\rm eq}$ for $\Gamma_{\rm ann}/H \gg 1$, and $Y > Y_{\rm eq}$ for $G_{\rm ann}/H \ll 1$. See also (9.41) in Peacock.

## 10.2 Hot and cold relics

To continue one uses a simple parametrization of the annihilation rate. Inspection of the momentum dependence of $\overline{|\mathcal{M}|^2}$ shows that the temperature dependence of a thermally averaged $\langle \sigma v \rangle$ is approximately just a power behavior, $\langle \sigma_{\rm ann} v \rangle \propto T^n \propto x^{-n}$, with $n = 0, 1, \ldots$ (typically $n = 0$). Then, with $H \propto x^{-2}$ and $s \propto x^{-3}$, we can rewrite (342) in the form

$$\frac{dY}{dx} = -\lambda x^{-n-2} \left(Y^2 - Y_{\rm eq}^2\right), \tag{344}$$

with

$$\lambda = \left[ s x \frac{\langle \sigma_{\rm ann} v \rangle}{H} \right]_{x=1} \tag{345}$$

The behavior of $Y_{\rm eq}$ follows easily from the results in section 7,

$$\begin{aligned} Y_{\rm eq} &= \frac{45\zeta(3)}{2\pi^4} \frac{g_{\rm eff}}{g_{*S}} = 0.278 \frac{g_{\rm eff}}{g_{*S}}, & x \ll 3, \qquad (346) \\ &= \frac{45}{4\pi^3 \sqrt{2\pi}} \frac{g_\psi}{g_{*S}} x^{3/2} e^{-x} = 0.145 \frac{g_\psi}{g_{*S}} x^{3/2} e^{-x}, & x \gg 3, \qquad (347) \end{aligned}$$

where $x = m_\psi/T$ and $g_{\rm eff} = g_\psi$ (bosons) and $g_{\rm eff} = 3g_\psi/4$ (fermions). So $Y_{\rm eq}$ is time-independent in the relativistic regime $x \ll 3$ and it decreases exponentially fast in the non-relativistic regime $x \gg 3$. Depending on the value of $\lambda$, a species $\psi$ may decouple 'early' or 'late'.

A *hot relic* decouples when it is still relativistic. Let $x_f$ represent the time of freeze out. Then we expect $Y(\infty) \approx Y_{\rm eq}(x_f)$, and since $Y_{\rm eq}(x)$ is almost independent of $x$ in the relativistic domain, the precise value of $x_f$ does not matter very much. It may be estimated from the criterion $\Gamma_{\rm ann} = H$. Assuming adiabatic expansion (no entropy production) after freeze out of the species under study, we have today

$$n_{\psi 0} = s_0 Y(\infty) \approx 2900 Y(\infty) \approx 800 \frac{g_{\text{eff}}}{g_{*S}(x_f)} \text{ cm}^{-3}, \tag{348}$$

with today's $g_{*S} = 2 + (7/8) \times 2 \times 3 \times (4/11) = 3.91$ in photons and three massless neutrino/antineutrino species at $T_\nu = (4/11)^{1/3} T_\gamma$, $T_\gamma = 2.73$ K. The energy density today in the massive $\psi$s is given by

$$\rho_{\psi 0} = n_{\psi 0} m_\psi, \tag{349}$$

or for the fractional density

$$\Omega_\psi h^2 = h^2 \frac{\rho_{\psi 0}}{\rho_c} = 8.00 \times 10^{-2} \frac{g_{\text{eff}}}{g_{*S}(x_f)} \frac{m}{\text{eV}}, \tag{350}$$

where $h$ is 'little Hubble constant' and we used $\rho_c = 8.0992\, h^2 \times 10^{-47}$ GeV$^4$.

The above equation sets a strong upper bound on $m_\psi$ by requiring $\Omega_\psi$ to be less than one. Suppose the $\psi$ are light-mass neutrinos. Using dimensional analysis, their annihilation rate is of order $G_F^2 T^5$, with $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$ the Fermi constant governing the low-energy weak interactions. Comparing $\Gamma_{\text{ann}}$ with $H = O(T^2/m_P)$ gives a decoupling temperature $T$ of order 1 MeV. At that time $g_{*S}(x_f) = 2 + (7/8)(2 \times 2 + 3 \times 2) = 10.75$ in photons (anti)electrons and (anti)neutrinos. For a massive, left-handed neutrino plus its antiparticle, $g_{\text{eff}} = 2 \times (3/4) = 1.5$, $g_{\text{eff}}/g_{*S} = 0.140$, and we obtain the bound

$$\Omega_\nu < 1 \Rightarrow m_\nu < 89\, h^2 \text{ eV}, \tag{351}$$

or, with today's values $h \approx 0.7$, $\Omega_{\text{matter}} \approx 0.3$ and three neutrino species,

$$\sum_\nu m_\nu < 89\, h^2\, \Omega_{\text{matter}} < 13 \text{ eV}. \tag{352}$$

The reasoning leading to this bound was first given by Gerstein and Zeldovich[7]

*Cold relics* decouple when they have become non-relativistic. In this case the precise value of $x_f$ is important because $Y_{\text{eq}}$ is rapidly varying in this regime, and the same will be true for $Y$ itself.

Kolb & Turner apply equations (342) and (344) to various examples in sections 5.2 (Freeze Out) and 5.4 (Recombination Revisited). There is no substitute for their skillful and entertaining exposition, and the reader is urged to take a good look at these sections. Note in particular a formula for cold relics:

$$Y_{x=\infty} \approx \frac{3.8(n+1)(g_*^{1/2}/g_{*S})\, x_f}{m_\psi m_{\text{P}} \langle \sigma_{\text{ann}} v \rangle}, \quad m_{\text{P}} = G^{-1/2}, \tag{353}$$

[7] See the discussion in A.D. Dolgov, *Nuetrinos in cosmology*, arXiv:hep-ph/0202122.

where the right-hand side is evaluated at freeze out. There is also a simple equation for the inverse freeze-out temperature $x_f$, obtained by matching simple asymptotic solutions to numerical solutions. Note that the abundance $Y$ is inversely proportional to the magnitude of the annihilation cross section at decoupling. Weaker interactions lead to earlier decoupling and larger abundance ('the weak prevail').

## 10.3 Problem

1. The chemical potentials in the radiation dominated era are believed to be very small (e.g. for baryon number $\mu/T = O(10^{-10})$). Neclecting them, the entropy density in local thermal equilibrium is given by $s = (\rho + p)/T$, where $\rho$ and $p$ are functions of the instantaneous temperature (cf. (215)).

   Show, using (65), that the entropy in a unit comoving volume $V = a^3$ is conserved in time, i.e. $sa^3 =$ constant.

   Verify that

   $$s = \frac{2\pi^2}{45}\, g_{*S}\, T^3, \tag{354}$$

   with

   $$g_{*S} = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3. \tag{355}$$

# 11 The Standard Model

The Standard Model describes the strong, electromagnetic and weak interactions currently known and verified by experiment, ignoring gravity.

## 11.1 Spontaneous symmetry breaking

[8] The principle of local gauge invariance works beautifully for the strong and electroweak interactions. It gives a method for determining the couplings and as 't Hooft, Veltman and others showed in the early seventies, even spontaneously-broken non-abelian gauge theories are renormalizable. The application to weak interactions was more subtle because gauge invariance forbids mass terms for gauge fields and whereas the photon and gluons are indeed massless, the W's and Z's certainly are not. Gauge fields as it turned out can be given a mass, exploiting spontaneous symmetry breaking and the Higgs mechanism. This is a subtle procedure and it pays to begin by thinking about how one identifies a mass term in a Lagrangian.

Consider the following Lagrangian for a scalar field $\phi$:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4. \tag{356}$$

Here $\mu^2 < 0$ and $\lambda > 0$. The second term looks like a mass and the third like an interaction. However, the sign of $\mu^2$ is wrong! If that's a mass term, the mass would be imaginary, which is nonsense. The answer on how to interpret this Lagrangian comes from understanding the perturbation procedure, in which we start from the ground state (the 'vacuum') and treat the fields as fluctuations about that state. For the Lagrangian (356) the trivial field configuration $\phi = 0$ is not the ground state! To determine the 'true' ground state, the field configuration of minimum energy, we write $\mathcal{L}$ as a free 'kinetic' term $\mathcal{T} = (\frac{1}{2}\partial_0\phi)(\partial^0\phi)$ minus a 'potential' term $\mathcal{U}$ (that is, writing it as $T^{00}$, cf. equation (102):

$$\mathcal{L} = \mathcal{T} - \mathcal{U}, \quad \mathcal{U} = V + \frac{1}{2}\partial_k\phi\partial_k\phi, \tag{357}$$

and look for the minimum of $\mathcal{U}$. This occurs for constant fields, $\partial_k\phi = 0$, and consequently $\mathcal{U} = V$. In the present case:

$$V = -\frac{1}{2}|\mu|^2\phi^2 + \frac{1}{4}\lambda\phi^4. \tag{358}$$

The minimum of $V$ then occurs at:

$$\phi = v = \pm|\mu|/\sqrt{\lambda}. \tag{359}$$

[8] The following three sections were contributed by Stefan Nobbenhuis.

For this Lagrangian the Feynman calculus must be formulated in terms of deviations from one or the other of these ground states. This suggests that we introduce a new field variable $\eta$, defined by:

$$\eta \equiv \phi - v. \tag{360}$$

a) Sketch the potential, give the Lagrangian in terms of $\eta$ and interpret the different terms.

The example above illustrates an important phenomenon. The original Lagrangian is even in $\phi$: It is invariant as $\phi \to -\phi$. But the reformulated Lagrangian is not even in $\eta$; the symmetry has been 'broken'. This happened because the 'vacuum', whichever of the two ground states we care to work with, does not share the symmetry of the Lagrangian. The collection of all ground states of course does, but to set up the perturbation formalism we are obliged to work with one or the other of them and that spoils the symmetry. We call this 'spontaneous' symmetry breaking, because no external agency is responsible (as for example gravity breaks the the three-dimensional symmetry in this room explicitly, making 'up' and 'down' quite different from 'left' and 'right'). The true symmetry of the system in other words, is hidden.

## 11.2 The Higgs mechanism

Now we apply the idea of spontaneous symmetry breaking to local gauge theories. We begin with an Abelian example and end with the Glashow-Weinberg-Salam theory of electroweak interactions.

Consider a complex scalar field coupled to both itself and to an electromagnetic field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |D_\mu\phi|^2 - V(\phi), \tag{361}$$

with $D_\mu = \partial_\mu - ieA_\mu$ the covariant derivative and $A_\mu$ is the vectorpotential with $e$ the coupling to the scalar field.

a) Show that this Lagrangian is invariant under the local $U(1)$ transformation

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \qquad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x). \tag{362}$$

We now choose the potential to be of the form:

$$V(\phi) = \mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, \tag{363}$$

where $\lambda > 0$.

b) Show that when

$\mu^2 < 0$, the field $\phi$ will acquire a vacuum expectation value and that the $U(1)$ global symmetry will be spontaneously broken. What happens if $\mu^2 > 0$?

We assume now $\mu^2 < 0$. Let us expand the Lagrangian (361) about the vacuum state, $\phi = v$ and decompose it as:

$$\phi(x) = v + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)). \tag{364}$$

c) Rewrite the Lagrangian in terms of $\phi_i, i = 1, 2$ and determine the apparent masses of the different fields.

You should find that the gauge fields have acquired mass terms.

One of the fields, say $\phi_1$, has acquired a mass, while the other field $\phi_2$ remains massless. This massless field is called a Goldstone boson field. It can be shown (Goldstone's theorem) that spontaneous breaking of a continuous global symmetry is always accompanied by the appearance of one or more massless scalar particles, which are accordingly called Goldstone bosons. The number of such Goldstone bosons is always equal to the difference between the order (i.e. the number of generators) of the original symmetry group and the order of the surviving symmetry group. They can be understood physically as being excitations along the symmetry directions in which the potential is unchanged. Suppose for example we would have started with the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}\mu^2\phi_i^2 - \frac{1}{4}\lambda\phi_i^2, \tag{365}$$

describing a theory of $N$ real scalar fields $\phi_i$ where summation over $i = 1, \ldots, N$ is implied. In this example the original group of symmetry transformations is $O(N)$, with $\frac{1}{2}N(N-1)$ generators and the final group after rewriting the Lagrangian as fluctuations about one of its ground states,

say $\phi_1 = v$, $\phi_2 = \cdots = \phi_N = 0$, is $O(N-1)$ with $\frac{1}{2}(N-1)(N-2)$ generators, so there will be $(N-1)$ massless particles (which you can check explicitly).

To show the generality of this result, suppose $\phi = \{\phi_i\}$ forms a multiplet of symmetrygroup $G$, such that $V(\phi)$ and hence $\mathcal{L}(\phi)$ is invariant under:

$$\phi \to \phi + \delta\phi = (1 + i\alpha_a T^a)\phi \tag{366}$$

where $T^a$, with $a = 1, \ldots, N$, are the generators of $G$. If the minimum of the potential corresponds to $\langle 0|\phi_i|0\rangle = v_i$, then:

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi=v} = 0, \qquad \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\phi=v} = M_{ij}^2 > 0. \tag{367}$$

where $M$ is the mass matrix for the fields $\phi$. Now suppose that some of the generators satisfy:

$$T^a v = 0, \qquad a = 1, \ldots, n, \tag{368}$$

while the remaining generators break the symmetry of the vacuum because:

$$T^a v \neq 0, \qquad a = (n+1), \ldots, N. \tag{369}$$

The $n$ unbroken generators form a subgroup $H$ of $G$ since they are closed under commutation, $(T^a T^b - T^b T^a)v = 0$. To determine the effect of the symmetry breaking on the mass matrix, note that invariance of $V(\phi)$ under (366) gives:

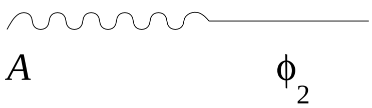
$$0 = V(\phi + \delta\phi) - V(\phi) = \frac{\partial V}{\partial \phi_i}\delta\phi_i = i\alpha_a \frac{\partial V}{\partial \phi_i}(T^a)_{ij}\phi_j. \tag{370}$$

To express this in terms of $M$, we differentiate with respect to $\phi_k$ and evaluate it at $\phi = v$ using (367) to obtain:

$$M^2_{ki}(T^a v)_i = 0. \tag{371}$$

For the unbroken subgroup $H$ this is trivially satisfied, but for each broken generator $T^a v$ is an eigenvector of $M^2$ with zero eigenvalue. Thus the number of massless Goldstone bosons is simply the number $(N - n)$ of broken generators.

Returning to our example with the complex scalar field, we still seem to have a massless Goldstone boson and there is a somewhat strange looking quantity: $\sqrt{2}\, ev A_\mu \partial^\mu \phi_2$. If we read it as an interaction it leads to a vertex of the form:




This suggests that the massless Goldstone boson turns into the massive gauge boson $A$. Any such term, bilinear in two different fields, indicates that there is some form of mixing between the fields in the theory and this would mean that neither one exists as an independent 'physical' field. The physical fields are those for which the mass matrix is diagonal. Although the Goldstone boson plays an important formal role in this theory, it does not appear as an independent physical particle. The easiest way to see this is to make a particular choice of gauge, called the *unitarity gauge*. Using the local $U(1)$ gauge symmetry (362), we can choose $\alpha(x)$ in such a way that $\phi(x)$ becomes real-valued at every point $x$.

d) Show that with this choice the field $\phi_2$, the Goldstone boson field, is removed from the theory and that the Lagrangian (361) becomes:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - (\partial_\mu \phi_1)^2 - e^2 \phi_1^2 A_\mu A^\mu - V(\phi_1). \tag{372}$$

If the potential $V(\phi)$ favors a non-zero vacuum expectation value of $\phi$, the gauge field acquires a mass; it also retains a coupling to the remaining physical field $\phi_1$.

This mechanism, by which spontaneous symmetry breaking generates a mass for a gauge boson, was explored and generalized to the non-Abelian case by Higgs, Kibble, Guralnik, Brout and Englert and is now known as the *Higgs mechanism*. The role of the Goldstone thus is intricate and seems somewhat mysterious at this level of the discussion. We saw that the involvement of the Goldstone boson is necessary, as a matter of principle, in order for the gauge boson to acquire a mass. But we also saw that the Goldstone boson field can be formally eliminated from the theory! However, we might argue that the Goldstone boson has not completely disappeared. A massless vector boson has only two physical polarization states, as is the case for the photon, the longitudinal polarization state cannot be produced and appears in the formalism only to cancel other unphysical contributions. However, a massive vector boson must have three physical polarization states: In its rest frame, it is a spin-1 object, which can make no distinction between transverse and longitudinal polarizations. It is therefore tempting to say that the gauge boson acquired its extra degree of freedom by 'eating' the Goldstone boson.

## 11.3 Standard Model electroweak theory

We will now discuss the spontaneously broken gauge theory of the weak interactions, an important part of today's Standard Model, introduced by Glashow, Weinberg and Salam. This model incorporates (as not to say: unifies) a description of weak and electromagnetic interactions, in which, as we shall see, the massless photon will correspond to a particular combination of symmetry generators that remain unbroken. This will require some group theoretic slang to make our steps explicit, however, the important aspects should become clear without to much worrying about that.

We begin with a theory with $SU(2)$ gauge symmetry. To break the symmetry spontaneously, we introduce a scalar field in the spinor representation of $SU(2)$. However, this theory will lead to a system with no massless gauge bosons. Therefore, we introduce an additional $U(1)$ gauge symmetry. We assign the scalar field a charge $+1/2$ under this $U(1)$ symmetry, so that its complete gauge transformation is:

$$\phi \to e^{i\beta^a I^a} e^{i\alpha/2}\phi, \tag{373}$$

where the $I^a = \sigma^a/2$ and the $\sigma$-matrices are the familiar Pauli spin matrices. If the field $\phi$ acquires a vacuum expectation value of the form:

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \tag{374}$$

then a gauge transformation with:

$$\beta^1 = \beta^2 = 0, \qquad \beta^3 = \alpha, \tag{375}$$

leaves $\langle\phi\rangle$ invariant. Thus the theory will contain one massless gauge boson, corresponding to this particular combination of generators. The remaining three gauge bosons will acquire masses from the Higgs mechanism.

It is in principle rather straightforward to work out the details of the mass spectrum. The covariant derivative of $\phi$ is:

$$D_\mu\phi = \left(\partial_\mu - igA_\mu^a\frac{\sigma^a}{2} - i\frac{1}{2}g'B_\mu\right)\phi, \tag{376}$$

where $A_\mu^a$ and $B_\mu$ are respectively, the $SU(2)$ and $U(1)$ gauge bosons. Since the $SU(2)$ and $U(1)$ factors of the gauge group commute with one another, they can have different coupling constants, which we have called $g$ and $g'$.

The gauge boson mass terms come from the square of (376), evaluated at the scalar field expectation value (374).

a) Find the eigenstates of the mass matrix and corresponding eigenvalues, the masses. These are the famous $W^\pm$ and $Z^0$ vector bosons of weak interactions and we will identify the massless field $A^\mu$ with the electromagnetic vector potential.

From particle masses and interaction strengths it is found that $v$, the symmetry breaking scale in this model is about 246 GeV.

In many cases it will turn out to be easier to work with the mass eigenstate fields. For a fermion field belonging to a general representation of $SU(2)$ and $U(1)$ charge $Y$, the covariant derivative takes the form:

$$D_\mu = \partial_\mu - igA_\mu^a I^a - ig'\frac{Y}{2}B_\mu \tag{377}$$

b) Show that in terms of the mass eigenstate fields this becomes:

$$\begin{aligned} D_\mu &= \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ I^+ + W_\mu^- I^-) - i\frac{1}{\sqrt{g^2+g'^2}}Z_\mu(g^2 I^3 - g'^2 Y) \\ &- i\frac{gg'}{\sqrt{g^2+g'^2}}A_\mu(I^3+Y), \end{aligned} \tag{378}$$

where $I^\pm = I^1 \pm iI^2$ and $W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2)$ and the normalizaion is chosen so that, in the spinor representation of $SU(2)$, $I^\pm = \frac{1}{2}(\sigma^1 \pm \sigma^2) = \sigma^\pm$.
The last term makes explicit that the massless gauge boson $A_\mu$ couples to the gauge generators $(I^3 + Y/2)$, which generates precisely the symmetry operation (375).

The massless gauge boson $A$ is naturally interpreted as the photon. Therefore we identify the electric charge quantum number as:

$$Q = I^3 + Y/2. \tag{379}$$

and use this relation to specify the eigenvalues of the $U(1)$ generator, $\frac{1}{2}Y$ (the factor $\frac{1}{2}$ being purely conventional). $I^i$ and $Y$ are referred to as the weak isospin and weak hypercharge generators respectively.

The covariant derivative (378) can be cast in easier form by also identifying the coefficient of the electromagnetic interaction as the electron charge:

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}, \tag{380}$$

The notation can be simplified further by defining the weak mixing angle $\theta_W$, to be the angle that appears in the change of basis from $(A^3, B)$ to $(Z^0, A)$ (see also p.234 from Peacock):

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3 \\ B \end{pmatrix} \tag{381}$$

c) Using (378), (379), (381) rewrite the covariant derivative once more in terms of $e$ and $\sin\theta$. This shows that the couplings of all the weak bosons are described by two parameters: the well measured electron charge $e$ and a new parameter, $\theta_W$.

d)Verify that we can write $m_W = m_Z \cos\theta_W$.

All effects of $W$ and $Z$ exchange processes, at least at tree level, can be written in terms of the three basic parameters $e$, $\theta_W$ and $m_W$. Experiments have shown that $\sin^2\theta_W \simeq 0.23$.

## 11.4 The action of the (Extended) Standard Model

The Standard Model is a gauge theory with gauge group $U(1)\times SU(2)\times SU(3)$. A gauge transformation in the defining respresentation of this group can be written as (the $\frac{1}{2}$ is conventional)

$$(\Omega)_{ac,a'c'} = e^{i\frac{1}{2}\alpha} \left(e^{i\beta_k \frac{1}{2}\sigma_k}\right)_{aa'} \left(e^{i\omega_p \frac{1}{2}\lambda_p}\right)_{cc'}, \tag{382}$$

acting on complex fields, say $\psi^{ac}$, where $a, a' = 1, 2$ or $u, d$ ('up', 'down'), and $c, c' = 1, 2, 3$ or 'red', 'green', 'blue'. The $\lambda_p$ are a complete set of hermitian traceless $3 \times 3$ matrices, normalized as $\mathrm{Tr}\,(\lambda_p\lambda_q) = 2\delta_{pq}$, eight in total, $p = 1, \cdots, 8$, similar to the Pauli matrices, called 'Gell-Mann matrices'. They satisfy the commutation relations $[\lambda_p, \lambda_q] = 2if_{pqr}\,\lambda_r$, where the real $f_{pqr}$ are the structure constants of the group SU(3). Note that $\sigma_k$ and $\lambda_p$ commute, since they act on different indices in the tensor product representation. The lagrangian of the Standard Model is invariant under these space-time dependent gauge transformations.

The basic ingredients for constructing the lagrangian $\mathcal{L}$ are (a) hermiticity, (b) symmetry (Lorentz and gauge invariance), (c) field representation and (d) renormalizability. The latter property is garanteed by allowing only constants in the lagrangian with mass dimension $\geq 0$, and all possible couplings allowed by symmetry have to be present. Renormalizability makes it possible to absorb the effects of the regularization (e.g. a cutoff), needed to define loop integrals which would otherwise be divergent, into the bare constants of the lagrangian. There may still be non-perturbative effects which limit the validity of the model to momenta smaller than some effective scale (the 'Landau ghost', 'triviality scale'), but this scale can be unobservably high, even beyond the Planck scale, depending on the parameters of the model.

The lagrangian consists of several terms,

$$S = \int d^4x\, \mathcal{L}, \tag{383}$$

$$\mathcal{L} = \epsilon + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_{\theta_1} + \mathcal{L}_{\theta_2} + \mathcal{L}_{\theta_3} + \mathcal{L}_{\mathrm{F}} + \mathcal{L}_{\mathrm{FH}} + \mathcal{L}_{\mathrm{M}}, \tag{384}$$

in which $\epsilon$ controlls the cosmological constant, with Lagrange densities $\mathcal{L}_1$, ..., $\mathcal{L}_{\mathrm{M}}$ which will now be described.

The Bose fields in the model are the $U(1)$-gauge field $B_\mu$, the $SU(2)$-gauge fields $A_\mu^k$, $k = 1, 2, 3$, and the $SU(3)$-gauge fields $G_\mu^p$, $p = 1, \cdots, 8$. Their standard derivative parts in the lagrangian are given by (cf. section 8.11)

$$-\mathcal{L}_1 = \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{385}$$

$$-\mathcal{L}_2 = \frac{1}{4} A_{\mu\nu}^k A^{\mu\nu k}, \quad A_{\mu\nu}^k = \partial_\mu A_\nu^k - \partial_\nu A_\mu^k + g_2\, \epsilon_{klm} A_\mu^l A_\nu^m, \tag{386}$$

$$-\mathcal{L}_3 = \frac{1}{4} G_{\mu\nu}^p G^{\mu\nu p}, \quad G_{\mu\nu}^p = \partial_\mu G_\nu^p - \partial_\nu G_\mu^p + g_3\, f_{pqr} G_\mu^q G_\nu^r. \tag{387}$$

Here $g_2$ and $g_3$ are the $SU(2)$ and $SU(3)$ coupling constants. The $\mathcal{L}_\theta$'s are given by

$$\mathcal{L}_{\theta_1} = \theta_1 \frac{1}{64\pi^2} \epsilon^{\kappa\lambda\mu\nu} B_{\kappa\lambda} B_{\mu\nu}, \tag{388}$$

$$\mathcal{L}_{\theta_2} = \theta_2 \frac{1}{64\pi^2} \epsilon^{\kappa\lambda\mu\nu} A_{\kappa\lambda}^k B_{\mu\nu}^k, \tag{389}$$

$$\mathcal{L}_{\theta_3} = \theta_3 \frac{1}{64\pi^2} \epsilon^{\kappa\lambda\mu\nu} G_{\kappa\lambda}^p B_{\mu\nu}^p, \tag{390}$$

with parameters $\theta_1$, $\theta_2$ and $\theta_3$. It can be shown (cf. problem 1) that these $\mathcal{L}_\theta$ are total derivatives, so they have no influence on the field equations that follow from the stationary action principle. Classically they are irrelevant. However, the world is quantum mechanical and the $\theta$s may still influence the quantum evolution in time, e.g. through their effect on the hamiltonian. In addition they may influence the ground state (vacuum).

The set of Bose fields is completed with the Higgs doublet

$$\phi = \begin{pmatrix} \varphi_u \\ \varphi_d \end{pmatrix}. \tag{391}$$

where $\varphi_u$ and $\varphi_d$ are complex scalar fields. The Higgs part of the lagrangian is

$$-\mathcal{L}_{\mathrm{H}} = (D_\mu\phi)^\dagger D^\mu\phi + \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{392}$$

where $\mu^2 < 0$ and $\lambda > 0$ to invoke spontaneous symmetry breaking. The covariant derivative for the Higgs doublet is given by

$$D_\mu\phi = \left(\partial_\mu - ig_1\frac{1}{2}B_\mu - ig_2\frac{1}{2}\sigma_k A^k_\mu\right)\phi, \tag{393}$$

where $g_1$ is the $U(1)$ gauge coupling constant. Note that the Higgs doublet is invariant under $SU(3)$ transformations, it is a singlet under $SU(3)$.

The Fermi fields in the action may be described by Dirac fields $\psi$ and their conjugates $\bar{\psi} = \psi^\dagger\beta$.[9] As far as we know the fermions come in three generations with identical transformation properties. The fields of the first generation may be denoted by $\psi_1 = (\psi_{\nu_e}, \psi_e, \psi_u, \psi_d)$, but it is customary to write simply $(\nu_e, e, u, d)$, with the conjugate fields $\bar{\psi}_1 = (\bar{\nu}_e, \bar{e}, \bar{u}, \bar{d})$. The fields of the second and third generation are $\psi_2 = (\nu_\mu, \mu, c, s)$, $\psi_3 = (\nu_\tau, \tau, t, b)$, and their conjugates.

However, these are not irreducible representations of the Lorentz group, whereas in the Standard Model the representation of the gauge group carried by the fields depends on the type of Lorentz irrep (irreducible representation). The Dirac fields can split into irreps $L$ and $R$ of the Lorentz group with the help of the left- and right-handed projectors $P_L$ and $P_R$,

$$P_L = \frac{1}{2}(1-\gamma_5), \quad P_R = \frac{1}{2}(1+\gamma_5), \quad P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = 0, \quad P_L + P_R = 1. \tag{394}$$

The left- and right-handed fields are obtained from the Dirac fields as

$$\psi_L = P_L\psi, \quad P_L(\bar{\psi}C)^T = (\bar{\psi}_R C)^T = (\psi^\dagger P_R\beta C)^T, \quad \text{left}, \tag{395}$$

$$\psi_R = P_R\psi, \quad P_R(\bar{\psi}C)^T = (\bar{\psi}_L C)^T = (\psi^\dagger P_L\beta C)^T, \quad \text{right}. \tag{396}$$

Here $C$ is the charge conjugation matrix and $T$ denotes transposition.

In general, a field transforms under the gauge group as

$$\psi \to e^{i\alpha\frac{1}{2}Y}\, e^{i\beta_k I_k}\, e^{i\omega_p T_p}\,\psi, \tag{397}$$

[9]Our Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\gamma_0 = -\gamma^0$, $\gamma_0^\dagger = -\gamma_0$, $\gamma_k^\dagger = \gamma_k$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger$, $\beta = i\gamma^0$, $\alpha_k = -\gamma^0\gamma^k$. Furthermore, $\bar{\psi} \equiv \psi^\dagger\beta$ and the charge-conjugation matrix $C$ has the properties $C = -C^T$, $C^\dagger = C^{-1}$, $\gamma_\mu^T = -C^\dagger\gamma_\mu C$. The charge conjugates of $\psi$ and $\bar{\psi}$ are $\psi^{(c)} = (\bar{\psi}C)^T$ and $\bar{\psi}^{(c)} = -(C^\dagger\psi)^T$.

| $\psi_1$ | $Y/2$ | $I_k$ | $T_p$ |
|---|---|---|---|
| $\nu_{eR}$ | 0 | 0 | 0 |
| $e_R$ | –1 | 0 | 0 |
| $\ell_L$ | –1/2 | $\sigma_k/2$ | 0 |
| $u_R$ | 2/3 | 0 | $\lambda_p/2$ |
| $d_R$ | –1/3 | 0 | $\lambda_p/2$ |
| $q_L$ | 1/6 | $\sigma_k/2$ | $\lambda_p/2$ |

Table 1: $U(1) \times SU(2) \times SU(3)$ representation of the fermion fields. Shown are fields in the first generation. The other generations $\psi_2 = (\nu_\mu,\, \mu,\, c,\, s)$ and $\psi_3 = (\nu_\tau,\, \tau,\, t,\, b)$ transform identically. The representation of the $\bar{\psi}$ fields follows from $\bar{\psi} = \psi^\dagger \beta$.

where $I_k$ and $T_p$ are a representation of $\frac{1}{2}\sigma_k$ and $\frac{1}{2}\lambda_p$. In the Standard Model only the trivial representation ($I_k \to 0$ or $T_p \to 0$) and the fundamental representations of $SU(2)$ ($I_k \to \frac{1}{2}\sigma_k$) and $SU(3)$ ($T_p \to \frac{1}{2}\lambda_p$, $T_p \to -\frac{1}{2}\lambda_p^*$) occur. For $SU(2)$, $-\sigma_k^*$ is equivalent to $\sigma_k$, because

$$-\sigma_k^* = i\sigma_2 \sigma_k (-i\sigma_2)^\dagger, \qquad \left(e^{i\beta_k \frac{1}{2}\sigma_k}\right)^* = i\sigma_2 \left(e^{i\beta_k \frac{1}{2}\sigma_k}\right)(-i\sigma_2)^\dagger. \tag{398}$$

The representation of the first generation is given in table 1. Note that

$$\ell_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \tag{399}$$

is a doublet under $SU(2)$ and a singlet under $SU(3)$, whereas

$$u_R = \begin{pmatrix} u_R^{\text{red}} \\ u_R^{\text{green}} \\ u_R^{\text{blue}} \end{pmatrix} \tag{400}$$

is a singlet under $SU(2)$ and a triplet under $SU(3)$. The symbol $q_L$ represents six fields: the $SU(2)$-doublet

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{401}$$

in which $u_L$ and $d_L$ are $SU(3)$-triplets. The singlets under $SU(3)$ are called leptons. They have only electroweak interactions. The triplets under $SU(3)$ are called quarks, which have also strong interactions.

The values of the *weak hypercharge* $Y$ in the table look strange. However, their ratios are almost unique upon imposing the condition that the model is *anomaly free.*[10] In particular, both leptons and quarks are needed for a consistent theory, the anomalies of the leptons cancel out against those of the quarks.

[10] See for example Weinberg's book *The quantum theory of fields II*, section 22.4

The fermion terms in the lagrangian involving the covariant derivative

$$D_\mu\psi = \left(\partial_\mu - ig_1\frac{1}{2}YB_\mu - ig_2 I_k A^k_\mu - ig_3 T_p G^p_\mu\right)\psi, \tag{402}$$

follow directly from the table,

$$\begin{aligned}
-\mathcal{L}_\mathrm{F} &= \bar\psi\gamma^\mu D_\mu\psi \\
&= \bar\nu_{eR}\gamma^\mu\partial_\mu\nu_{eR} + +\bar e_R\gamma^\mu\left(\partial_\mu + ig_1 B_\mu\right)e_R \\
&\quad + \bar\ell_L\gamma^\mu\left(\partial_\mu + ig_1\frac{1}{2}B_\mu - ig_2 A_\mu\right)e_L \\
&\quad + \bar u_R\gamma^\mu\left(\partial_\mu - ig_1\tfrac{2}{3}B_\mu - ig_3 G_\mu\right)u_R + \bar d_R\gamma^\mu\left(\partial_\mu + ig_1\frac{1}{3}B_\mu - ig_3 G_\mu\right)d_R \\
&\quad + \bar q_L\gamma^\mu\left(\partial_\mu - ig_1\tfrac{1}{6}B_\mu - ig_2 A_\mu - ig_3 G_\mu\right)q_L \\
&\quad + \text{2nd generation} + \text{3rd generation},
\end{aligned} \tag{403}$$

where

$$A_\mu = A^k_\mu\frac{1}{2}\sigma_k, \quad G_\mu = G^p_\mu\frac{1}{2}\sigma_p. \tag{404}$$

It would have been more systematic to exhibit only left-handed fields in table 1 (i.e. exhibit the $(\bar\psi_R C)^T$ in stead of the $\psi_R$), but then we would have to make some conjugations in order to arrive at the conventional form (403) of the lagrangian.

The neutrino fields $\nu_R$ and $\bar\nu_R$ are singlets under the gauge group, and they are often not considered as part of the Standard Model. The model without them is called the Minimal Standard Model. Nowadays, $\nu_R$ and $\bar\nu_R$ are usually included because of the evidence that the neutrinos have masses, and the model including them is called the Extended Standard Model. The $\nu_R$ and $\bar\nu_R$ can have only interactions via so-called Yukawa couplings with the Higgs doublet, to which we now turn.

The fermion–Higgs part of the lagrangian has terms of the form $(\bar\ell_L\phi)e_R + \bar e_R(\phi^\dagger\ell_L)$, and $(\bar q_L\phi)d_R + \bar d_R(\phi^\dagger q_L)$, which are gauge invariant. However, because of (398), the field

$$\tilde\phi = i\sigma_2\phi^* = \begin{pmatrix}\phi_d^* \\ -\phi_u^*\end{pmatrix} \tag{405}$$

also transforms like a doublet under $SU(2)$, but with opposite hypercharge $Y_{\tilde\phi} = -1$ as compared with $Y_\phi = 1$. So we may also consider $(\bar\ell_L\tilde\phi)\nu_{eR} + \bar\nu_{eR}(\tilde\phi^\dagger\ell_L)$, and $(\bar q_L\tilde\phi)u_R + \bar u_R(\tilde\phi^\dagger q_L)$, since they are also gauge invariant. In addition to these forms we have the possibility of coupling constants which mix generations, for example couplings like $(\bar u_L, \bar d_L)\phi s_R$ and $(\bar u_L, \bar d_L)\phi b_R$. Such generation mixing was already possible in $\mathcal{L}_\mathrm{F}$, but we can always make transformations on the fields such that $\mathcal{L}_\mathrm{F}$ has the standard form (403), which is just a straightforward sum over generations without mixing.

The generation mixing is is most easily written down if we use a notation in which we add a generation index to the fields of the first generation, such that

$$\ell_{L1} = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad \ell_{L2} = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \quad \ell_{L3} = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \tag{406}$$

$$q_{L1} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad q_{L2} = \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad q_{L3} = \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \tag{407}$$

and similarly for the right-handed fields,

$$\nu_{R1} = \nu_{eR}, \quad \nu_{R2} = \nu_{\mu R}, \quad \nu_{R3} = \nu_{\tau R}, \quad e_{R1} = e_R, \quad e_{R2} = \mu_R, \cdots, \; d_{R3} = b_R. \tag{408}$$

Then the fermion–Higgs part of the lagrangian can be written as

$$-\mathcal{L}_{\rm FH} = (\bar{\ell}_{Lg}\tilde{\phi})F^{u,\ell}_{gg'}\nu_{Rg'} + (\bar{\ell}_{Lg}\phi)F^{d,\ell}_{gg'}e_{Rg'} + (\bar{q}_{Lg}\tilde{\phi})F^{u,q}_{gg'}u_{Rg'} + (\bar{q}_{Lg}\phi)F^{d,q}_{gg'}d_{Rg'} + \text{h.c.}, \tag{409}$$

where h.c. means hermitian conjugate. The coupling constants $F^{u,\ell}$, $F^{d,\ell}$, $F^{u,q}$ and $F^{d,q}$ are matrices 'in generation space'.

Finally, for the neutral fields $\nu_{Rg}$ and $\bar{\nu}_{Rg}$ we can still add *Majorana mass terms* of the form

$$\begin{aligned} -\mathcal{L}_{\rm M} &= \frac{1}{2}(-C^\dagger \nu_{Rg})^T M_{gg'}\nu_{Rg'} + \frac{1}{2}\bar{\nu}_{Rg} M^*_{gg'} (\bar{\nu}_{Rg'} C)^T \\ &= \frac{1}{2}\nu_R^T C^\dagger M \nu_R + \frac{1}{2}\nu_R^{*T} C M^* \nu_R^*, \end{aligned} \tag{410}$$

where $M = M^T$ is symmetric.

This completes our first introduction of the lagrangian (382) of the Extended Standard Model.

## 11.5 Parameters, CKM matrix and Fermi constant

Apart from parameter $\epsilon$, which serves to set the cosmological constant to zero[11], there is only one dimensional parameter in the Minimal Standard Model, $\mu^2$, whereas there are more in the extended model, $\mu^2$ and the matrix elements $M_{gg'}$. We shall first limit ourselves to the Minimal Standard Model (MSM), in which the right-handed neutrino fields $\nu_{Rg}$ and their conjugates $\bar{\nu}_{Rg}$ are absent. Hence the Majorana mass matrix $M$ and Yukawa coupling matrix $F^{u,\ell}$ are then absent too. Alternatively, we may include the $\nu_{Rg}$ and $\bar{\nu}_{Rg}$ in the MSM with $M = 0$ and $F^{u,\ell} = 0$. Then, without gravity the $\nu_{Rg}$ and $\bar{\nu}_{Rg}$ are simply free fields and irrelevant for particle physics. With gravity they could have effects in cosmology.

Before going to discuss the values of the parameters in the MSM we need to brush up the coupling matrices $F$. These can be arbitray complex matrices, which

[11] Or, possibly, to the current tiny value of $\Lambda = \Omega_\Lambda \rho_c \approx 0.7 \times 8.0992\, h^2 \times 10^{-47}$ GeV$^4 \approx (2 \times 10^{-3}$ eV$)^4$.

seems to imply many parameters. However, we can still make field transformations to bring them into a more restricted standard form. The transformations

$$q_{Lg} \to X_{gg'}\, q_{Lg'}, \quad u_{Rg} \to Z^u_{gg'}\, u_{Rg'}, \quad d_{Rg} \to Z^d_{gg'}\, d_{Rg'}, \tag{411}$$

have the effect that

$$F^{u,q} \to X^\dagger F^{u,q} Z^u, \quad F^{d,q} \to X^\dagger F^{d,q} Z^d. \tag{412}$$

Such transformations can be used to bring $F^{u,d}$ into the standard form (cf. problem 2)

$$F^{u,q} = \begin{pmatrix} f_u & 0 & 0 \\ 0 & f_c & 0 \\ 0 & 0 & f_t \end{pmatrix} > 0, \quad F^{d,q} = \left(F^{d,q}\right)^\dagger > 0, \tag{413}$$

where the diagonal elements $f_u$, ..., $f_t$ and the eigenvalues of the hermitian $F^{d,q}$ are real and positive, as in dicated by the $> 0$ sign, with six real parameters and one phase (cf. problem 2). We could also have chosen a standard form in which $F^d$ is diagonal and $F^u$ hermitian and positive. This latter choice will be made for the leptons, for which similar transformations can be carried out to bring $F^{u,\ell}$ and $F^{d,\ell}$ into the form

$$F^{u,\ell} = \left(F^{u,\ell}\right)^\dagger > 0, \quad F^{d,\ell} = \begin{pmatrix} f_e & 0 & 0 \\ 0 & f_\mu & 0 \\ 0 & 0 & f_\tau \end{pmatrix} > 0. \tag{414}$$

This form is very convenient in the MSM for which $F^{u,\ell} = 0$.

We should mention here that the above transformations on the fermion fields may influence the values of the parameters $\theta_1$, $\theta_2$ and $\theta_3$, through intricate quantum effects called anomalies. The $\theta$ parameters may have observable effects in the quantum theory, although, as mentioned earlier, classically they are irrelevant because the $\mathcal{L}_\theta$ are total derivatives. They have no effect on propagators or interaction vertices in Feynman diagrams. Nevertheless, they may have non-perturbative effects. According to current understanding only $\theta_3$ is observable.

The parameters of the MSM are now $g_1$, $g_2$, $g_3$, $\lambda$, $f_e$, $f_\mu$, $f_\tau$, $f_u$, $f_c$, $f_t$, the seven real parameters in $F^{d,q}$, $\theta_3$, and the dimensional $\mu^2$ and $\epsilon$.

We have seen already that spontaneous symmetry breaking

$$\langle\phi\rangle \equiv \phi_{\rm vac} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v = \sqrt{-\mu^2/\lambda}, \tag{415}$$

leads to non-zero masses for the $W$, $Z$ and Higgs bosons,

$$m_W^2 = \frac{1}{2} g_2 v^2, \quad m_Z = m_W/\cos\theta_W, \quad m_H^2 = 2\lambda v^2 \quad \tan\theta_W = g_1/g_2. \tag{416}$$

and zero mass for the photons. The gluon fields also remain massless as they are not coupled to the Higgs doublet. The couplings of the photon field $A_\mu$ and the $Z$ boson field to the fermions follow from

$$A^3_\mu = Z_\mu \cos\theta_W + A_\mu \sin\theta_W, \quad B_\mu = -Z_\mu \sin\theta_W + A_\mu \cos\theta_W, \tag{417}$$

such that

$$g_1 \frac{Y}{2} B_\mu g_2 I_3 A^3_\mu \;=\; eQA_\mu + \frac{e}{\cos\theta_W \sin\theta_W}(I_3 - \sin^2\theta_W\, Q) Z_\mu \tag{418}$$

$$Q \;=\; I_3 + \frac{Y}{2}, \quad e = g_1 \cos\theta_W = g_2 \sin\theta_W. \tag{419}$$

Substituting $\phi = \phi_{\rm vac}$ into the Yukawa-coupling part of the lagrangian results in terms of the form $\bar\psi_L m \psi_R + \text{h.c.} = \bar\psi_L m \psi_R + \bar\psi_R m \psi_L = \bar\psi m \psi$, which are mass terms for Dirac fields $\psi = \psi_L + \psi_R$, with $m = m^\dagger = Fv/\sqrt{2}$. So the fermions also get their masses from spontaneous symmetry breaking,

$$m_{\nu_e,\nu_\mu,\nu_\tau} = 0, \quad m_{u,c,t} = \frac{1}{\sqrt{2}} g_{u,c,t} v, \quad m_{e,\mu,\tau} = \frac{1}{\sqrt{2}} g_{e,\mu,\tau} v, \tag{420}$$

with a mass matrix for the $d, s, b$ quarks (cf. (409)),

$$(\bar q_{Lg} \phi_{\rm vac})\, F^{d,q}_{gg'}\, d_{Rg'} + \text{h.c.} = \bar d_g m^d_{gg'} d_{g'}, \quad m^d = \frac{1}{\sqrt{2}}\, v\, F^{d,q}. \tag{421}$$

The hermitian matrix $F^{d,q}$ can be diagonalized by a unitary transformation,

$$F^{d,q} = V \begin{pmatrix} f_d & 0 & 0 \\ 0 & f_s & 0 \\ 0 & 0 & f_b \end{pmatrix} V^\dagger \tag{422}$$

so, on making the transformation

$$d_g \to V_{gg'} d_{g'}, \tag{423}$$

we obtain a diagonal mass matrix for the transformed fields,

$$\bar d_g m^d_{gg'} d_{g'} \to m_d \bar d d + m_s \bar s s + m_b \bar b b, \tag{424}$$

with

$$m_{d,s,b} = \frac{1}{\sqrt{2}} f_{d,s,b}\, v. \tag{425}$$

The transformation (423) can be written in 'up-down space' as

$$\begin{pmatrix} u_g \\ d_g \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & V_{gg'} \end{pmatrix} \begin{pmatrix} u_{g'} \\ d_{g'} \end{pmatrix} = \left( \frac{1+\sigma_3}{2} + \frac{1-\sigma_3}{2} V_{gg'} \right) \begin{pmatrix} u_{g'} \\ d_{g'} \end{pmatrix}. \tag{426}$$

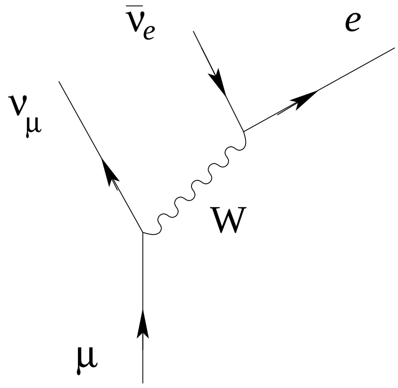



Figure 1: Diagram for the decay $\mu^- \to \nu_\mu + e^- + \nu_e$.

Hence after the transformation the matrix $V$ pops up again in those quark parts of the lagrangian that contain $\sigma_1$ and $\sigma_2$, as these do not commute with $\sigma_3$. So these involve the $W$ fields

$$W_\mu^+ = \frac{1}{\sqrt{2}}(A_\mu^1 - iA_\mu^2), \quad W_\mu^- = \frac{1}{\sqrt{2}}(A_\mu^1 + iA_\mu^2), \tag{427}$$

and the quark part of the $W$ current. The interaction terms in the lagrangian of the fermions with the $W$-boson field can be written as

$$\mathcal{L}_{FW} \to \frac{g_2}{2\sqrt{2}}(j^\mu W_\mu^+ + j^{\mu\dagger} W_\mu^-), \tag{428}$$

with

$$\begin{aligned}
j^\mu &= j_\ell^\mu + j_q^\mu, && (429)\\
j_\ell^\mu &= \bar{e}\, i\gamma^\mu(1-\gamma_5)\nu_e + \bar{\mu}\, i\gamma^\mu(1-\gamma_5)\nu_\mu + \bar{\tau}\, i\gamma^\mu(1-\gamma_5)\nu_\tau, && (430)\\
j_q^\mu &= \bar{u}_g\, i\gamma^\mu(1-\gamma_5)V_{gg'}d_{g'} && (431)\\
&= V_{ud}\,\bar{u}\, i\gamma^\mu(1-\gamma_5)d + V_{us}\,\bar{u}\, i\gamma^\mu(1-\gamma_5)s + V_{ub}\,\bar{u}\, i\gamma^\mu(1-\gamma_5)b \\
&\quad + V_{cd}\,\bar{c}\, i\gamma^\mu(1-\gamma_5)d + \cdots, && (432)
\end{aligned}$$

where we used the conventional notation $V_{ud} = V_{11}$, $V_{us} = V_{12}$, etc. The unitary matrix $V$ is called the *Cabbibo–Kobayashi–Maskawa* (CKM) matrix. It contains four real parameters: $7(F^{d,q}) - 3(f_{d,s,b}) = 4$, and it can be parametrized in terms of three angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and a phase $\delta_{12}$[12]

Once we know the expectation value $v$, the diagonal Yukawa couplings $f$ are determined in terms of the fermion masses by $f = \sqrt{2}m/v$. The value of $v$ can be found from the mass and lifetime of the muon, which is unstable through the decay $\mu^- \to \nu_\mu + e^- + \bar{\nu}_e$, as illustrated by the Feynman diagram in figure 1. The

[12]See the website of the Particle Data Group, http://pdg.lbl.gov.

invariant amplitude for the decay can be written as

$$\mathcal{M} = \left(\frac{g_2}{2\sqrt{2}}\right)^2 \langle p_e\lambda_e, p_{\bar{\nu}_e}|j^{\nu\dagger}|0\rangle \, D^W_{\mu\nu}(k) \, \langle p_{\nu_\mu}|j^\mu_\ell|p_\mu\lambda_\mu\rangle, \quad k = p_\mu - p_{\nu_\mu}, \tag{433}$$

where the $\lambda$'s are spin labels (e.g. for the electron and muon it is the $z$ component of the spin in rest frame of the particles, the massless neutrinos have only one spin polarization in the MSM), and $D^W_{\mu\nu}$ is the $W$ propagator

$$D^W_{\mu\nu}(k) = \frac{\eta_{\mu\nu} - k_\mu k_\nu/m_W^2}{k^2 + m_W^2 - i\epsilon}. \tag{434}$$

Actually, the momentum transfer $k^\mu$ is only of the order of the muon mass $m_\mu =$ 106 MeV, which is much smaller than the $W$ mass $m_W = 80.4$ GeV. So we may approximate

$$D^W_{\mu\nu}(k) \approx \frac{\eta_{\mu\nu}}{m_W^2}, \tag{435}$$

and under such circumstances the effective Fermi interaction

$$\mathcal{L}_{FW} \to \frac{G_F}{\sqrt{2}} j^\mu j^\dagger_\mu, \quad \frac{G_F}{\sqrt{2}} = \left(\frac{g_2}{2\sqrt{2}}\right)^2 \frac{1}{m_W^2} = \frac{1}{2v^2}, \tag{436}$$

parametrized in terms of the *Fermi constant* $G_F$, gives an accurate description of the weak interactions with charged currents. There is a corresponding form involving neutral currents, obtained by making an approximation similar to (435) for the $Z$ boson, which is even heavier than the $W$ boson.

The decay rate turns out to be accurately given by neglecting also the electron mass $m_e = 0.511$ MeV (see e.g. Veltman's book 'Diagrammatica'),

$$\Gamma(\mu \to \nu_\mu e \bar{\nu}_e) = \frac{G_F^2}{192\pi^2} m_\mu^5, \tag{437}$$

The observed lifetime $\Gamma^{-1}$ of $2.197 \times 10^{-6}$ sec. and muon mass $m_\mu = 105.66$ MeV give $G_F = 1.164 \times 10^{-5}$ GeV$^{-2}$. Including higher order corrections leads to $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$, and correspondingly $v = 246$ GeV.

The Yukawa couplings follow now from the ratio of fermion masses with $v$, $f = \sqrt{2}\, m_f/v$. The fermion masses vary over five orders of magnitude,

$$\begin{aligned} &m_e = 0.000511, \quad m_\mu = 0.106, \quad m_\tau = 1.78, \\ &m_u \approx 0.003, \quad m_d \approx 0.006, \quad m_s \approx 0.1, \quad m_c \approx 1.3, \quad m_b \approx 4.1, \quad m_t \approx 175, \end{aligned} \tag{438}$$

in GeV units, and so the Yukawa couplings vary from $f_e = 0.29 \times 10^{-5}$ to $f_t = 1.0$. The elements of the CKM matrix $V$ are currently under active experimental scrutiny, and some of them are known already to considerable precision.

The ratios of the masses $m_W = 80.4$ GeV and $m_Z = 91.2$ GeV with $v$ determine the gauge couplings $g_1$ and $g_2$,

$$g_2 = \frac{2m_w}{v} = 0.65, \quad g_1 = g_2 \tan\theta_W = g_2\sqrt{m_Z^2/m_W^2 - 1} = 0.35, \tag{439}$$

with $\sin^2\theta_W = 1 - m_W^2/m_Z^2 = 0.22$. The implied value for $e = g_1 \cos\theta_W = 0.31$ is close to the experimental value $e = \sqrt{4\pi\alpha} = \sqrt{4\pi/137} = 0.30$. Taking into account higher order corrections in perturbation theory has led to surprisingly accurate agreement with all experimental results so far, see http://pdg.lbl.gov. They are so accurate that the mass of the Higgs boson, which has currently not yet been found, can be bounded to be lower than about 230 GeV (90% confidence limit), see http://pdg.lbl.gov.

There is good evidence for neutrino oscillations, which implies that the neutrinos are massive. The experimental results can be incorporated in the Extended Standard Model (ESM), which includes the fields $\nu_R$ and $\bar{\nu}_R$, together with the Yukawa coupling matrix $F^{u,\ell}$ and the Majorana mass matrix $M$. These constitute many more parameters which are very hard to determine. Luckily, their influence on much of particle physics is very limited, because the neutrino masses are so small. On the other hand, they also break symmetries and may very well have important cosmological consequences.

## 11.6 Global symmetries

Apart from the local invariance related to gauge-field dynamics, the lagrangian may also be invariant under global (space-time independent) symmetry transformations. The MSM lagrangian is invariant under

$$q_{Lg} \to e^{i\alpha}\, q_{Lg}, \quad u_{Rg} \to e^{i\alpha}\, u_{Rg}, \quad d_{Rg} \to e^{i\alpha}\, d_{Rg}, \qquad \text{(baryon number } B) \tag{440}$$

$$\ell_{Lg} \to e^{i\alpha_g}\, \ell_{Lg}, \quad e_{Rg} \to e^{i\alpha_g}\, e_{Rg}, \qquad \text{(lepton number } L_g) \tag{441}$$

with the complex conjugate transformation for the conjugate fields. In parenthesis we have indicated the conserved quantities as a consequence of the Noether theorem. The conserved quantities are given by

$$B = \frac{1}{3}\int d^3x \sum_g \left(q_{Lg}^\dagger q_{Lg} + u_{Rg}^\dagger u_{Rg} + d_{Rg}^\dagger d_{Rg}\right), \tag{442}$$

$$L_g = \int d^3x \left(\ell_{Lg}^\dagger \ell_{Lg} + e_{Rg}^\dagger e_{Rg}\right), \tag{443}$$

where the factor 1/3 is inserted because there are three quarks in a baryon (without it we would speak of quark number). Note that because of generation mixing there is only one conserved baryon number and there are three conserved lepton numbers.

The situation is actually more complicated because of the anomaly phenomenon. It turns out that in principle the sum $B + \sum_g L_g$ is not coserved because of an anomaly, although at energies substantially below the $W$ mass the $B$ violation is negligible. However, this anomaly plays an important role in theories of *baryogenesis*.

In the ESM all three lepton number symmetries are broken on the lagrangian level.

Last but not least, without going into details we mention that the MSM and ESM are not invariant under $C$ (charge conjugation) and $P$ (parity). The MSM is almost invariant under the product $CP$: it is broken by the fact that with three or more generations, the CKM matrix, or equivalently, the matrix $F^{d,q}$ is necessarily complex. The ESM has the possibility of additional $CP$ breaking in the $F^{u,\ell}$ and $M$. The $CP$ violation may be a driving mechanism for baryogenesis. Finally, any local lagrangian $\mathcal{L}$ that is Lorentz and translation invariant, and hermitian, is invariant under the product $CPT$ of $C$, $P$ and $T$ (time reversal).

## 11.7 QED and QCD

In the approximation that neglects $1/m_{W,Z,H}$ effects altogether, we can delete the $W$, $Z$ and Higgs fields from the MSM lagrangian. This gives the lagrangian for quantum electrodynamics (QED) plus quantum chromodynamics (QCD). The free fermion action contains all the (generation-diagonal) mass terms found so far,

$$\mathcal{L}_F^{\rm free} = \bar{e}_g(\gamma^\mu\partial_\mu + m_\ell)e_g + \bar{u}_g(\gamma^\mu\partial_\mu + m^u)u_g + \bar{d}_g(\gamma^\mu\partial_\mu + m^d)d_g \tag{444}$$

with an implied sum over all generations, as before. The interaction with the fermions has the form

$$e j^\mu_{\rm em} A_\mu + g_3 j^\mu_{p\,{\rm color}} G^p_\mu, \tag{445}$$

with

$$j^\mu_{\rm em} = \bar{e}_g\, i\gamma^\mu Q\, e_g + \bar{u}_g\, i\gamma^\mu Q u_g + \bar{d}_g\, i\gamma^\mu Q d_g, \tag{446}$$

$$j^\mu_{p\,{\rm color}} = \bar{u}_g\, i\gamma^\mu \frac{1}{2}\lambda_p\, u_g + \bar{d}_g\, i\gamma^\mu \frac{1}{2}\lambda_p\, d_g. \tag{447}$$

Note that $\gamma_5$ has dropped out of the lagrangian. This corresponds to the fact that QED and QCD are invariant under $C$ and $P$ separately, as well as under $T$.

## 11.8 Further problems

1. Verify that $\mathcal{L}_{\theta_1}$ is a total derivative.

2. An arbitray non-singular matrix $M$ can be written in polar decomposition form

$$M = PU, \quad P = P^\dagger > 0, \quad U^\dagger = U^{-1}. \tag{448}$$

where $> 0$ means that it is a positive matrix, i.e. any 'expectation value' $c^\dagger P c > 0$. We can define $P$ by $P = \sqrt{MM^\dagger}$ and then find $U$ as $U = MP^{-1}$. A hermitian matrix can be diagonalized by a unitary transformation, $P = VDV^\dagger$, where $D$ is diagonal and $V^\dagger V = 1$. How would you take the square root to compute $P$?

Verify (413).

Verify that $F^{d,q}$ can be brought into the form

$$F^{d,q} = \begin{pmatrix} a & b & c \\ b & d & e \\ c^* & e & f \end{pmatrix}, \tag{449}$$

with real $a$, $b$, $d$, $e$, $f$, such that $F^{d,q}$ depends only on seven real parameters. Hint: after establishing (413) we can still make phase transformations with $X = Z^u = Z^d$ that are diagonal. This can be used to limit the number of free parameters in $F^{d,q}$.

Verify that the CKM matrix $V$ can be limited to contain only four real parameters.

Suppose there are only three generations. Verify that in this case $F^{d,q}$ and $V$ can be chosen real without loss of generality.

3. Verify that the MSM is invariant under (440) and (441). Use the Noether method (186) to obtain the corresponding currents, and the charges (442) and (443).

# 12 Decoupling temperature at nucleosynthesis

Nucleosynthesis started when the weak interactions, which are capable of changing protons into neutrons through processes such as $e^- + p \leftrightarrow n + \nu_e$, became ineffective, because of the expansion of the universe. A simple criterion is to compare the Hubble rate, $H$, with the rate of scatterings, $\Gamma$, that e.g. a proton experiences. Suppose $\Gamma$ has a temperature dependence of the form $\Gamma = cT^n$. In the era of radiation dominance the scale factor $a \propto t^{1/2}$, $H = 1/2t$ and the temperature falls with time like $T \propto t^{-1/2}$ (cf. (98). Then the total number of scatterings a proton experiences after time $t$ is given by

$$N_{\rm scatt} = \int_t^\infty dt'\, \Gamma(t') = \frac{1}{n-2}\, \frac{\Gamma(t)}{H(t)}, \tag{450}$$

provided that $n > 2$. This number becomes smaller than one when $\Gamma(t)$ drops sufficiently below $H(t)$, and the scatterings effectively stop. The proton distribution function then 'freezes' (apart from the Hubble expansion) and does not drop to zero anymore $\propto \exp(-m_p/T)$. This is the 'origin of species' described in more detail by Kolb and Turner (chapter 5) using the Boltzmann equation.

In the following we shall make the above estimate more concrete by evaluating the rate $\Gamma$ for the process $e^- + p \leftrightarrow n + \nu_e$.

Consider a proton in a thermal bath of neutrons, electrons and electron-neutrinos. The temperature is of the order of 1 MeV, so the protons and neutrons are nonrelativistic. We want to calculate the rate (per proton) for the process $p + e \to n + \nu_e$. Recall equation (273) in section 9.1, and make the identification $j = p$ (proton), $i = e$ (electron), $1 = n$ (neutron) and $2 = \nu_e$. The densities $n_{i,j}$ in section 9.1 were for fixed momentum of the colliding particles. We now take $n_p = \delta(\mathbf{x})$ and replace $n_e$ by the distribution function $f_e(\mathbf{p}_e)$ for electrons, while integrating over the electron momenta $\mathbf{p}_e$. We also add Pauli-blocking factors $[1 - f_n(\mathbf{p}_n)][1 - f_{\nu_e}(\mathbf{p}_{\nu_e})]$, which disallow a fermion state to be occupied more than once. We encounteed such factors earlier with the introduction of the Boltzmann equation and the arguments for their appearence were given in (210)–(212) for bosons and (225) for fermions.

However, since the neutrons are nonrelativistic, $f_n \ll 1$, and Pauli blocking does not play a role here for neutrons.

a. Verify that the rate for the process $p + e \to n + \nu_e$ per proton is given by

$$\Gamma_{pe\to n\nu_e} = \frac{4}{2p_p^0} \int d\omega_n\, d\omega_{\nu_e}\, d\omega_e\, f_e(\mathbf{p}_e)[1 - f_{\nu_e}(\mathbf{p}_{\nu_e})]\, (2\pi)^4 \delta^4(p_n + p_{\nu_e} - p_p - p_e)\, \overline{|\mathcal{M}|^2}. \tag{451}$$

Note that neutrinos taking part in ordinary weak interactions have only one helicity state (left handed for neutrinos, right handed for antineutrinos), and we ignored the other helicity state.

From the theory of weak interactions one finds (see below for a derivation), in the non-relativistic limit for the nucleons,

$$\overline{|\mathcal{M}|^2} = 4G_F^2|V_{ud}|^2 m_N^2[(E_e E_{\nu_e} + \mathbf{p}_e \cdot \mathbf{p}_{\nu_e}) + g_A^2(3E_e E_{\nu_e} - \mathbf{p}_e \cdot \mathbf{p}_{\nu_e})], \tag{452}$$

where $m_N$ is the nucleon mass (ignoring the difference between $m_n$ and $m_p$), $E_e = \sqrt{\mathbf{p}_e^2 + m_e^2}$, $E_{\nu_e} = |\mathbf{p}_{\nu_e}|$, $G_F$ is the Fermi weak interaction constant, $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$, $g_A = 1.267$ is the axial vector coupling constant of the nucleon, and $|V_{ud}|$ is the absolute value of the Kobayashi-Moskawa matrix element, $|V_{ud}| = 0.9735$ (we left out the experimental uncertainties, see the Particle Data Group website: http://pdg.lbl.gov). This spin-averaged invariant amplitude also enters in the neutron decay rate (inverse life time)

$$\tau_n^{-1} \equiv \Gamma_{n\to pe\bar{\nu}_e} = \frac{4}{2m_n} \int d\omega_p\, d\omega_e\, d\omega_\nu\, (2\pi)^4 \delta^4(p_p + p_e + p_\nu - p_n)\, \overline{|\mathcal{M}|^2}. \tag{453}$$

Here it is of course crucial that $m_n > m_p + m_e$, otherwise the neutron could not decay because of energy conservation. However, in $\overline{|\mathcal{M}|^2}$ we may still ignore the difference between $m_n$ and $m_p$. Note that the contribution of the terms $\propto \mathbf{p}_e \cdot \mathbf{p}_{\nu_e}$ in $\overline{|\mathcal{M}|^2}$ will vanish upon integration over angles. So effectively

$$\overline{|\mathcal{M}|^2} \to 4G_F^2|V_{ud}|^2 m_N^2(1 + 3g_A^2)\, E_e E_{\nu_e}. \tag{454}$$

b. Verify that the rate $\Gamma_{n\to pe\bar{\nu}_e}$ can be written in the form

$$\Gamma_{n\to pe\bar{\nu}_e} = G_F^2|V_{ud}|^2(1 + 3g_A^2)\frac{m_e^5}{2\pi^3}\lambda_0, \tag{455}$$

$$\lambda_0 = \int_1^q d\epsilon\, \epsilon\sqrt{\epsilon^2 - 1}\,(q - \epsilon)^2 = 1.632, \tag{456}$$

where $q = Q/m_e = (m_n - m_p)/m_e$ and where in the rest of the kinematic factors we have approximated $m_n = m_p = m_N$. Hint: first integrate over $\mathbf{p}_p$, then over $\mathbf{p}_\nu$, then convert the remaining integral over $\mathbf{p}_e$ into an integral over $\epsilon = E_e/m_e = \sqrt{\mathbf{p}_e^2 + m_e^2}/m_e$. We have $Q = 1.292$ MeV, $m_e = 0.5110$ MeV.

Verify that the rate $\Gamma_{pe\to n\nu_e}$ can be written in the form

$$\Gamma_{pe\to n\nu} = \frac{1}{\tau_n\lambda_0}\int_q^\infty d\epsilon\, \frac{\epsilon\sqrt{\epsilon^2 - 1}\,(q - \epsilon)^2}{[1 + e^{\epsilon z}][1 + e^{(q-\epsilon)z_\nu}]}, \tag{457}$$

where $z = m_e/T$, $z_\nu = m_e/T_\nu$, with neutrino temperature $T_\nu$ possibly differing from the electron temperature $T$.

c. Assuming $T_\nu = T$, verify

$$\Gamma_{pe\to n\nu} \approx \frac{2q\sqrt{q^2 - 1}}{\tau_n\lambda_0}\left(\frac{T}{m_e}\right)^3 e^{-Q/T}, \quad T \ll Q,\, m_e, \tag{458}$$

$$\approx \frac{7\pi}{60}(1 + 3g_A)|V_{ud}|^2\, G_F^2 T^5, \quad T \gg Q,\, m_e. \tag{459}$$

d. Assuming the $T \gg Q$, $m_e$ form (459), for what temperature is $\Gamma_{pe\to n\nu} = H$, with $H = 0.331\sqrt{g_*}\,T^2/m_{\rm P}$ the Hubble rate for an effective number of degrees of freedom $g_* = 2 + (7/8)(4+6) = 10.75$ corresponding to photons, electrons and neutrinos (and their antiparticles)?

e. Answer the above question more accurately by evaluating the integral (457) numerically using e.g. Mathematica, and $T_\nu/T = (4/11)^{1/3}$.

We shall now sketch the calculation of the above spin-averaged invariant amplitude $\overline{|\mathcal{M}|^2}$. The effective Fermi interaction lagrangian for this process reads

$$\mathcal{L}_{\rm int} = \frac{G_F}{\sqrt{2}}\, j^\mu j_\mu^\dagger, \tag{460}$$

where the charged current has the form

$$\begin{aligned} j^\mu &= j^\mu_{\rm had} + j^\mu_{\rm lep}, && (461)\\ j^\mu_{\rm had} &= V_{ud}\,\bar{u}\, i\gamma^\mu(1-\gamma_5)d + \cdots, && (462)\\ j^\mu_{\rm lep} &= \bar{\nu}_e\, i\gamma^\mu(1-\gamma_5)e + \cdots. && (463) \end{aligned}$$

Here $u$, $d$, $e$ and $\nu_e$ are the up-quark, down-quark, electron and electron-neutrino fields, respectively. For the process $e^- + p \to \nu_e + n$ we need the matrix elements of the currents between the in- and outgoing states, which separates, to first order in $G_F$, into separate matrix elements for the leptons and hadrons. For the lepton current this is

$$\langle \nu_e(p_{\nu_e}, \lambda_{\nu_e})|j^{\mu\dagger}_{\rm lep}(x)|e(p_e, \lambda_e)\rangle = \bar{u}(p_{\nu_e}, \lambda_{\nu_e}) i\gamma^\mu(1-\gamma_5)u(p_e, \lambda_e)\, e^{i(p_{\nu_e}-p_e)x}, \tag{464}$$

where $u(p, \lambda)$ is the Dirac polarization spinor corresponding to four-momentum $p$ and spin index $\lambda$, covariantly normalized as $\bar{u}(p, \lambda')i\gamma^\mu u(p, \lambda) = 2p^\mu \delta_{\lambda',\lambda}$. The spin index takes values $\lambda = \pm$ and may be taken to indicate the value of the $z$ component of the spin in the rest frame of the particle. We treat the neutrinos as massless and for them we use instead the $z$ component of the spin in the direction of momentum – the helicity. The matrix element of the hadron current is not easy to evaluate, because the proton and neutron are bound states of quarks and gluons. However, we can parametrize this matrix element on grounds of symmetry. Things greatly simplify if we take the nonrelativistic limit for the nucleons, which is appropriate for momenta of the order of 1 MeV, since the nucleons have a mass of nearly 1 GeV. Then it is known that

$$\langle n(p_n, \lambda_n)|j^\mu_{\rm had}(x)|p(p_p, \lambda_p)\rangle \approx \bar{u}(p_n, \lambda_n)i\gamma^\mu(1-g_A\gamma_5)u(p_p, \lambda_p)e^{i(p_n-p_p)x}, \tag{465}$$

where all the complicated bound-state structure of the nucleon is buried in the constant $g_A$. The invariant amplitude is now given by

$$\begin{aligned} \mathcal{M} &= \frac{G_F}{\sqrt{2}}\,\bar{u}(p_{\nu_e}, \lambda_{\nu_e})i\gamma^\mu(1-\gamma_5)u(p_e, \lambda_e)\,\bar{u}(p_n, \lambda_n)i\gamma_\mu(1-g_A\gamma_5)u(p_p, \lambda_p)\\ &\equiv \frac{G_F}{\sqrt{2}}\, l^\mu h_\mu. && (466) \end{aligned}$$

In the nonrelativistic limit we set $\mathbf{p}_p = \mathbf{p}_n = 0$ in the nucleon spinors and we also approximate $m_p = m_n \equiv m_N$. Then for the vector current there is a non-zero contribution only for $\mu = 0$,

$$\bar{u}(p_n, \lambda_n) i\gamma^\mu u(p_p, \lambda_p) \to 2m_N \delta_0^\mu\, \delta_{\lambda_n,\lambda_p}. \tag{467}$$

For the axial vector current we get zero for $\mu = 0$, and for $\mu = k = 1, 2, 3$:

$$\bar{u}(p_n, \lambda_n) i\gamma^k \gamma_5 u(p_p, \lambda_p) \to 2m_N\, \sigma^k_{\lambda_n,\lambda_p}, \tag{468}$$

where the $\sigma^k$ are the three Pauli matrices. After multiplying by $\mathcal{M}^*$ the average over spins can now be performed,

$$\overline{|\mathcal{M}|^2} = \frac{G_F^2}{2} \frac{1}{2^3} \sum_{\lambda_p \lambda_n \lambda_{\nu_e} \lambda_e} l^\mu l^{\nu *} h_\mu h_\nu^*. \tag{469}$$

For the nucleons we get

$$\sum_{\lambda_n \lambda_p} h^\mu h^{\nu *} = 8m_N^2, \quad \mu = \nu = 0, \tag{470}$$

$$= 4m_N^2 \mathrm{Tr}\,(\sigma^k \sigma^l) = 8m_N^2\, \delta_{kl}, \quad \mu = k, \;\; \nu = l, \tag{471}$$

and zero for $(\mu, \nu) = (0, l),\ (k, 0)$. For the leptons we use a standard trace technique. First observe that $\bar{u} = u^\dagger \beta$, $\beta \equiv i\gamma^0 = \beta^\dagger$, $\bar{u}_1 M u_2^* = \bar{u}_2 \beta M^\dagger \beta u_1$, where $M$ is a combination of Dirac matrices. Next, observe that $|\bar{u}_1 M u_2|^2 = \bar{u}_1 M u_2 \bar{u}_2 \beta M^\dagger \beta u_1 = \mathrm{Tr}\,(M u_2 \bar{u}_2 \beta M^\dagger \beta u_1 \bar{u}_1)$, where Tr denotes taking the trace of the matrix. Summing over the spins polarizations we encounter $\sum_\lambda u(p, \lambda)\bar{u}(p, \lambda) = m - ip^\kappa \gamma_\kappa$. For the neutrinos we can also use this formula with $m = 0$, because the factor $(1 - \gamma_5)$ has the effect that only the left-handed spin-polarization contributes. So, using the properties of the Dirac matrices we get ($m_{\nu_e} = 0$)

$$\sum_{\lambda_{\nu_e} \lambda_e} l^\mu l^{\nu *} = -\mathrm{Tr}\,[\gamma^\mu (1 - \gamma_5)(m_e - ip_e^\alpha \gamma_\alpha)\gamma^\nu (1 - \gamma_5)(m_{\nu_e} - ip_{\nu_e}^\beta \gamma_\beta)]$$

$$= 2p_e^\alpha p_{\nu_e}^\beta\, \mathrm{Tr}\,[\gamma^\mu \gamma_\alpha \gamma^\nu \gamma_\beta (1 + \gamma_5)] \tag{472}$$

where we also used $(1 - \gamma_5)^2 = 2(1 - \gamma_5)$, $(1 - \gamma_5)(1 + \gamma_5) = 0$. The traces are given by

$$\mathrm{Tr}\,(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu}, \tag{473}$$

$$\mathrm{Tr}\,(\gamma_\kappa \gamma_\lambda \gamma_\mu \gamma_\nu) = 4(\eta_{\kappa\lambda}\eta_{\mu\nu} - \eta_{\kappa\mu}\eta_{\lambda\nu} + \eta_{\kappa\nu}\eta_{\lambda\mu}), \tag{474}$$

$$\mathrm{Tr}\,(\gamma_\kappa \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_5) = 4i\epsilon_{\kappa\lambda\mu\nu}. \tag{475}$$

The $\gamma_5$ term in (472) actually does not contribute, because the hadronic part only picks out combinations with $\mu = \nu$. Putting things together leads to (452).

# 13 Inflation

## 13.1 Slow roll

[13] We assume a flat universe, i.e. $k = 0$. Then the Friedmann equation is given by

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi\rho}{3M_P^2} \tag{476}$$

where $G = 1/M_P^2$.

Now we investigate an inflationary model with one real scalar field $\phi$. Its Lagrangean density and action are given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), & (477) \\ S &= \int \mathrm{d}^4 x \sqrt{-g}\, \mathcal{L}. & (478) \end{aligned}$$

Apart from the Friedmann equation, we also need the equation of motion for $\phi$ to determine the evolution of the universe during inflation.

a. Derive the equation of motion for $\phi$,

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = \Box\phi = \frac{\partial V}{\partial \phi}, \tag{479}$$

using the action principle. Here the d'Alembertian operator $\Box$ is defined as

$$\Box\phi \equiv g^{\mu\nu} D_\mu D_\nu \phi = g^{\mu\nu} \left( \partial_\mu \partial_\nu - \Gamma^\alpha_{\mu\nu} \partial_\alpha \right) \phi. \tag{480}$$

b. Show that for a Robertson-Walker metric this reduces to

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2} \nabla^2 \phi = -\frac{\partial V}{\partial \phi}, \tag{481}$$

where $\nabla^2/a^2$ is defined in the same way as $\Box$, but with all Greek indices replaced by Roman ones. The factor $1/a^2$ has been taken out because then, for a flat universe, $\nabla^2$ is just the usual three-dimensional Laplace operator.

Now we assume that the field $\phi$ is very homogeneous, so that we can neglect the spatial derivative term. Moreover, we also assume that there are no other sources of energy than this scalar field. In that case the energy density $\rho$ in equation (476) is equal to

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{482}$$

[13] The following two section were contributed by Bartjan van Tent.

So we have the following two equations:

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \\ H^2 = \dfrac{8\pi}{3M_P^2}\left(\dfrac{1}{2}\dot{\phi}^2 + V(\phi)\right). \end{cases} \tag{483}$$

The slow roll approximation now says that $\phi$ evolves very slowly so that we can neglect $\ddot{\phi}$ compared with $3H\dot{\phi}$ and $\frac{1}{2}\dot{\phi}^2$ compared with $V(\phi)$. Then we have:

$$\text{Slow roll:} \begin{cases} \dot{\phi} = -\dfrac{V'(\phi)}{3H}, \\ H^2 = \dfrac{8\pi V(\phi)}{3M_P^2}. \end{cases} \tag{484}$$

c. Show that the consistency conditions for the slow roll approximation are:

$$|V''(\phi)| \ll 9H^2 \approx \frac{24\pi V(\phi)}{M_P^2}, \tag{485}$$

$$\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll \frac{48\pi}{M_P^2}. \tag{486}$$

Conventionally one writes the slow roll conditions as

$$\epsilon \ll 1, \tag{487}$$

$$|\eta| \ll 1, \tag{488}$$

with

$$\epsilon \equiv \frac{M_P^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \tag{489}$$

$$\eta \equiv \frac{M_P^2}{8\pi}\frac{V''(\phi)}{V(\phi)}. \tag{490}$$

The number of e-folds of inflation that occur as $\phi$ rolls down the potential from $\phi_1$ to $\phi_2$ is given by $N(\phi_1 \to \phi_2)$:

$$N(\phi_1 \to \phi_2) \equiv \ln\frac{a_2}{a_1} = \int_{t_1}^{t_2} H\mathrm{d}t. \tag{491}$$

d. Show that

$$N(\phi_1 \to \phi_2) = \frac{2\sqrt{\pi}}{M_P}\left|\int_{\phi_1}^{\phi_2} \epsilon^{-1/2}\mathrm{d}\phi\right|. \tag{492}$$

## 13.2 Chaotic inflation

We consider the chaotic inflationary scenario with a scalar inflaton field $\phi$ with potential

$$V(\phi) = \frac{\lambda\phi^n}{nM_P^{n-4}} \tag{493}$$

with $n \geq 2$ and even.

a. Compute the slow roll parameters $\epsilon$ and $\eta$, and the number of e-folds of inflation $N(\phi_1 \to \phi_2)$.

b. Assuming slow roll (484) derive the solution for $\phi(t)$, and the solution for $a(t)$,

$$a(t) = a_i \exp\left[\frac{4\pi}{nM_P^2}\left(\phi_i^2 - \phi(t)^2\right)\right], \tag{494}$$

where the subscript $_i$ denotes quantities at the start of inflation.

To be able to put some constraints on the parameters of the potential (i.e. $\lambda$), we need observational input. We will not go into the details of this process, but it boils down to the following: inflation theory predicts a spectrum of density perturbations, which can be indirectly measured in the cosmic background radiation. This spectrum is, among other things, characterized by its amplitude at the scale of the observable universe, i.e. the scale of that part of the universe that we can *now* observe, or in other words the *present* horizon distance. Note that this is not equal to the horizon distance during inflation. This amplitude is called $\delta_H$. The expression for $\delta_H$ is:

$$\delta_H^2 = \frac{32}{75}\frac{V_*}{M_P^4}\frac{1}{\epsilon_*}, \tag{495}$$

where $V$ and $\epsilon$ have to be evaluated at the time $t_*$ during inflation when the scale of the observable universe leaves the horizon. This last sentence may need some explanation. During slow roll inflation the Hubble parameter changes very slowly, and hence the horizon distance $1/H$ remains more or less the same. On the other hand, because of the rapid expansion, fluctuations of a certain scale will be inflated and after some time become larger than the horizon distance ('the scale leaves the horizon'). After inflation the horizon grows faster than the expansion and hence the scales will one by one reenter the horizon. The scale of the observable universe is that scale that is just now reentering the horizon.

c. Inflation ends when $\max\{\epsilon, |\eta|\} = 1$. Compute $\phi_{\text{end}}$.

d. $N_* \equiv N(\phi_* \to \phi_{\text{end}})$ is the number of e-folds of inflation between the time when the scale of our observable universe leaves the horizon and the end of inflation. Compute $\phi_*$ in terms of $N_*$.

e. Show that $\delta_H$ is given by the expression

$$\delta_H^2 = \frac{512\pi}{75}\frac{\lambda}{n^3}\left[\frac{n}{4\pi}\left(N_* + \frac{n-1}{2}\right)\right]^{\frac{n}{2}+1}. \tag{496}$$

The observational value for $\delta_H$ is: $\delta_H \approx 2 \cdot 10^{-5}$.

f. Taking $N_* = 60$ and $n = 4$, calculate the observationally determined value for $\lambda$. This very small value illustrates the problem of fine-tuning in inflationary model building.

In the chaotic inflationary scenario inflation starts right at the Planck scale. The initial conditions are assumed to be chaotic (hence the name chaotic inflation) in the sense that all different initial conditions for fields etc. occur in different regions of the universe. Only the regions with favourable initial conditions will inflate, and they will come to dominate all non-inflated regions. At the Planck scale $V(\phi)$ can take all values from 0 to $\mathcal{O}(M_P^4)$. Only regions with high $V$ will inflate, but values of order $M_P^4$ may be expected to occur naturally.

g. Assume $V(\phi_i) = M_P^4$. Compute the initial value of $\phi$, $\phi_i$. Now take $n = 4$ and calculate the total amount of inflation. So getting sufficient inflation is not a problem at all!

# 14 Cosmic strings and the abelian Higgs model

Consider the action (271) for the U(1) gauge theory,

$$\begin{aligned} S &= -\int d^4x \left[\eta^{\mu\nu}(D_\mu\phi)^* D_\nu\phi + V(\phi) + \frac{\eta^{\kappa\mu}\eta^{\lambda\nu}}{4}F_{\kappa\lambda}F_{\mu\nu}\right], && (497) \\ D_\mu\phi &= \partial_\mu\phi - ieA_\mu\phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, && (498) \\ V(\phi) &= \epsilon + \kappa\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \quad \lambda > 0. && (499) \end{aligned}$$

When the gauge field is identified with the Maxwell field, the theory is called scalar electrodynamics. The quantum theory of gauge fields is complicated and we shall only deal with it in leading order in the semiclassical description, i.e the classical description interpreted as an approximation to the quantum theory. We first need to determine the ground state.

The energy-momentum tensor may be obtained by coupling to the gravitational field, replacing $\eta^{\mu\nu}$ by $g^{\mu\nu}$ and $d^4x$ by $d^4x\sqrt{-g}$, varying the action with respect to $g_{\mu\nu}$ and setting $g_{\mu\nu} = \eta_{\mu\nu}$ afterwards.

a. Verify that in general relativity the electromagnetic field $F_{\mu\nu}$ may still be written as $\partial_\mu A_\nu - \partial_\nu A_\mu$, i.e. with ordinary derivatives in place of covariant derivatives.

b. Obtain $T^{\mu\nu}$ and verify that the energy density is given by

$$\begin{aligned} T^{00} &= (D_0\phi)^* D_0\phi + (D_k\phi)^* D_k\phi + V(\phi) + \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \qquad (500) \\ E_k &= F^{0k}, \quad B_k = \frac{1}{2}\epsilon_{klm} F_{lm}. \qquad (501) \end{aligned}$$

c. Reason that the classical ground state is given by

$$D_0\phi = 0, \quad D_k\phi = 0, \quad E^k = 0, \quad B^k = 0, \quad V(\phi) = \text{minimal}. \qquad (502)$$

For $\kappa > 0$ a solution is simply $\phi = 0$, $A_\mu = 0$, up to equivalence by gauge transformations. The masses of the particles in the theory can be read off from the action by expanding in the field deviations from the classical ground state and identifying the quadratic terms without derivatives. For $\kappa > 0$ the scalars have mass $m_\phi = \sqrt{\kappa}$ and the photons are massless, $m_A = 0$.

For $\kappa < 0$ the potential $V(\phi)$ has a continuous set of minima at $|\phi|^2 = -\kappa/2\lambda$, which is a circle in the complex $\phi$-plane. The ground states are given by $\phi = \sqrt{-\kappa/2\lambda}\,\exp(i\alpha_0)$, $0 \leq \alpha_0 < 2\pi$, $A_\mu = 0$, up to equivalence by gauge transformations. We pick one of these ground states, say $\alpha_0 = 0$. Before expanding the fields about the ground state, let us choose $\epsilon$ such that the ground state energy is zero and change the notation to conform to Kolb and Turner's

$$V = \lambda\left(\phi^*\phi - \frac{\sigma^2}{2}\right)^2. \qquad (503)$$

Furthermore we transform to polar coordinates for $\phi$, writing

$$\phi(x) = \frac{\rho(x)}{\sqrt{2}}\, e^{i\alpha(x)}. \qquad (504)$$

The action then takes the form

$$S = -\int d^4x \left[\frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\rho^2(eA_\mu - \partial_\mu\alpha)(eA^\mu - \partial^\mu\alpha) + V + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \qquad (505)$$

Writing

$$\rho = \sigma + H, \qquad (506)$$

the expansion about the ground state then takes the form:

$$\begin{aligned} S &= -\int d^4x \left[\frac{1}{2}\partial_\mu H\partial^\mu H + \frac{1}{2}(2\lambda\sigma^2)H^2 \right. \\ &\quad \left. + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sigma^2(eA_\mu - \partial_\mu\alpha)(eA^\mu - \partial^\mu\alpha) + \cdots\right], \qquad (507) \end{aligned}$$

where the $\cdots$ are of third and higher order in the deviations from the ground state. The combination

$$V_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha \tag{508}$$

can be treated as a new variable, which is gauge invariant. The action can be easily expressed in it since $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. We can now read off the masses of the particles to be

$$m_H^2 = 2\lambda\sigma^2, \quad m_V^2 = e^2\sigma^2. \tag{509}$$

Note that there is only one scalar particle (corresponding to $H$), called the Higgs particle (of this model). Since the $\alpha$ degree of freedom has been absorbed into $V_\mu$, we expect the $V$-particles to have a degree of freedom more than the photon. A more detailed investigation of the quantum theory indeed shows that they are massive spin one particles, which have three independent spin states, one more than the photon.[14]

The model with the 'mexican hat potential' (503) is a relativistic generalization of the Landau-Ginsburg description of a superconductor. The corresponding $\phi$ is the Cooper pair field. Cooper pairs are bound states of two electrons with spin zero, which actually have charge $\approx 2e$. These pairs can effectively be described by the field $\phi$. Here, in the relativistic setting, $A_\mu$ is not necessarily identified with the electromagnetic field (the model may be part of a 'grander' model), and the generic model is known under the name 'abelian Higgs model'.

In a type II superconductor magnetic flux can penetrate the system in the form of quantized flux lines, the Abrikosov vortex lines. Their relativistic analogues are called Nielsen-Oleson strings. If they exist they have cosmological relevance as discussed by Kolb and Turner. The basic object is a single infinitely long string along the z-direction. The configuration is static, so $E_k$ and $D_0\phi$ are zero. In polar coordinates, $(x, y, z) = (r\cos\theta, r\sin\theta, z)$, the fields are independent of $z$. At each point in the $(x, y)$ plane we have the unit vectors

$$\hat{r} = (\cos\theta, \sin\theta), \quad \hat{\theta} = \partial\hat{r}/\partial\theta = (-\sin\theta, \cos\theta). \tag{510}$$

In the string field configuration, the scalar field and the vector potential make one winding about the z-axis,

$$\phi = \frac{1}{\sqrt{2}}\rho(r)e^{i\theta}, \quad eA_k = f(r)\hat{\theta}_k. \tag{511}$$

For large $r$ the energy density should go to zero such that the energy per unit length is finite,

$$\rho \to \sigma, \quad D_k\phi \to 0, \quad B_k \to 0, \quad \text{for} \quad r \to \infty. \tag{512}$$

[14] It is better not to use variables $\rho, \alpha$ in the previous case $\kappa > 0$. The reason is that polar coordinates are singular at $\rho = 0$ (since $\alpha$ is undefined there) and they cannot be used for expanding about a ground state at $\rho = 0$.

Using

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}, \tag{513}$$

we get

$$\mathbf{D}\phi = (\nabla - ie\mathbf{A})\phi = \frac{1}{\sqrt{2}}\left[\hat{r}\rho' + i\hat{\theta}\rho\left(\frac{1}{r} - f\right)\right] e^{i\theta}, \tag{514}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{e}\left(f' + \frac{f}{r}\right)\hat{z}, \tag{515}$$

where the prime denotes differentiation with respect to $r$. Requiring $\mathbf{D}\phi \to 0$ as $r \to \infty$ it follows that $f \to 1/r$ and the magnetic flux of the string is quantized:

$$\int d^2x\, B_z = \int d\theta\, [r\hat{\theta} \cdot \mathbf{A}]_{r\to\infty} = \frac{2\pi}{e}. \tag{516}$$

For winding number $n$ ($\phi \propto \exp(in\theta)$) the magnetic flux would be $2\pi n/e$. The energy per unit length, which is the tension in the string, can be written as

$$\mu \equiv \int d^2x\, T^{00} = 2\pi \int_0^\infty dr\, r\left[\frac{1}{2}\rho'^2 + \frac{1}{2}\frac{\rho^2 K^2}{r^2} + \frac{1}{2e^2}\frac{K'^2}{r^2} + \frac{1}{4}\lambda(\rho^2 - \sigma^2)^2\right], \tag{517}$$

$$K(r) = 1 - rf(r) \tag{518}$$

(this is a more detailed version of (7.50) in Kolb & Turner). Minimizing $\mu$ with respect to $\rho(r)$ and $K(r)$ leads to coupled nonlinear differential equations for these quantities, which cannot be solved in closed form in general. Kolb and Turner use a simple variational method (eqs. (7.49)). It is also possible to obtain a numerical solution by discretizing the integral and minimizing with respect to $\rho$ and $K$ at each discrete $r$, sweeping through all $r$'s in succession many times till convergence is reached.

d. Verify the above integral expression for the string tension $\mu$.

e. Show that requiring $\mu$ to be stationary leads to

$$-\rho'' - \frac{\rho'}{r} + \frac{\rho K^2}{r^2} + \lambda\rho(\rho^2 - \sigma^2) = 0, \tag{519}$$

$$-K'' + \frac{K'}{r} + e^2\rho^2 K = 0. \tag{520}$$

f. Show that for $r \to \infty$,

$$\rho = \sigma + O(e^{-m_H r}), \quad K = O(e^{-m_V r}). \tag{521}$$

So the radial size of the magnetic flux tube is of order $m_V^{-1}$ (c.f. (515) and (518)). It turns out that

$$\rho = O(r), \quad f = O(r), \quad (K = 1 - O(r^2)), \quad r \to 0, \tag{522}$$

which implies that the energy density is finite at the center of the string.

g. Using the two component notation with two real scalar fields,

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), \quad \phi_k = \rho\,\hat{r}_k(-\theta), \tag{523}$$

verify that

$$D_\mu \phi_k = \partial_\mu \phi_k - ieA_\mu (T_3)_{kl}\phi_l, \quad k, l = 1, 2, \tag{524}$$

with $T_3$ the generator in (253).[15] Note that $-iA_\mu T_3$ is real. Make a drawing of how $\phi_k$ rotates as $\theta$ goes from zero to $2\pi$. What string configuration has $\phi_k \propto \hat{r}_k(+\theta)$? Make a drawing of the energy density and $|\mathbf{B}|$.

# 15 Magnetic monopoles in the SO(3) Higgs model

In this section we present the basic features of the 't Hooft-Polyakov monopole in the SO(3) Higgs model. The model is given by (270) with the scalar field in the defining representation of SO(3), which is the vector representation of SU(2):

$$\begin{aligned}
S &= -\int d^4x \left[\frac{1}{2}(D_\mu\phi)^T D^\mu\phi + V(\phi) + \frac{1}{4}F^p_{\mu\nu}F^{p\mu\nu}\right], &(525)\\
D_\mu\phi &= \partial_\mu\phi - igA_\mu\phi, \quad \phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, &(526)\\
A_\mu &= A^p_\mu T_p, \quad (T_p)_{kl} = -i\epsilon_{pkl}, &(527)\\
V(\phi) &= \epsilon + \frac{1}{2}\kappa\phi^T\phi + \frac{1}{4}\lambda(\phi^T\phi)^2. &(528)
\end{aligned}$$

Since the scalar field is real we have replaced the † by the transposition symbol $T$ (which should of course not be confused with the generators $T_p$) and used the conventional normalization factors of 1/2 and 1/4 in the scalar gradient and potential terms. The energy density of the model is similar to the expression in the abelian Higgs model (501),

$$\begin{aligned}
T^{00} &= \frac{1}{2}(D_0\phi)^T D_0\phi + \frac{1}{2}(D_k\phi)^T D_k\phi + V(\phi) + \frac{1}{2}(\mathbf{E}^p\cdot\mathbf{E}^p + \mathbf{B}^p\cdot\mathbf{B}^p), &(529)\\
E^p_k &= -F^p_{0k}, \quad B^p_k = \frac{1}{2}\epsilon_{klm}F^p_{lm}. &(530)
\end{aligned}$$

In the ground state each of these terms has to be minimal. As in the U(1) case the system described by the model can be in one of two phases, depending on whether $\kappa$ is positive or negative in the classical approximation (recall that $\lambda > 0$ in any case for stability of the ground state). The physics is very different in these phases.

[15] The minus sign in (523) is chosen such that in (524) $T_3$ appears and not $-T_3$.

For $\kappa > 0$ the minimum of $V$ is at $\phi = 0$ and the classical vacuum is given by

$$\phi = 0, \quad A^p_\mu = 0, \tag{531}$$

or any gauge transformation of this. The expansion of the action about the classical ground state suggests the presence of three species of spinless particles with mass $m = \sqrt{\kappa}$ ($\leftrightarrow \phi_k$, $k = 1, 2, 3$), coupled to three massless photon-like particles ($\leftrightarrow A^p_\mu$, $p = 1, 2, 3$). However, in non-abelian gauge models the physics in the quantum theory can be dramatically different. It turns out that due to the specific self interactions among the $A^p_\mu$ the quantum model is *confining.* This means that the 'original particles' (corresponding to the scalar and vector fields) cannot be separated at large distances, they remain close together in composite bound states. The original particles can only be observed (e.g. in high energy scattering experiments) as constituents of the composites. In fact, the model is similar to QCD in this respect, with the scalars playing the role of quarks, the vectors playing the role of gluons and the bound states similar to baryons and mesons. Perturbation theory about the classical ground state is useless here for determining the properties of the outcoming composite particles, but it still useful for describing the theory at high energies and short distances. The system is said to be in a confining phase.

For $\kappa < 0$ the theory the minimum of the potential $V$ is at nonzero $\phi$. Let us write the potential in the form of eq. (7.70) in Kolb & Turner,

$$V(\phi) = \frac{1}{4}\lambda(\phi^T\phi - \sigma^2)^2 \tag{532}$$

(our $\lambda$ is half that of K & T). The ground state is now given by

$$\phi = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix}, \quad A^p_\mu = 0, \tag{533}$$

or any gauge transformation of this. The vacuum vector $\phi_0 = (0, 0, \sigma)$ is still invariant under rotations about the 3-axis in internal space: the SO(3) symmetry has spontaneously broken to SO(2).

a. Verify

$$e^{-i\omega T_3} \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix}. \tag{534}$$

Below we shall see that the breaking of SO(3) symmetry has the consequence that some of the gauge bosons become massive and the residual SO(2)=U(1) symmetry implies that one gauge boson remains massless. Perturbation about the classical ground state works well in this case, the physics of which is similar to that of the electroweak bosons $W$, the photon and the Higgs particle. The system is said to be in a Higgs phase.

To find the masses of the particles in the theory we expand the action in terms of the deviation of the fields from the ground state. It is convenient to generalize the polar decomposition used in the abelian Higgs model and write the Higgs field in spherical coordinates $(\rho, \alpha, \beta)$ in internal space. It can be written as a rotation from a vector pointing in the internal 3-direction:

$$\phi = \begin{pmatrix} \rho \sin\alpha \cos\beta \\ \rho \sin\alpha \sin\beta \\ \rho \cos\alpha \end{pmatrix} = R(\alpha, \beta) \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}, \quad R(\alpha, \beta) = e^{-i\beta T_3}\, e^{-i\alpha T_2}. \tag{535}$$

b. By expanding the exponentials, show that

$$e^{-i\alpha T_2} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \quad e^{-i\beta T_3} = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{536}$$

The transformation looks like a gauge transformation and we shall make a similar transformation of variables for the vector potentials,

$$gA_\mu = R(\alpha, \beta) g V_\mu R(\alpha, \beta)^{-1} - i\partial_\mu R(\alpha, \beta) R(\alpha, \beta)^{-1}, \tag{537}$$
$$gV_\mu = R(\alpha, \beta)^{-1} g A_\mu R(\alpha, \beta) - i\partial_\mu R(\alpha, \beta)^{-1} R(\alpha, \beta). \tag{538}$$

Evaluating the covariant derivative of the scalar field,

$$D_\mu \phi = R(\alpha, \beta) \left[ \partial_\mu \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix} - iV_\mu \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix} \right] = R(\alpha, \beta) \begin{pmatrix} gV^2_\mu \rho \\ -gV^1_\mu \rho \\ \partial_\mu \rho \end{pmatrix}, \tag{539}$$

leads to

$$\frac{1}{2}(D_\mu \phi)^T D^\mu \phi == \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{1}{2} g^2 \rho^2 (V^1_\mu V^{1\mu} + V^2_\mu V^{2\mu}). \tag{540}$$

Because of gauge invariance, the $F^p_{\mu\nu} F^{p\mu\nu}$ term in the action can be expressed in terms of $V_\mu$ by the simple replacement $A^p_\mu \to V^p_\mu$. The masses of the particles now follow upon writing

$$\rho = \sigma + \Delta\rho, \tag{541}$$

and inspecting the nonderivative terms in the action which are quadratic in the deviation from the ground state,

$$m_H^2 = \sqrt{2\lambda}\,\sigma, \quad m_{V^1} = m_{V^2} = g\sigma, \quad m_{V^3} = 0. \tag{542}$$

The Higgs particle of this model corresponds to the $\Delta\rho$ field. Note that the $V^3$ mass is zero because of (534).

It is very suggestive to interpret $V_\mu^3$ as the vector potential of the electromagnetic field. This makes sense because in the new variables the theory is stil invariant under the U(1) gauge transformations corresponding to rotation about the 3-axis in internal space. Under such transformations $V_\mu^1$ and $V_\mu^2$ transform into each other like a charged vector field, with covariant derivative coupling similar to (524). The vector bosons have charge one in units of $g$ and gyromagnetic ratio two, like the electron. The Higgs particle has zero electric charge. So we interprete

$$V_\mu^3 = A_\mu^{\rm em}, \quad g = e, \tag{543}$$

with $A_\mu^{\rm em}$ the electromagnetic vector potential and $e$ the elementary charge unit. We realize that the Standard Model way of describing the electromagnetic interactions is not exactly like the SO(3) Higgs model, but it is possible that it is embedded in a grand unified theory such that the electromagnetic interactions emerge in a way similar to the SO(3) Higgs model.

The SO(3) Higgs theory has magnetic monopoles, to which we turn our attention now. The monopole at rest is a static field configuration of finite energy of the form

$$\phi_p(x) = \rho(r)\hat{r}_p, \quad eA_k^p(x) = f(r)\epsilon_{pkl}\hat{r}_l, \quad A_0^p = 0, \quad r = |\mathbf{x}|, \quad \hat{r}_k = \frac{x_k}{r}. \tag{544}$$

It is sometimes called 'the hedgehog' because of the way the internal vector of the scalar field is pointing outwards along the radial direction in ordinary space. The covariant derivative of the scalar field becomes

$$D_k\phi_p = \rho\left(\frac{1}{r} - f\right)(\delta_{kp} - \hat{r}_k\hat{r}_p) + \rho'\hat{r}_k\hat{r}_p, \tag{545}$$

where the prime denotes $\partial/\partial r$. Similarly, the nonabelian 'magnetic' field strength (cf. (530)) is given by

$$eB_k^p = \frac{K^2-1}{r^2}\,\hat{r}_p\hat{r}_k + \frac{K'}{r}(\delta_{pk} - \hat{r}_p\hat{r}_k), \quad K(r) = 1 - rf(r). \tag{546}$$

c. Verify these expressions. Hint: verify first $\epsilon_{plm}\epsilon_{pkn} = \delta_{kl}\delta_{mn} - \delta_{km}\delta_{ln}$, $\partial_k\hat{r}_l = (\delta_{kl} - \hat{r}_k\hat{r}_l)/r$.

Requiring the energy density to go to zero at large $r$ leads to

$$\rho \to \sigma, \quad f \to \frac{1}{r}, \quad r \to \infty. \tag{547}$$

The first condition follows from requiring the potential $V(\phi) = \lambda(\rho^2 - \sigma^2)^2/4$ to vanish: $\phi$ should approach a locally ground state configuration. The second condition corresponds to $D_k\phi_p \to 0$. Of course, $\rho'$ and $K'$ should also vanish sufficiently fast. Substitution into the expression for the energy (cf. (7.74) in Kolb & Turner) and requiring this to be stationary leads to nonlinear differential equations for $\rho$ and $f$ which can be solved numerically, similar to the cosmic string case. In general $\rho$ and $K$ approach their asymptotic values $\sigma$ and 0 exponentially fast, with length scales set by $1/m_H$ and $1/m_V$, while the behavior at the origin is such that the energy density is finite. For $\lambda \to 0$ an analytical solution is known (cf. eqs. (7.75) and fig. 7.13 in Kolb & Turner), but this so-called Prasad-Sommerfield limit has the unphysical feature that the Higgs mass vanishes.

We get a first indication of the magnetic monopole interpretation of the field configuration by examining it at distances $r \gg m_H^{-1}, m_V^{-1}$ where $\rho_p \approx \sigma \hat{r}_p$, $eA_k^p \approx \epsilon_{pkl}\hat{r}_l/r$ and $eB_k^p \approx -\hat{r}_p\hat{r}_k/r^2$, to a very good approximation. In this region $\phi$ is very close to its local ground state, which is invariant under gauge transformations of the form $\exp(-i\omega\hat{r}_p T_p)$, so it is natural to interpret $\hat{r}_p$ as the local electromagnetic direction in internal space. Then $\hat{r}_p B_k^p \approx -\hat{r}_k/er^2$, which is the Coulomb-like field of a point magnetic charge $-4\pi/e$. We note furthermore that, without any electromagnetic interpretation, the energy density of the configuration is approximately that of a point charge:

$$T^{00} \approx \frac{1}{2e^2 r^4} \qquad r \gg m_H^{-1}, m_V^{-1}. \tag{548}$$

However, the above magnetic field $\hat{r}_p B_k^p$ is not the rotation of the would-be electromagnetic vector potential $\hat{r}_p A_k^p$, since the latter is zero for the hedgehog configuration. To sharpen our magnetic monopole interpetation of the hedgehog, we shall comb its hairs such that they all point in the $\hat{3}$-direction in internal space: $\phi_p = \rho\hat{r}_p \to \rho\delta_{p3}$. In terms of spherical coordinates:

$$\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad \hat{\theta} = \partial\hat{r}/\partial\theta, \quad \hat{\phi} = \partial\hat{r}/\partial\phi, \tag{549}$$

the hedgehog scalar field is given by (535) with $\alpha = \theta$ and $\beta = \phi$, so the 'combing' can be done by applying the inverse rotation $R(\theta,\phi)^{-1}$ as in (538). Having done this, it is convenient to perform an additional rotation $\exp(-i\phi T_3)$ about the 3-axis, such that the combined rotation is regular at $\theta = 0$:

$$R^{-1} \equiv e^{-i\phi T_3} R(\theta,\phi)^{-1} = e^{-i\phi T_3}\, e^{i\theta T_2}\, e^{i\phi T_3}, \tag{550}$$

which equals the unit matrix at $\theta = 0$. (N.B. so we re-defined $R$.) This additional rotation does of course not affect the combed scalar field, but it does influence the final form of electromagnetic field $A_\mu^{\rm em}$ in a way which will be commented upon later. The transformed gauge field is now given by

$$eA_k^{\rm em} = eV_k^3 = eR_{3p}^{-1} A_k^p - \frac{1}{2}{\rm Tr}\,[T_3 i\partial_k R^{-1}\, R], \tag{551}$$

where we used (255) and (256). To evaluate $A_k^{\rm em}$ we first rewrite $A_k^p$ using spherical coordinates by contraction with $\delta_{kl} = \hat{r}_k\hat{r}_l + \hat{\theta}_k\hat{\theta}_l + \hat{\phi}_k\hat{\phi}_l$:

$$eA_k^p = f(r)(\hat{\theta}_p\hat{\phi}_k - \hat{\phi}_p\hat{\theta}_k). \tag{552}$$

Rotating $\hat{r}$ into the $\hat{3}$ direction, the corresponding vectors $\hat{\theta}$ and $\hat{\phi}$ come to lie in the 1-2 plane. Hence, the first term on the right hand side of eq. (551) is zero and $V_k^3$ is given entirely by the third internal component of $-i\partial_k R^{-1} R$. Using

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{553}$$

we find

$$eA_\phi^{\rm em} = \frac{\cos\theta - 1}{r\sin\theta}, \quad A_r^{\rm em} = A_\theta^{\rm em} = 0. \tag{554}$$

d. Verify this.

The potential $\mathbf{A}^{\rm em}$ is regular along the positive $z$-axis, $\theta = 0$, which is why we introduced the additional rotation along the 3-axis in (550). It is singular along the negative $z$-axis, $\theta = \pi$, and at the origin $r = 0$. We shall now show that it is the vector potential of an infinitely long, infinitely thin solenoid along the negative $z$-axis ending at $z = 0$, with magnetic flux equal to $4\pi/e$. The magnetic flux of the solenoid can be found by calculating the line integral of the vector potential along a loop in the $x$-$y$ plane around the negative $z$-axis, of radius $h = \sqrt{x^2 + y^2} = r\sin\theta$:

$$\int_{\sqrt{x^2+y^2}<h} d^2x\, \hat{z}\cdot\mathbf{B}^{\rm em} = h\int d\phi A_\phi^{\rm em} = -\frac{4\pi}{e} + O((\pi-\theta)^2). \tag{555}$$

For fixed $z$ the result holds for $h \to 0$; for $z \to -\infty$ the result is independent of the radius $h$. The solenoid is infinitely thin and we may write for its magnetic field: $e\mathbf{B}^{\rm em} = -4\pi\delta(x)\delta(y)\Theta(-z)\hat{z}$. Using standard formulas for the rotation of a vector field in spherical coordinates gives the magnetic field away from the singularities: $B^{\rm em} = \nabla\times e\mathbf{A}^{\rm em} = -\hat{r}/r^2$. Putting things together we have

$$e\mathbf{B}^{\rm em} = -\frac{\hat{r}}{r^2} - 4\pi\delta(x)\delta(y)\Theta(-z)\hat{z}. \tag{556}$$

Note that $\nabla\cdot\mathbf{B}^{\rm em} = 0$ in accordance with $\mathbf{B}^{\rm em} = \nabla\times\mathbf{A}^{\rm em}$.

Infinitely thin solenoids with flux quantized in units of $2\pi/e$ are called Dirac strings. The remarkable fact discovered by Dirac is that they do not lead to observable effects in quantum mechanics. For example, such a string does not give an interference pattern in Aharonov-Bohm scattering. Furthermore, the string can be moved about by electromagnetic gauge transformations.

e. Suppose we take away the factor $\exp(-i\phi T_3)$ in (550). Discuss the properties of the resulting $\mathbf{A}^{\rm em}$ and $\mathbf{B}^{\rm em}$.

In the present SO(3) theory the Dirac string is clearly an artefact of the 'combed hedgehog gauge', because there is no trace of it in the energy density. Note that it best to evaluate the energy density in a gauge in which the gauge field is regular, e.g. in the original hedgehog configuration.

The physically observable magnetic field is the Coulomb-like field corresponding to a magnetic monopole, as if $\nabla \cdot \mathbf{B}^{\mathrm{em}} = -(4\pi/e)\delta^3(\mathbf{x})$. For the 't Hooft-Polyakov monopole the magnetic charge is two Dirac units. This is very large compared to $e$: their ratio is $4\pi/e^2 \approx 137$.

# 16 Phase transitions and the effective potential

To investigate possible phase transitions, we are going to study the thermodynamic potential $(-\ln Z/\beta L^3)$ by means of the effective potential. We shall calculate it in the semiclassical approximation and find that phase transitions may occur as a function of temperature.

## 16.1 Spontaneous symmetry breaking

Consider again the simple scalar field model given by the classical hamiltonian

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\partial_k\phi\partial_k\phi + V(\phi)\right], \quad V(\phi) = \frac{1}{2}\kappa\phi^2 + \frac{1}{4}\lambda\phi^4 + \epsilon. \tag{557}$$

The classical ground state corresponds to the minimum value of $H$, i.e. $\pi = 0$ and $V(\phi)$ minimal. Suppose now that $\kappa < 0$. Then the minimum $V(\phi)$ is not at $\phi = 0$ but at

$$\phi = \phi_0, \quad \phi_0 \equiv \pm\sqrt{-\kappa/\lambda}. \tag{558}$$

The quantum system may now be approximately treated by expanding about one of the minima, say $+\sqrt{-\kappa/\lambda}$. To lowest order the system describes free particles with a mass

$$m_0^2 = \left[\frac{\partial^2 V(\phi)}{\partial\phi^2}\right]_{\phi_0} = -2\kappa = 2\lambda\phi_0^2. \tag{559}$$

a. Plot $V(\phi)$ and verify (558), (559). Determine the energy momentum tensor of the classical ground state. Note that the contribution to the cosmological constant depends on $\kappa$, $\lambda$, as well as $\epsilon$ when $\kappa$ is negative.

The action and hamiltonian have a symmetry $\phi \to -\phi$. In the present situation the ground state is evidently not symmetric since $\langle 0|\phi|0\rangle = \phi_0 +$ quantum corrections $\neq 0$. This phenomenon is called spontaneous symmetry breaking: the ground state is not symmetric under a symmetry of the action. We shall see that at sufficiently high temperatures the order parameter

$$\bar\phi \equiv \langle\phi\rangle = \mathrm{Tr}\,\rho\phi \tag{560}$$

vanishes and the symmetry gets restored.

## 16.2 Effective potential

The effective potential can be introduced by probing the system with a spatially constant external source $J$, such that the density matrix is

$$\rho = Z^{-1}(J)\, e^{-\beta H + \beta J\int d^3x\,\phi}, \quad Z(J) = \mathrm{Tr}\, e^{-\beta H + \beta J\int d^3x\,\phi}, \tag{561}$$

where we have explicitly indicated the dependence on $J$. Then

$$\frac{-\partial\Omega(J)}{\partial J} = \frac{\partial \ln Z(J)}{\beta L^3 \partial J} = \frac{1}{L^3}\langle \int d^3x\, \phi\rangle = \bar{\phi}, \tag{562}$$

where $\bar{\phi}$ depends on $J$. Assuming that this dependence can be inverted such that we may consider $J$ to be a function of $\bar{\phi}$, we make a Legendre transformation:

$$\Omega(J) = V_{\text{eff}}(\bar{\phi}) - J\bar{\phi}, \tag{563}$$

which defines the effective potential $V_{\text{eff}}$ as a function of $\bar{\phi}$. We can then attempt to find the thermal ground state by minimizing the effective potential (for $J = 0$ this gives the minimum of $\Omega(0)$). For nonzero $J$ we have from (562) and (563),

$$\frac{\partial V_{\text{eff}}(\bar{\phi})}{\partial \bar{\phi}} = J. \tag{564}$$

## 16.3 Semiclassical approximation

In the classical approximation the effective potential is just $V$, which is temperature independent. However, in the full quantum theory it does depend on temperature. We now calculate $\Omega(J)$ in the semi-classical approximation. This means we expand $\phi$ about its classical solution $\phi_c$, writing $\phi = \phi_c + \phi'$ and keep only the terms quadratic in $\phi'$ in the hamiltonian. This reduces the problem formally to that of free fields, which we have already discussed in the previous sections. Before going ahead we have to realize that the quantum corrections may be infinite. We have seen this already in the case $\lambda = 0$ for the correction to $\epsilon$. This was dealt with by starting with a so-called bare parameter $\epsilon_0$ which absorbs the divergence leaving a finite renormalized $\epsilon$. Here in the nonzero $\lambda$ case the same procedure will be necessary for $\kappa$ and $\lambda$. The idea is to regularize the theory, replace $\epsilon \to \epsilon_0$, $\kappa \to \kappa_0$, $\lambda \to \lambda_0$ and tune these bare parameters in such a way that the physical results come out finite as the regularization is removed. In perturbation theory this is implemented by writing

$$V_0 \equiv \epsilon_0 + \frac{1}{2}\kappa_0\phi^2 + \frac{1}{4}\lambda_0\phi^4 = \epsilon + \frac{1}{2}\kappa\phi^2 + \frac{1}{4}\lambda\phi^4 + \Delta V = V + \Delta V, \tag{565}$$

making an expansion in $\lambda$ and and treating $\Delta V$ also as a perturbation such that it cancels the divergencies coming up in the $\lambda$ expansion. Doing this order by order determines the bare parameters ($\epsilon_0$, $\kappa_0$, $\lambda_0$) in terms of the renormalized ones (i.e. $\epsilon$, $\kappa$, $\lambda$). The terms in $\Delta V$ are called counterterms[16]. In this way the perturbation is kept small (instead of diverging).

In fact, when corrections get large, as they turn out to be for large temperature, we can improve the expansion by using even finite counterterms to obtain

[16]In higher orders also a divergent rescaling of $\phi$ is needed (to keep $\bar{\phi}$ finite).

redefined corrections that remain small. We shall do this with a finite mass counterterm by choosing

$$V + \frac{1}{2} m_1^2 \phi^2 \tag{566}$$

as the zero order potential and

$$\Delta V - \frac{1}{2} m_1^2 \phi^2 = \epsilon_0 - \epsilon + \frac{1}{2}(\kappa_0 - \kappa)\phi^2 + \frac{1}{4}(\lambda_0 - \lambda)\phi^4 - \frac{1}{2} m_1^2 \phi^2 \tag{567}$$

as the counterterms. The mass $m_1$ will be determined self-consistently later.

After these preparations we now expand $\phi$ about the classical solution $\phi_c$ determined by $V_0$,

$$V_0'(\phi_c) \equiv \frac{\partial V_0(\phi_c)}{\partial \phi_c} = J, \tag{568}$$

writing $\phi = \phi_c + \phi'$ and keeping only the terms quadratic in $\phi'$ in the hamiltonian:

$$\begin{aligned}
H - J \int d^3x\, \phi &= L^3[V_0(\phi_c) - J\phi_c] && (569)\\
&\quad + \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\partial_k\phi'\partial_k\phi' + \frac{1}{2}V_0''(\phi_c)\phi'^2\right] + O(\phi'^3), \\
&= L^3\left[V(\phi_c) + \frac{1}{2}m_1^2\phi_c^2 - J\phi_c\right] && (570)\\
&\quad + L^3[\Delta V - \frac{1}{2}m_1^2\phi_c^2] + \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}\partial_k\phi'\partial_k\phi' + \frac{1}{2}m^2(\phi_c)\phi'^2\right] \\
&\quad + \cdots, \\
m^2(\phi_c) &= V''(\phi_c) + m_1^2 = \kappa + 3\lambda\phi_c^2 + m_1^2, && (571)\\
\cdots &= \int d^3x\, \frac{1}{2}[\Delta V''(\phi_c) - m_1^2]\phi'^2 + O(\phi'^3). && (572)
\end{aligned}$$

The linear terms drop out because $\phi_c$ satisfies the classical equation (568). The first line in (570) is leading, it is of order $1/\lambda$. For example, for $J = 0$, $\phi_c^2 = -\kappa_0/\lambda_0 \approx -\kappa/\lambda$, $V(\phi_c) \approx \epsilon - \kappa^2/4\lambda$ The second line in (570) is $O(\lambda^0) = O(1)$, this is going to give give the quantum corrections. The $\cdots$ in (572) are $O(\lambda)$ and are neglected.

The commutation rules between $\pi$ and $\phi'$ as operators are unchanged, and we can immediately use our previous result (204) for $\ln Z$, adapted to the present case with only one scalar field:

$$\Omega(J) = V_0(\phi_c) - J\phi_c + \int \frac{d^3p}{(2\pi)^3} \left\{\frac{1}{2}\omega_{\mathbf{p}}(\phi_c) + \frac{1}{\beta}\ln[1 - e^{-\beta\omega_{\mathbf{p}}(\phi_c)}]\right\}, \tag{573}$$

where

$$\omega_{\mathbf{p}}(\phi_c) = \sqrt{\mathbf{p}^2 + m^2(\phi_c)}. \tag{574}$$

We now need to carry out the Legendre transform. Eq. (562) leads to

$$\bar{\phi} = \phi_c - \frac{6\lambda\phi_c}{m^2(\phi_c)} \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}(\phi_c)} \left[\frac{1}{2} + \frac{1}{e^{\beta\omega_{\mathbf{p}}(\phi_c)} - 1}\right], \tag{575}$$

where we used

$$\frac{\partial \omega_{\mathbf{p}}(\phi_c)}{\partial J} = \frac{6\lambda\phi_c}{2\omega_{\mathbf{p}}(\phi_c)}\frac{\partial \phi_c}{\partial J}, \quad V_0''(\phi_c)\frac{\partial \phi_c}{\partial J} = 1 \to \frac{\partial \phi_c}{\partial J} \approx \frac{1}{m^2(\phi_c)} \tag{576}$$

(cf. (568) for the second equation). Now consider eqs. (563), (564) and (573). Eliminating $\Omega(J)$ and $J$ gives

$$\begin{aligned} & V_0(\phi_c) + \int \frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2}\omega_{\mathbf{p}}(\phi_c) + \frac{1}{\beta}\ln[1 - e^{-\beta\omega_{\mathbf{p}}(\phi_c)}]\right\} && (577)\\ = \; & V_{\text{eff}}(\bar\phi) - \frac{\partial V_{\text{eff}}(\bar\phi)}{\partial\bar\phi}(\bar\phi - \phi_c) && (578)\\ = \; & V_{\text{eff}}(\phi_c) + O((\phi_c - \bar\phi)^2). && (579) \end{aligned}$$

Since $\phi_c$ is arbitrary and $O((\phi_c - \bar\phi)^2) = O(\lambda^2)$ (cf. (575), this gives $V_{\text{eff}}(\bar\phi)$ to order $\lambda$. Regrouping into a vacuum (zero temperature) contribution and a temperature dependent addition we have ($T = 1/\beta$)

$$\begin{aligned} V_{\text{eff}}(\bar\phi) &= V_{\text{eff}}^{\text{vac}}(\bar\phi) + V_{\text{eff}}^{T}(\bar\phi), && (580)\\ V_{\text{eff}}^{\text{vac}}(\bar\phi) &= V(\bar\phi) + \Delta V(\bar\phi) + \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\omega_{\mathbf{p}}(\bar\phi), && (581)\\ V_{\text{eff}}^{T}(\bar\phi) &= T\int \frac{d^3p}{(2\pi)^3}\ln[1 - e^{-\omega_{\mathbf{p}}(\bar\phi)/T}]. && (582) \end{aligned}$$

The finite temperature contribution $V_{\text{eff}}^{T}(\bar\phi)$ is given by a finite integral over the momenta $\mathbf{p}$. However, the zero temperature contribution $V_{\text{eff}}^{\text{vac}}$ has the form of the free field vacuum energy and it is divergent. This divergence can be dealt with by renormalization.

## 16.4 Renormalization

Regularizing the integral in (581) with a simple cutoff as in (139) gives

$$\begin{aligned} V_{\text{eff}}^{\text{vac}} &= \epsilon_0 + \frac{1}{2}\kappa_0\bar\phi^2 + \frac{1}{4}\lambda_0\bar\phi^4 && (583)\\ &\quad + \frac{1}{64\pi^2}\left[4\Lambda^4 + 4\Lambda^2 m^2(\bar\phi) + \frac{1}{2}m^4(\bar\phi) - m^4(\bar\phi)\ln\left(\frac{2\Lambda}{m(\bar\phi)}\right)^2 + O(\Lambda^{-2})\right] \end{aligned}$$

(one way to do the integral is to substitute $p = mx$, $x = \sinh\chi$, $2\sinh\chi\cosh\chi = \sinh 2\chi$). A crucial observation is now that the divergent terms proportional to $\Lambda^4$, $\Lambda^2$ and $\ln\Lambda$ are polynomials in $\bar\phi^2$, so we can cancel the divergent terms by a suitable choice of the parameters $\epsilon_0$, $\kappa_0$ and $\lambda_0$, writing

$$V_{\text{eff}}^{\text{vac}} = \epsilon + \frac{1}{2}\kappa\bar\phi^2 + \frac{1}{4}\lambda\bar\phi^4 + \frac{1}{64\pi^2}m^4(\bar\phi)\ln\frac{m(\bar\phi)^2}{\mu^2}, \tag{584}$$

where $\mu$ is an arbitrary mass scale and we dropped the terms vanishing as $\Lambda \to \infty$. Comparing the divergent coefficients of $\bar\phi^n$, $n = 0, 2, 4$ gives the relations between the bare and renormalized parameters:

$$\epsilon_0 = \epsilon - \frac{1}{64\pi^2}\left(4\Lambda^4 + 4\Lambda^2\kappa - \kappa^2 \ln\frac{c^2\Lambda^2}{\mu^2}\right), \quad c^2 = 4e^{-1/2}, \tag{585}$$

$$\kappa_0 = \kappa - \frac{\lambda}{32\pi^2}\left(12\Lambda^2 - 6\kappa\ln\frac{c^2\Lambda^2}{\mu^2}\right), \tag{586}$$

$$\lambda_0 = \lambda + \frac{9\lambda^2}{16\pi^2}\ln\frac{c^2\Lambda^2}{\mu^2}. \tag{587}$$

We could of course also have used a covariant regularization but this would only change the relation between bare and renormalized parameters and not the renormalized form of $V_{\rm eff}^{\rm vac}(\bar\phi)$. Typically it would lead to a change in the mass scale $\mu^2$ because it would have been introduced in a different way. However, a change in $\mu^2$ can be compensated by a change in $\epsilon$, $\kappa$ and $\lambda$. For $m_1 = 0$ eq. (584) is the potential given in Kolb & Turner, eq. (7.7).

The miracle of renormalizable field theory (such as the $\phi^4$ model) is that all other physical quantities like scattering amplitudes (or the effective potential) are also finite when re-expressed in terms of the renormalized parameters. This can be proved to all orders in perturbation theory. It has also been understood non-perturbatively through Renormalization Group theory and checked by numerical Monte Carlo computations using the lattice regularization.

## 16.5 Quasiparticles

Minimizing the effective potential should give us the best approximation to the partition function (cf. (563,564)), which leads to

$$0 = \frac{\partial V_{\rm eff}}{\partial\bar\phi} = \frac{\partial V_{\rm eff}^{\rm vac}}{\partial\bar\phi} + \frac{\partial V_{\rm eff}^{T}}{\partial\bar\phi} \tag{588}$$

$$= \bar\phi\left[\kappa + \lambda\bar\phi^2 + \frac{6\lambda m^2(\bar\phi)}{16\pi^2}\left(\ln\frac{m^2(\bar\phi)}{\mu^2} + 1\right) + 6\lambda I(m^2(\bar\phi))\right], \tag{589}$$

where

$$I(m^2(\bar\phi)) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf p}(\bar\phi)}\,\frac{1}{e^{\omega_{\mathbf p}(\bar\phi)/T} - 1}. \tag{590}$$

Suppose we choose $m_1 = 0$ and look for a solution $\bar\phi_0$ near the classical $\phi_0 = \sqrt{-\kappa/\lambda}$. Then $m^2(\bar\phi_0) \approx -2\kappa$ and we get a reasonable solution for low temperature where the integral $I$ is small. For high temperatures

$$I(m^2(\phi)) \approx \frac{1}{4\pi^2}\zeta(2)T^2 = \frac{1}{24}T^2, \tag{591}$$

which can lead to a substantial correction to $\kappa$ such that $\kappa+6\lambda I \approx \kappa+\lambda T^2/4 \gg 0$.[17] Under such circumstances we expect symmetry restoration with the only solution of (589) being $\bar{\phi}_0 = 0$. However, for $\bar{\phi}_0 = 0$ the mass $m^2(\bar{\phi}_0) = \kappa < 0$ if $m_1 = 0$. This does not make sense because the argument of $\ln[m^2(\bar{\phi})/\mu^2]$ becomes negative and the effective energy $\omega_{\mathbf{p}}(\phi)$ becomes complex. Going back we see that the starting point of our approximation, the quantum hamiltonian in (570) does not have a ground state, because $m^2(\bar{\phi}_0)$ would be negative. This is the problem with the usual form of the effective potential with $m_1 = 0$.

The resolution lies in the concept of quasiparticles. We have approximated the partition function by that of a system of particles with mass $m(\phi)$. This may be a reasonable description at weak coupling, but to keep the corrections small the particles should be interpreted as having an effective mass $m(\bar{\phi})$ which changes with temperature. They are called quasiparticles. The optimal choice for $m_1$ is such that the $m(\bar{\phi})$ is equal to the effective mass

$$
\begin{aligned}
m^2_{\text{eff}}(\bar{\phi}) &\equiv \frac{\partial^2 V_{\text{eff}}(\bar{\phi})}{\partial\bar{\phi}^2} && (592)\\
&= \kappa + 3\lambda\bar{\phi}^2 + \frac{(6\lambda\bar{\phi})^2}{16\pi^2}\left(\ln\frac{m^2(\bar{\phi})}{\mu^2}+2\right) + \frac{6\lambda m^2(\bar{\phi})}{16\pi^2}\left(\ln\frac{m^2(\bar{\phi})}{\mu^2}+1\right)\\
&\quad + 6\lambda I(m^2(\bar{\phi})). && (593)
\end{aligned}
$$

With this choice the corrections to the quasiparticle mass vanish and we get an equation for $m^2(\bar{\phi})$:

$$m^2(\bar{\phi}) = m^2_{\text{eff}}(\bar{\phi}) \tag{594}$$

Since $m^2(\bar{\phi})$ depends on $m_1^2$ this is sometimes called a self-consistent equation (for $m_1^2$ in terms of $\epsilon$, $\kappa$, $\lambda$ and $T$). We are particularly interested in its solution for $\bar{\phi} = \bar{\phi}_0$:

$$\bar{m}^2 \equiv m^2(\bar{\phi}_0). \tag{595}$$

If eq. (589) has a solution $\bar{\phi}_0 \neq 0$ we can use the terms in the square bracket in (589) to simplify (594), (595) to

$$\bar{m}^2 = 2\lambda\bar{\phi}_0^2 + \frac{(6\lambda\bar{\phi}_0)^2}{16\pi^2}\left(\ln\frac{\bar{m}^2}{\mu^2}+2\right). \tag{596}$$

We should then check that the solution with nonzero $\bar{\phi}_0$ has a lower effective potential that that with $\bar{\phi}_0 = 0$.

[17]Higher order corrections to the leading $T^2$ behavior should have the form $\kappa + [\lambda + O(\lambda^2)]T^2/4$, and even at very weak coupling the $T^2$ 'correction' may dominate over $\kappa$ at sufficiently high temperature.

## 16.6 The phase transition

Consider now eq. (589) at temperatures so high that the sum of the terms in the square bracket is positive (cf. the high temperature approximation (591) to $I(\bar{m}^2)$). Then the only solution is $\bar{\phi}_0 = 0$, and for sufficiently high $T$ the solution to (594) is

$$\bar{m}^2 \approx \frac{6\lambda T^2}{24}. \tag{597}$$

Since this $\bar{m}^2$ is positive there is no problem with imaginary square roots etc. So we have symmetry restoration at high temperatures: $\bar{\phi}_0 = 0$. Conversely, if we start at $\bar{\phi}_0 \neq 0$ and raise the temperature from zero upwards, eq. (589) in the high temperature approximation (591) shows that $\bar{\phi}_0^2$ decreases:

$$\lambda\bar{\phi}_0^2 \approx -\kappa - \frac{6\lambda T^2}{24}. \tag{598}$$

Then also $\bar{m}^2$ decreases according to (596). The phase transition occurs at the critical temperature where $\bar{\phi}_0^2 = 0$,

$$T_{\rm crit} = \sqrt{\frac{2}{\lambda}}\, m_c, \qquad m_c = \sqrt{-2\kappa}, \tag{599}$$

where $m_c$ is the classical mass, which is approximately the effective mass at zero temperature. Eq. (596) shows that then also $\bar{m}^2 = 0$, and we have a second order phase transition[18].

A good description of the critical region $\bar{m} \approx 0$ is actually quite difficult and involves the theory of critical phenomena.

a. To get a feeling for the size of various terms in the equations, let's assume that $\phi$ is the Higgs field of the Standard Model. At zero temperature we then have $\bar{\phi}_0 = 246$ GeV. The Higgs mass is not known at the time of this writing, but let's assume it is about the value implied by fitting the Standard Model parameters to current results of high precision experiments, say $\bar{m} = 200$ GeV. Let us choose $\mu^2$ such that the order $\lambda^2$ correction in eq. (596) vanishes at zero temperature. Then $\bar{m}^2 = 2\lambda\bar{\phi}_0^2$, which gives $\lambda \approx 0.4$, $T_{\rm crit} \approx 450$ GeV. A good project is now to solve the equations set up so far numerically and to plot the effective potential for various temperatures. A useful approximation is the high temperature expansion[19]. Note that the critical region is delicate. For instance, eq. (596) has no solution with positive $\bar{m}^2$ and $\bar{\phi}_0^2$ in a tiny (how large?) region near $\bar{m}^2 = 0$. For such small effective masses we need improved calculational tools.

[18] The correlation length diverges at the transition. It can be shown that the correlation length is the inverse of $\bar{m}$.

[19] See e.g. J.I, Kapusta, *Finite-temperature field theory*, appendix A2.