Guide to the course Particles and Fields (2007)

PM = 'Introduction to Quantum Field Theory', lecture notes by Piet Mulders.

JS = 'Introduction to Relativistic Quantum Fields', lecture notes by Jan Smit.

The book by Peskin & Schroeder appears to be not entirely consistent in its use of the electron charge. According to p. xxi e is negative, but in the bulk of the book it appears to be positive, as far as I can judge from the application of Feynman rule on p. 123 and eq. (6.38), which are correct in the convention $e > 0$, with $-e$ the charge of the of the electron. We will use this convention.

The correct Feynman rule for the vertex in QED is then given in (A.6), but on p. 124 there is then a sign error in the Feynman rule for the vertex: $-iQ|e|\gamma^\mu \to +iQe\gamma^\mu$, with $e > 0$ and $Q = -1$ for the electron.

Course Material covered so far:

Without further specification, ch. 5 or (6.39) means chapter 5 or equation (6.39) in the book by Peskin & Schroeder, etc.

- Chapter 5

  Sects. 5.1 and 5.5 .

  Supporting the material for the summary given in the first lecture can also be found in JS sects. 4.1, 4.2, 4.3, 4.6, 6.6, and Problem 2 in sect. 8.7.

- Chapter 6

  Sects. 6.1, 6.2, 6.3, 6.4. plus conclusion of 6.5, eq. (6.84) and further.

  Note for p. 186: the Ward identity is valid only between spinors $\bar{u}(p')$ and $u(p)$, so $q_\mu \Gamma^\mu = 0 \to q_\mu \bar{u}' \Gamma^\mu u = 0$. Similarly, (6.33) holds only between spinors.

  For the interpretation of the form factors $F_1(0)$ and $F_2(0)$ a reasoning was given along the lines of JS sect. 8.4 may also be useful.

  A derivation of 'Feynman parameter formulas' like (6.39) is given in JS sect. 4.8 .

- Chapter 7

  Sect. 7.1 and the material in sect. 9.3 of JS. In discussion the position of the pole and residue in the electron propagator an alternative presentation was given as follows:

The diagram for the free electron propagator was denoted by $-iS_0$; in momentum space

$$S_0(p) = (m_0 - \not{p} - i\epsilon)^{-1}.$$

The sum of diagrams for the electron two-point function was denoted by $-iS$, and the sum of all 1PI self-energy diagrams $-i\Sigma$, such that (suppressing the $i\epsilon$)

$$S(p) = (S_0^{-1} + \Sigma)^{-1} = (m_0 + \Sigma - \not{p})^{-1}.$$

Writing

$$\Sigma(p) = A(p^2) - B(p^2)\not{p},$$

we have

$$S = (1+B)^{-1}\frac{M+\not{p}}{M^2-p^2}, \quad M(p^2) \equiv \frac{m_0 + A(p^2)}{1+B(p^2)}.$$

The position of the pole in p^2 then satisfies the equation

$$m^2 = M^2(m^2).$$

and expansion around $p^2 = m^2$ gives

$$M^2(p^2) - p^2 = m^2 - p^2 + \left[\frac{2(m_0+A)A'}{(1+B)^2} - \frac{(m_0+A)^2 2B'}{(1+B)^3}\right](p^2 - m^2) + O((p^2-m^2)^2),$$

where $A' = dA/dp^2$ etc. and the expression in aquare brackets is to be evaluated at $p^2 = m^2$. Near the pole

$$S \approx Z_2 \frac{m+\not{p}}{m^2-p^2}$$

with the so-called 'wave-function renormalization-constant' Z_2 given by

$$Z_2^{-1} = 1 + B(m^2) - 2mA'(m^2) + 2m^2 B'(m^2).$$

Writing formally Σ as a function of $\not{p}$ one gets the same results from (7.24) and (7.26) in Peskin and Schroeder.

The LSZ formula (sect. 7.2) was derived along the lines of sects. 9.3 and 9.4 of JS. See also Piet Mulders' lecture notes, the text leading to eq. (10.66).

Read through sect. 7.2; the end of this section is important for the course. Note that (7.46), the equation below it and (7.47) are only valid between spinors $\bar{u}'$ and u.

We skipped section 7.3.

Read through section 7.4 (W.T. identity). In class we applied (7.65) to (6.38) and showed that (7.69) is correct to one-loop order. We also derived

(7.70).

Section 7.5

The Final Project 'Radiation of Gluon Jets' is to be handed in.

- Chapter 8: read through

- Chapter 9

  The path integral has aleady been introduced in PM, and we skipped sections 9.1, 9.2, 9.3, 9.5.

  Section 9.4. (See also sects. 7.5 – 7.7 and the beginning of sect. 8.2 in JS, which I more or less followed in my lectures.)

  Sect. 9.6 was essentially covered in class (skipped equations of motion, but will mentioned them as leading to the Dyson-Schwinger hierachy in connection with the effective action – the generating functional of 1PI vertex functions in sect. 11.5).

- Chapter 10

  Sect. 10.1, read sect. 10.2, sect. 10.3, read sect. 10.4

  Skipped sect. 10.5.

  Problem 10.2 is to be handed in.

- Chapter 11

  Sect. 11.1 (covered already in PM), beginning of 11.2, read 11.3, 11.5 Skipped rest of 11.2, 11.4, 11.6.

  We skipped the material of section 11.5, which is an important subject, so reading is strongly encouraged. A convenient notation (due to Bryce DeWitt) is useful here, see my lecture notes 'Quantum Field Theory (1995)', section 2.6. I also recommend reading sect. 2.7 of these notes, on the hierarchy of Dyson-Schwinger equations.

  Note some (to my eyes) confusing notation in sect. 11.4: functional derivatives of Lagrange densities (as in (11.55), 11.58) etc.) are expected to give Dirac delta functions. What is meant is presumably $\mathcal{L}_1 \to S_1 = \int d^4x\, \mathcal{L}_1$.

- Chapter 12. Read sect. 12.1.

  Sect. 12.2.

  Note: to make sense out of the eqs. directly above (12.31), replace it by: $\int d^4x\, d^4x'\, e^{ipx-ip'x'}\, \langle 0|T\phi(x)\phi(x')|0\rangle = (2\pi)^4\delta^{(4)}(p-p')\, i/(p^2+i\epsilon)$. Similarly for the equation below (12.31).

  Note: In eqs. (12.53) and (12.58), the $\partial/\partial M$ does not act on g or e.

Read sect. 12.3; in class we covered (12.81), (12.82) (using a different notation: $M \to M_1$, $p \to M$), and some of the 'Alternatives for the running coupling constants'.

Skipped sects. 12.4, 12.5.

- Chapter 13: Skipped.

- Chapter 14: Read.

- Chapter 15: Read. The Yang-Mills and QCD actions were derived 'in class'.

- Chapter 16.

  Section 16.1. Know how to derive the Feynman rules of QCD as given on p. 507. We skipped the equality of coupling constants and the flaw in the argument (worth reading).

  Section 16.2. The material was covered 'in class'.

  Section 16.3: Read.

  Section 16.4: Skipped.

  Section 16.5: Read.

  Section 16.6: Skipped.

  Section 16.7: Read.

- Chapter 17.

  Sections 17.1 - 17.3 sketched in the lectures.

  Sections 17.4 - 17.5: Skipped, except figures 17.22.

  Section 17.6: Read. Note that $\alpha_{\rm s}$ has also been postdicted from hadron-spectrum computations using the lattice regularization. This can be understood from an equation of the form $m_{\rm hadron} = C_{\rm hadron}\,\Lambda$, in which the coefficient $C_{\rm hadron}$ is computed using non-perturbative methods and Λ is fitted by comparing with the actual hadron mass; then $\alpha_{\rm s}$ is determined by (17.17) and its more accurate analogs.

- Chapter 18: Skipped.

- Chapter 19

  Read the introduction up to section 19.1.

  We skipped section 19.1 and and the beginning of section 19.2 (but not equation (19.45)), and 'joined P & S with de subsection 'Triangle Diagrams' on page 661. However, we used the euclidean cutoff method to analyse the polar- and axial-vector Ward-Takahashi Identities. See below where we derive (19.60). The rest of section 19.2 (Fujikawa's method) was skipped again.

Section 19.3. We went into somewhat more detail on chiral symmetry and the relation with the pseudosalar mass spectrum, the extension of (19.94) to three flavors and effect of the chiral anomaly through the Witten-Veneziano relation. The material was taken from chapter 8 of J. Smit, *Introduction to quantum fields on a lattice*, Lecture Notes in Physics, Cambridge University Press 2002. See below for a shorter summary.

Read also the presentation of the Goldberger-Treiman relation (19.102), that starts above (19.95) (which we had to skip because of lack of time), and the 'Anomalies of Chiral Currents' in the rest of section 19.2 (which was only sketched in the last lecture). See below for the pion-nucleon sigma model.

Note: a factor τ^a needs to be inserted on the right-hand-side of (19.95); a factor $(2\pi)^4\delta^{(4)}(p+k-q)$ is lacking on the right-hand-side of (19.109); the polarization vectors epsilon in (19.109) are in general not the same.

Section 19.4: read.

Section 19.5: skipped.

Problem 19.2 is to be handed in.

- Chapter 20: Much of this was already treated in PM. We mentioned the Anomaly Cancellation (pages 705-707).

The triangle anomaly

We start with the expression (19.47) in Peskin & Schroeder, to which we add its permutation $(p,\nu) \leftrightarrow (k,\lambda)$ and denote the sum by $\Gamma_5^{\mu\nu\lambda}$:

$$\begin{aligned}\Gamma_5^{\mu\nu\lambda}(p,k) \;=\; & -(-ie)^2 i^3 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{tr}\,\Big[\gamma^\mu\gamma^5 S(\ell-k)\gamma^\lambda S(\ell)\gamma^\nu S(\ell+p) \\ & \gamma^\mu\gamma^5 S(\ell-p)\gamma^\nu S(\ell)\gamma^\lambda S(\ell+k)\Big]_{\mathrm{norm}},\end{aligned}$$

where the subscript 'norm' means 'normalized', i.e. it has to be regularized and the resulting arbitrariness fixed by imposing renormalization conditions. Above, S is the electron propagator

$$S(\ell) = \gamma^\mu \ell_\mu/(\ell^2 + i\epsilon) = (\gamma^\mu\ell_\mu)^{-1}.$$

The vertex function $\Gamma_5^{\mu\nu\lambda}(p,k)$ appears in the correlation function

$$\langle 0|T j_5^\mu(x) A_\nu(y) A_\lambda(z)|0\rangle, \quad j_5^\mu(x) = \bar\psi(x)\gamma^\mu\gamma_5\psi(x),$$

in one-loop order (after removal of the external-line photon-propagators). On-shell we get the matrix element of the axial current between the vacuum and a two-photon state:

$$\int d^4x\, e^{-iqx}\, \langle p\epsilon, k\epsilon'|j_5^\mu(x)|0\rangle = (2\pi)^4\delta^{(4)}(p+k-q)\,\epsilon^*_\nu(p)\epsilon'^*_\lambda(k)\Gamma_5^{\mu\nu\lambda}(p,k). \qquad (1)$$

The vertex function $\Gamma_5^{\mu\nu\lambda}$ is linearly divergent. Dimensional regularization leads to problems with the extension of γ_5 and the Levy-Civita tensor $\epsilon^{\kappa\lambda\mu\nu}$ to arbitrary dimensions. Pauli-Villars regularization introduces chiral-symmetry breaking by heavy fermion masses. Therefore we use a euclidean cutoff Λ to define the vertex function. This leads to some details in the following calculation that are instructive about the way renormalized perturbation theory works.[1]

Because of its linear divergence, $\Gamma_5^{\mu\nu\lambda}$ is ambiguous by a polynomial of first degree in the momenta, a polynomial of mass dimension one, depending on the details of the regularization. Due to the presence of γ_5, $\Gamma_5^{\mu\nu\lambda}$ is a pseudo tensor under parity and contains the Levy-Civita tensor. By Lorentz invariance and Bose symmetry the polynomial ambiguity can only be of the form

$$ce^2\epsilon^{\mu\nu\lambda\rho}(p-k)_\rho, \qquad (2)$$

where c is a dimension-less numerical constant. Because of the linear divergence of $\Gamma_5^{\mu\nu\lambda}$ one might expect a term linear in Λ in (2), but this would lead to a polynomial of mass dimension two, whereas $\Gamma_5^{\mu\nu\lambda}$ has dimension one.

The constant c is to be specified by renormalization conditions. In QED there is no counterterm for $\Gamma_5^{\mu\nu\lambda}$ and the only renormalization conditions are the Ward-Takahashi-Identities (WTIs)

$$p_\nu\Gamma_5^{\mu\nu\lambda}(p,k) = 0, \quad k_\lambda\Gamma_5^{\mu\nu\lambda}(p,k) = 0. \qquad (3)$$

Because of the Bose symmetry, if one of these conditions is satisfied, then also is the other. The *axial* WTI that is suggested by the Noether theorem following from the chiral symmetry of the classical action (with fermion mass equal to zero),

$$(p+k)_\mu\Gamma_5^{\mu\nu\lambda}(p,k) = 0,$$

will turn out to be violated by 'anomalous' terms in the quantum theory. This is not a problem, because the axial-vector current $\bar\psi\gamma^\mu\gamma_5\psi$ is not a dynamical current such as the electromagnetic current in QED.

The euclidean cutoff method can be formulated as follows:
(i) choose the external momenta such (e.g. space-like), that a Wick rotation can be made for the integral over ℓ^0. This amounts to making the substitution

[1] Actually, the Pauli-Villars method is also useful for the case of the triangle anomaly. For a discussion of dimensional regularization in this context, see F. Jegerlehner, *Facts of life with* γ_5, hep-th/0005255.

$\ell^0 = i\ell^0_{\rm E}$, and $\int_{-\infty}^{\infty} d\ell^0\, f(\ell^0) \to i\int_{-\infty}^{\infty} d\ell^0_{\rm E}\, f(i\ell^0_{\rm E})$

(ii) use a spherical cutoff for the four-dimensional integral over $\ell^0_{\rm E}$, ℓ^1, ℓ^1, ℓ^3.

So

$$-i\int \frac{d^4\ell}{(2\pi)^4}\, f(l) \to \int_{\sqrt{(\ell^0_{\rm E})^2+\boldsymbol{\ell}\cdot\boldsymbol{\ell}}<\Lambda} \frac{d\ell^0_{\rm E} d^3\ell}{(2\pi)^4}\, [f(l)]_{\ell^0 = i\ell^0_{\rm E}} \equiv \int_{|\ell_{\rm E}|<\Lambda} f(l).$$

The vertex function is now defined as

$$\begin{aligned}
\Gamma_5^{\mu\nu\lambda}(p,k) &= \Gamma_{5\Lambda}^{\mu\nu\lambda}(p,k) + ce^2\epsilon^{\mu\nu\lambda\rho}(p-k)_\rho, \\
\Gamma_{5\Lambda}^{\mu\nu\lambda}(p,k) &= -e^2\int_{|\ell_{\rm E}|<\Lambda} \mathrm{tr}\,\Big[\gamma^\mu\gamma^5 S(\ell-k)\gamma^\lambda S(\ell)\gamma^\nu S(\ell+p) \\
&\qquad \gamma^\mu\gamma^5 S(\ell-p)\gamma^\nu S(\ell)\gamma^\lambda S(\ell+k)\Big],
\end{aligned} \tag{4}$$

where contributions vanishing as $\Lambda \to \infty$ are to be dropped and the constant c is to be chosen to satisfy the WTIs (3).

Consider these vector WTIs. Using the tree-graph WTIs

$$\begin{aligned}
k_\lambda\gamma^\lambda &= \not\ell - (\not\ell - \not k) = (\not\ell + \not k) - \not\ell \\
&= S(\ell)^{-1} - S(\ell-k)^{-1} = S(\ell+k)^{-1} - S(\ell)^{-1},
\end{aligned}$$

we find

$$\begin{aligned}
k_\lambda \Gamma_{5\Lambda}^{\mu\nu\lambda}(p,k) &= -e^2\int_{|\ell_{\rm E}|<\Lambda} \mathrm{tr}\,\{\gamma^\mu\gamma_5\,[S(\ell-k)\gamma^\nu S(\ell+p) - S(\ell)\gamma^\nu S(\ell+p) \\
&\quad + S(\ell-p)\gamma^\nu S(\ell) - S(\ell-p)\gamma^\nu S(\ell+k)]\}\,.
\end{aligned}$$

We see that this has the form

$$\begin{aligned}
&-e^2\int_{|\ell_{\rm E}|<\Lambda} [f_1^{\mu\nu}(\ell+p-k,p,k) - f_1^{\mu\nu}(\ell,p,k)] \\
&-e^2\int_{|\ell_{\rm E}|<\Lambda} [f_2^{\mu\nu}(\ell-p,p) - f_2^{\mu\nu}(\ell,p)]\,,
\end{aligned}$$

with

$$\begin{aligned}
f_1^{\mu\nu}(\ell,p,k) &= \mathrm{tr}\,[\gamma^\mu\gamma_5 S(\ell-p)\gamma^\nu S(\ell+k)] \\
f_2^{\mu\nu}(\ell,p) &= \mathrm{tr}\,[\gamma^\mu\gamma_5 S(\ell)\gamma^\nu S(\ell+p)]\,.
\end{aligned}$$

If the integral representing $k_\lambda\Gamma_{5\Lambda}^{\mu\nu\lambda}$ were convergent we could shift the integration variable and find zero in the limit $\Lambda \to \infty$. However, the integral is linearly divergent (the individual integrals over f_1 and f_2 are even quadratically divergent) and this leads to a non-vanishing contribution from the surface of the integration region, the sphere with radius Λ, as $\Lambda \to \infty$.

Consider the f_1 integral. We use the Taylor expansion

$$\begin{aligned}
f_1^{\mu\nu}(\ell+p-k,p,k) - f_1^{\mu\nu}(\ell,p,k) &= (p-k)^\alpha \frac{\partial}{\partial\ell^\alpha}\, f_1(\ell,p,k) \\
&\quad + \frac{1}{2}(p-k)^\alpha(p-k)^\beta \frac{\partial}{\partial\ell^\alpha}\frac{\partial}{\partial\ell^\beta}\, f_1(l,p,k) + \cdots
\end{aligned}$$

and the Gauss formula

$$\int_{|\ell_{\rm E}|<\Lambda} \frac{\partial}{\partial \ell^\alpha}\, f(\ell) = \frac{\Lambda^3}{(2\pi)^4} \int d\Omega_4\, n^\alpha f(\Lambda n).$$

Here $\int d\Omega_4$ is the integral over angles in four-dimensional spherical coordinates, $\int d\Omega_4 = 2\pi^2$, and n^α is the outward pointing unit vector, except that its 'time' component n^0 has an extra factor i coming from $\ell^0 = i\ell^0_{\rm E}$. Symmetry arguments lead to

$$\langle n^\alpha n^\beta \rangle = -\frac{1}{4} g^{\alpha\beta}, \quad \langle n^\alpha n^\beta n^\gamma \rangle = 0, \quad \langle n^\alpha n^\beta n^\gamma n^\delta \rangle = \frac{1}{24}(g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}),$$

etc., where

$$\langle F \rangle = \frac{\int d\Omega_4\, F}{\int d\Omega_4}.$$

The function f_1 is given by

$$f_1^{\mu\nu}(\ell, p, k) = \frac{\mathrm{tr}\,[\gamma^\mu \gamma_5 (\not\ell - \not p) \gamma^\nu (\not\ell + \not k)]}{(\ell - k)^2 (\ell + k)^2} = 4i\epsilon^{\mu\rho\nu\sigma}(\ell - p)_\rho (\ell + k)_\sigma \frac{1}{(\ell - k)^2 (\ell + k)^2}.$$

We expand its values for $\ell = \Lambda n$ in descending powers of Λ,

$$f_1^{\mu\nu}(\Lambda n, p, k) = 4i\epsilon^{\mu\rho\nu\sigma} \left[\Lambda^{-2} n_\rho n_\sigma + \Lambda^{-3}(n_\rho k_\sigma - n_\sigma p_\rho + 2 n_\rho n_\sigma n_\beta (p^\beta - k^\beta)) + \cdots\right],$$

and similar for the second order terms in the Taylor expansion that involve $\partial f_1/\partial \ell^\beta$. The second order terms actually do not contribute because they turn out to depend only on one four-vector (the combination $p - k$), which give zero upon contraction with the indices ρ and σ of the ϵ tensor. For the same reason the f_2 contribution vanishes as it depends only on the one four-vector p. Applying the results for $\langle n^\alpha \cdots \rangle$ we find

$$ik_\lambda \Gamma^{\mu\nu\lambda}_{5\Lambda}(p, k) = \frac{e^2}{4\pi^2}\, \epsilon^{\mu\rho\nu\sigma} k_\rho p_\sigma + O(\Lambda^{-1}).$$

Note that the terms linear in Λ vanished upon angular integration.

It follows that we have to choose the constant

$$c = -i\frac{1}{4\pi^2}$$

in order to satisfy the vector WTI for the complete vertex function $\Gamma^{\mu\nu\lambda}_5(p, k) = \Gamma^{\mu\nu\lambda}_{5\Lambda}(p, k) + c\, e^2 \epsilon^{\mu\nu\lambda\rho}(p - k)_\rho$.

Having determined the constant c, we turn to the axial-vector WTI. The axial WTI at the tree-graph level (19.48) is ($q = p + k$)

$$q_\mu \gamma^\mu \gamma^5 = S(\ell + p)^{-1}\gamma^5 + \gamma^5 S(\ell - k)^{-1} = S(\ell + k)^{-1}\gamma^5 S(\ell - p)^{-1}.$$

Using this identity gives

$$
\begin{aligned}
q_\mu \Gamma^{\mu\nu\lambda}_{5\Lambda}(p,k) &= -e^2 \int_{|\ell_{\rm E}|<\Lambda} \Big\{ {\rm tr}\, \gamma_5 \gamma^\nu \left[S(\ell)\gamma^\lambda S(\ell+k) - S(\ell-k)\gamma^\lambda S(\ell) \right] \\
&\qquad + {\rm tr}\, \gamma_5 \gamma^\lambda \left[S(\ell)\gamma^\nu S(\ell+p) - S(\ell-p)\gamma^\nu S(\ell) \right] \Big\}.
\end{aligned}
$$

This is clearly zero, since the integration over ℓ yields a sum of terms, each involving only one four-vector that is twice contracted with the Levi-Civita tensor. For the complete vertex function this implies the celebrated anomaly in the axial WTI:

$$
i(p+k)_\mu \Gamma^{\mu\nu\lambda}_5(p,k) = i(p+k)_\mu c\, e^2 \epsilon^{\mu\nu\lambda\rho}(p-k)_\rho = -\frac{e^2}{2\pi^2}\, \epsilon^{\alpha\nu\beta\lambda} k_\alpha p_\beta,
$$

which is eq. (19.59) in P & S. The right-hand-side is called the anomaly. It is equal to the matrix element

$$
-\frac{e^2}{16\pi^2}\, \epsilon^{\alpha\nu\beta\lambda} \langle p\epsilon, k\epsilon' | F_{\alpha\nu}(0) F_{\beta\lambda}(0) | 0 \rangle,
$$

and comparing with (1) we can write

$$
\int d^4x\, e^{-iqx}\, \langle p\epsilon, k\epsilon' | \left[\partial_\mu j^\mu_5(x) - \frac{e^2}{16\pi^2}\, \epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda}(x) F_{\nu\nu}(x) \right] |0\rangle = 0.
$$

Studying other vertex functions and other matrix elements of $\partial_\mu j^\mu_5$ and $\epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda} F_{\mu\nu}$, and also using other methods mentioned in chapter 19 of P & S, has led to the conclusion that the situation may be summarized by the *operator equation*, for non-zero fermion mass m,

$$
\partial_\mu(\bar\psi \gamma^\mu \gamma_5 \psi) = 2m\, \bar\psi i\gamma_5 \psi - \frac{e^2}{16\pi^2}\, \epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda} F_{\nu\nu}.
$$

Further remarks:

- The calculation described above brings to the fore the arbitrariness of diagrams that are divergent, and how this can eliminated by imposing physical conditions.

- With the transformation of variable $\ell \to -\ell$ and using $\gamma^\mu = -C\gamma^{\mu T} C^\dagger$, where C is the charge-conjugation matrix, the second line in (4) can be shown to be equal to the first line after integration. This is important when there are non-abelian generators involved and it leads to the symmetric combination for the anomaly coefficient $\mathcal{A}$ in (19.132) in P & S.

- With the spherical cutoff method, the coefficient c depends on the choice of loop momentum in the loop integral. The total $\Gamma^{\mu\nu\lambda}_5$ is unambiguous after imposing the QED WTIs.

- It is evidently advantageous to use a regularization that allows shifts of integration variables in loop integrals. For comments on using the convenient dimensional regularization in case of anomalies, see F. Jegerlehner, *Facts of life with* γ_5, hep-th/0005255.

- A regularization of non-abelian chiral gauge theories to all orders in perturbation theory does not (yet?) exist 'in the continuum'. A perturbative lattice definition has recently been achieved in M. Lüscher, *Lattice regularization of chiral gauge theories to all orders of perturbation theory*, JHEP 006 (2000) 028, hep-lat/0006014. A non-perturbative definition using the lattice regularization has long been sought; for reviews see M. Golterman, *Lattice chiral gauge theories*, Nucl. Phys. Porc. Supll. 94 (2001) 189, hep-lat/0011027, and M. Lüscher, *Chiral gauge theories revisited*, hep-th/0102028. See also the publications by H. Neuberger mentioned in e.g. H. Neuberger, *An introduction to lattice chiral fermions*, hep-lat/0301040, and M. Golterman and Y. Shamir, *SU(N) chiral gauge theories on the lattice*, Phys. Rev. D70 (2004) 09506. Work in chiral gauge theories is still in progress, which is usually posted in the hep-lat archive.

Some more on chiral symmetry in QCD

The r.h.s. of eq. (19.91) lacks a γ_5, it should read

$$\partial_\mu j^{\mu 5a} = \bar{Q} i\gamma_5 \{m, \tau^a\} Q. \tag{5}$$

The analogue of this equation for the vector current is

$$\partial_\mu j^{\mu a} = \bar{Q} i [m, \tau^a] Q. \tag{6}$$

The one-particle states $|\pi^\pm\rangle$ and $|\pi^0\rangle$ for the charged and neutral pi-mesons can be created out of the vacuum by application of the composite field operators $\bar{Q}i\gamma_5\tau^\pm Q$ and $\bar{Q}i\gamma_5\tau^3$, where $\tau^\pm = \tau^1 \pm i\tau^2$. Note that $\bar{Q}i\gamma_5\tau^- Q = \bar{Q}_d i\gamma_5 Q_u$, which contains a creation operator for a an anti-u-quark and a creation operator for a d-quark. $\bar{Q}i\gamma_5\tau^- Q|0\rangle$ is a state with charge -1, with the internal quantum numbers of the π^- particle, and in addition it has also multi-particle components with these quantum numbers. We have by arguments similar to those leading to the spectral representation (7.9) (Källén-Lehmann representation)

$$\int d^4x\, e^{ipx}\, \langle 0|\bar{Q}(x) i\gamma_5\tau^+ Q(x) \bar{Q}(0) i\gamma_5 \tau^- Q(0)|0\rangle = \frac{iZ_{\pi^-}}{p^2 - m_{\pi^-}^2} + \dots \tag{7}$$

where ... denotes the contribution of multiparticle states. Here Z_{π^-} is a constant with dimension mass^4, which may be called the wave function renormalization constant for the composite 'π^- field' $\bar{Q}i\gamma_5\tau^- Q$. For small quark masses m we have approximate chiral $SU(2) \times SU(2)$ symmetry and then it is convenient to work with $\bar{Q} i\gamma_5 \tau^a Q$, $a = 1,2,3$ and call the corresponding particles π^a.

A derivation of (19.93) and (19.94) goes as follows. We start from (19.92), or equivalently

$$-p^2 f_\pi \delta_{ab} = \langle \pi^b(p) | \bar{Q} i\gamma_5 \tau^a Q | 0 \rangle. \tag{8}$$

By isospin symmetry (i.e. $SU(2) \times SU(2)$ transformations with $U_L = U_R$, which are a symmetry for $m = 0$) we have

$$\langle \pi^b(p) | \bar{Q} i\gamma_5 \tau^c Q | 0 \rangle = \sqrt{Z_\pi}\, \delta_{ab} + \mathcal{O}(m). \tag{9}$$

Isospin invariance also tells us that

$$\langle \pi^b | \bar{Q} i\gamma_5 Q | 0 \rangle = 0 + \mathcal{O}(m), \tag{10}$$

because $\bar{Q} i\gamma_5 Q |0\rangle$ is an isoscalar and $|\pi^b\rangle$ is an isovector when $m = 0$. Let f and g be flavor labels ($= u, d$) and let us write,

$$\langle \pi^b | \bar{Q}_f i\gamma_5 Q_g | 0 \rangle = 2\sqrt{Z_\pi}\, (X^b)_{gf}. \tag{11}$$

The matrix X^b can be determined from (9) and (10). We expand X^b in terms of the complete set of matrices $\tau^0 \equiv \frac{1}{2}\, \mathbb{1}$ and τ^c,

$$X^b = \sum_{\gamma=0}^{3} C^b_\gamma \tau^\gamma, \quad C^b_\gamma = 2\mathrm{tr}\, (X^b \tau_\gamma). \tag{12}$$

Eqs. (9) and (10) imply

$$(X^b)_{gf} \tau^c_{fg} = \mathrm{tr}\, (X^b \tau^c) = \frac{1}{2}\, \delta_{bc}, \quad \mathrm{tr}\, (X^b) = 0, \tag{13}$$

and it follows that

$$C^b_0 = 0, \quad C^b_c = \delta_{bc}, \qquad X^b = \tau^b. \tag{14}$$

Using this knowledge we now find for the r.h.s. of (8):

$$\begin{aligned} \bar{Q} i\gamma_5 \{m, \tau^a\} Q &= 2\sqrt{Z_\pi}\, (\tau^b)_{gf} \{m, \tau_a\}_{fg} + \mathcal{O}(m^2) = 2\sqrt{Z_\pi}\, \mathrm{tr}\, (\tau^b \{m, \tau^a\}) + \mathcal{O}(m^2) \\ &= 2\sqrt{Z_\pi}\, \mathrm{tr}\, (m \{\tau^a, \tau^b\}) + \mathcal{O}(m^2) = \sqrt{Z_\pi}\, \delta_{ab}\, \mathrm{tr}\, (m) + \mathcal{O}(m^2) \\ &= \sqrt{Z_\pi}\, (m_u + m_d) + \mathcal{O}(m^2). \end{aligned} \tag{15}$$

It follows that

$$m_\pi^2 = B(m_u + m_d), \quad B = -\frac{\sqrt{Z_\pi}}{f_\pi} \tag{16}$$

We can generalize this derivation to the case of the three light flavors u, d and s, as follows. The squared-mass of the pseudo Nambu-Goldstone (NG) boson with ('valence-quark') quantum numbers of $\bar{f}$, g is approximately given by

$$m_{fg}^2 = B(m_f + m_g) + \mathcal{O}(m^2), \tag{17}$$

where $f \neq g$. The case $f = g$ is more complicated because the axial current with $f = g$ is *anomalous*:

$$\begin{aligned} \partial_\mu(\bar{Q}_f \gamma^\mu \gamma_5 Q_g) &= (m_f + m_g)\bar{Q}_f i\gamma_5 Q_g + \delta_{fg}\, 2q, & (18) \\ q &= -\frac{g^2}{64\pi^2}\, \epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda}^a F_{\mu\nu}^a & (19) \end{aligned}$$

(this is a version of (19.103) that includes case of non-zero quark masses). The candidate Nambu–Goldstone (NG) bosons and their masses are

$$\begin{aligned} \pi^\pm &: \; m_{\pi^+}^2 = m_{ud}^2 = 0.0195 \text{ GeV}^2 \\ K^\pm &: \; m_{K^+}^2 = m_{us}^2 = 0.244 \text{ GeV}^2 \\ K^0, \bar{K}^0 &: \; m_{K^0}^2 = m_{ds}^2 = 0.248 \text{ GeV}^2 \\ \pi^0 &: \; m_{\pi^0}^2 = 0.0182 \text{ GeV}^2 \\ \eta &: \; m_\eta^2 = 0.301 \text{ GeV}^2 \\ \eta' &: \; m_{\eta'}^2 = 0.917 \text{ GeV}^2 \end{aligned} \tag{20}$$

For the unequal-flavor particles ($f \neq g$) we have indicated the quark labels. For the neutral π^0, η and η' the quark assignment turns out to be less straightforward.

Consider two light flavors, $n_{\mathrm{f}} = 2$. The mass formula (17) with $f = u, d$ and $g = u, d$ predicts four NG bosons in this case. The obvious candidates are $\pi^\pm$, π^0 and η, with

$$m_{\pi^+}^2 = m_{ud}^2 = B(m_u + m_d). \tag{21}$$

According to (17), the other eigenstates are $\bar{u}u$ and $\bar{d}d$. If we try to assign $\pi^0 \leftrightarrow \bar{u}u$, $\eta \leftrightarrow \bar{d}d$, the relation

$$m_{ud}^2 = \frac{1}{2}(m_{uu}^2 + m_{dd}^2) \tag{22}$$

cannot be fulfilled at all. If we assume that $m_u \approx m_d$ and π^0 an equal mixture of $\bar{u}u$ and $\bar{d}d$ to get $m_{\pi^0}^2 \approx m_{\pi^+}^2$, the orthogonal combination of $\bar{u}u$ and $\bar{d}d$ should have approximately the same mass as π^0: the η does not fit in.

Consider next three light flavors, $n = 3$. The mass formulas now predict nine NG bosons. We find

$$\frac{m_u + m_d}{m_u + m_s} = \frac{m_{\pi^+}^2}{m_{K^+}^2} \equiv R_1, \quad \frac{m_u + m_s}{m_d + m_s} = \frac{m_{K^+}^2}{m_{K^0}^2} \equiv R_2, \tag{23}$$

and from this

$$\frac{m_s}{m_u} = \frac{R_2(R_1 - 1)}{1 - R_2 - R_1 R_2} = 31, \quad \frac{m_s}{m_d} = \frac{R_2}{1 - R_2 + m_u/m_s} = 20. \tag{24}$$

Hence $m_u : m_d : m_s \approx 1 : 1.5 : 30$. The effective action furthermore predicts particles with masses

$$\begin{aligned} m_{uu}^2 &= \frac{2m_u}{m_u + m_d} m_{\pi^+}^2 = 0.0155\ \mathrm{GeV}^2, && (25)\\ m_{dd}^2 &= \frac{2m_d}{m_u + m_d} m_{\pi^+}^2 = 0.0235\ \mathrm{GeV}^2, && (26)\\ m_{ss}^2 &= \frac{2m_s}{m_u + m_s} m_{K^+}^2 = 0.473\ \mathrm{GeV}^2. && (27) \end{aligned}$$

The candidates π^0, η and η' do not fit into the $n = 3$ formulas either. The effective action obtained so far must be wrong.

This is an aspect of the notorious $U(1)$ problem. The problem is the chiral $U(1)$ invariance contained in $U(n_\mathrm{f}) \times U(n_\mathrm{f})$. These are the transformations of the type $U_\mathrm{L} = U_\mathrm{R}^\dagger = \exp(i\omega)\,\mathbb{1}$, or more generally, transformations $U_\mathrm{L} = U_\mathrm{R}^\dagger$ with $\det U_\mathrm{L} \neq 1$. We know that this invariance of the classical QCD action is broken in the quantum theory by 'anomalies': QCD has only approximate $SU(n_\mathrm{f}) \times SU(n_\mathrm{f})$ chiral symmetry, plus the flavor $U(1)$ symmetry $U_\mathrm{L} = U_\mathrm{R} = \exp(i\omega)\,\mathbb{1}$ corresponding to quark number conservation.

For the neutral ($f = g$) pseudoscalars the following mass formula can be derived (see e.g. Smit, *op. cit.*)

$$m_{ff,gg}^2 = 2Bm_f\delta_{fg} + \lambda, \tag{28}$$

or

$$m^2 = 2B \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} + \lambda \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{29}$$

We shall treat the quark-mass term as a perturbation to the λ term. For $m_f = 0$ we have the eigenvectors and eigenvalues

$$\begin{aligned} \phi_0 &= \frac{1}{\sqrt{3}}(1,1,1), \quad m^2 = 3\lambda, && (30)\\ \phi_3 &= \frac{1}{\sqrt{2}}(1,-1,0), \quad m^2 = 0, && (31)\\ \phi_8 &= \frac{1}{\sqrt{6}}(1,1,-2), \quad m^2 = 0. && (32) \end{aligned}$$

Using $m_{u,d,s}$ as a perturbation (in the way familiar from quantum mechanics) leads to the following mass formulas

$$m_{\eta'}^2 = 3\lambda + B(\tfrac{2}{3}m_u + \tfrac{2}{3}m_d + \tfrac{2}{3}m_s), \tag{33}$$

$$
\begin{aligned}
m_{\pi^0}^2 &= B(m_u + m_d), && (34)\\
m_\eta^2 &= B(\frac{1}{3}m_u + \frac{1}{3}m_d + \tfrac{4}{3}m_s), && (35)
\end{aligned}
$$

which hold for the mass ratios (24) up to tiny corrections. The eigenvectors are also interesting, but here we merely mention that π^0 and η are mainly ϕ_3 and ϕ_8, whereas the η' is predominantly ϕ_0. From (33) we can determine the chiral $U(1)$ breaking strength λ,

$$
3\lambda = m_{\eta'}^2 - \frac{1}{2}(m_{\pi^0}^2 + m_\eta^2) = 3(0.252)\ \mathrm{GeV}^2. \tag{36}
$$

The mass terms in the effective action depend on four parameters, Bm_u, Bm_d, Bm_s and λ. Hence we have two predictions for the five pseudoscalar masses:

$$
\begin{aligned}
m_{\pi^0}^2 &= m_{\pi^+}^2, && (37)\\
m_\eta^2 &= \tfrac{1}{6}(m_{uu}^2 + m_{dd}^2) + \tfrac{2}{3}m_{ss}^2 = 0.322\ \mathrm{GeV}^2, && (38)
\end{aligned}
$$

which agree reasonably well with experiment. It should be kept in mind that electromagnetic corrections, which affect in particular the electrically charged particles, are neglected.

In the early days the near equality of m_{π^0} and m_{π^+} was interpreted as an aspect of approximate flavor symmetry, $m_u \approx m_d$. Now we know that m_d is substantially larger than m_u and that the approximate flavor symmetry is due to approximate chiral symmetry, $Bm_{u,d} \ll 3\lambda$, the spontaneous-symmetry breaking pattern $U(n_\mathrm{f})\times U(n_\mathrm{f}) \to U(n_\mathrm{f})_\mathrm{flavor}$, and the flavor-singlet character of the chiral-anomaly.

The Witten-Veneziano relation

The Noether argument tells us that to each continuous symmetry of the action corresponds a 'conserved current' j^μ, $\partial_\mu j^\mu = 0$, and a conserved 'charge' $Q = \int d^3x\, j^0(x)$, $\partial_0 Q = 0$. This is true in the classical theory but not necessarily in the quantum theory, which needs more specification than merely giving the action, such as the precise definition of the path integral. In case the quantum analog of j^μ is not conserved, one speaks of an *anomaly* $\mathcal{A} \equiv \partial_\mu j^\mu$. In four space-time dimensions $\mathcal{A}$ is typically $\propto \epsilon^{\kappa\lambda\mu\nu}\,\mathrm{Tr}\,(G_{\kappa\lambda}G_{\mu\nu})$, where $G_{\mu\nu}$ is a gauge-field tensor. Relations like $\partial_\mu j^\mu = \mathcal{A}$ can be found in perturbation theory by studying correlation functions of j^μ and $\mathcal{A}$ with other fields.

Chiral anomalies correspond to triangle diagrams, and related diagrams, in which one vertex corresponds to a (polar) vector current, $\bar\psi i\gamma^\mu\psi$, or an axial vector current, $\bar\psi i\gamma^\mu\gamma_5\psi$, and the other two vertices to gauge fields. There must be an odd number of γ_5's in the trace over the Dirac indices ($\mathrm{Tr}\,(\gamma_5\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu) = 4i\epsilon_{\kappa\lambda\mu\nu}$), hence the name 'chiral anomalies'. These γ_5 may come from the gauge field vertices or from the current.

In QCD there is no γ_5 associated with the gauge field vertices and only axial vector currents can have an anomaly. In the Euclidean formulation their divergence reads

$$\partial_\mu(\bar\psi_f i\gamma_\mu\gamma_5\psi_g) = (m_f+m_g)\bar\psi_f i\gamma_5\psi_g + \delta_{fg}\, 2iq, \tag{39}$$

$$q = \frac{g^2}{32\pi^2}\,\epsilon_{\kappa\lambda\mu\nu}\,\mathrm{Tr}\,(G_{\kappa\lambda}G_{\mu\nu}). \tag{40}$$

For zero quark masses the right hand side of (39) is the anomaly. The vector currents have no such anomaly. Their divergence reads

$$\partial_\mu(\bar\psi_f i\gamma_\mu\psi_g) = i(m_f-m_g)\bar\psi_f\psi_g, \tag{41}$$

which is zero in the symmetry limit $m_f = m_g$, hence also in the chiral limit $m_f = m_g = 0$. The right-hand sides of the divergence equations (39) and (41) are zero for the currents corresponding to $SU(n_\mathrm{f})\times SU(n_\mathrm{f})$ symmetry, obtained by contraction of $\bar\psi_f i\gamma^\mu P_{\mathrm{L,R}}\psi_g$ with the n_f^2-1 flavor $SU(n_\mathrm{f})$ generators $(\lambda_k)_{fg}/2$, $\mathrm{Tr}\,\lambda_k = 0$. Hence, the anomaly in (39) breaks only chiral $U(1)$ invariance corresponding to $\lambda_0 \propto 1\!\!1$ with $\partial_\mu \sum_f \bar\psi_f i\gamma_\mu\gamma_5\psi_f = 2n_\mathrm{f} iq$.

The quantity q is called the topological charge density. Continuum gauge fields on topologically non-trivial manifolds (such as the torus T^4 which corresponds to periodic boundary conditions) fall into so-called Chern classes characterized by an integer, the Pontryagin index or topological charge Q_top:

$$Q_\mathrm{top} = \int d^4x\, q(x). \tag{42}$$

An important example of configurations with topological charge is given by superpositions of (anti)instantons. The latter are solutions of the Euclidean field equations (hence they are saddle points in the path integral) with localized action density, non-perturbative action $S = 8\pi^2/g^2$ and topological charge ± 1. In this context we mention also the Atiyah–Singer index theorem:

$$Q_\mathrm{top} = n_+ - n_-, \tag{43}$$

where $n_\pm$ are the number of zero modes (eigenvectors with zero eigenvalue) of the Dirac operator $\gamma_\mu D_\mu$ with chirality $\gamma_5 = \pm 1$.

The significance of all this for our pseudoscalar particle mass spectrum is that the phenomenologically required chiral $U(1)$ breaking is present indeed in *quantum* chromodynamics, provided that gauge-field configurations with topological charge density give sufficiently important contributions to the path integral. The analysis of this is complicated [1] but fortunately there is a simple approximate formula which expresses the effect of the chiral anomaly on the neutral pseudoscalar masses, the Witten–Veneziano formula [2, 3]:

$$\lambda \approx \frac{1}{2f_\pi^2}\,\chi_{\rm top}, \qquad \text{no quarks.} \tag{44}$$

Here λ is the $U(1)$-breaking mass term introduced in (28) and $\chi_{\rm top}$ is the *topological susceptibility*,

$$\chi_{\rm top} = \int d^4x\, \langle q(x)q(0)\rangle. \tag{45}$$

Note that in (44) $\chi_{\rm top}$ is to be computed in the pure gauge theory without quarks, although it can of course also be evaluated in the full theory with dynamical fermions. From (36) we have $\chi_{\rm top} \approx (180\,{\rm MeV})^4$.

Pion-nucleon σ model

Consider an effective nucleon field N that is a doublet in terms of Dirac proton (p) and neutron (n) fields

$$N(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}. \tag{46}$$

The effective action of the pion–nucleon sigma model is given by

$$S_{\rm eff} = -\int d^4x\, [\bar N\gamma^\mu\partial_\mu N + G\bar N(\phi P_{\rm R} + \phi^\dagger P_{\rm L})N] + S_{O(4)}, \tag{47}$$

where $S_{O(4)}$ is the scalar field action of the $O(4)$ model (section 11.1), with an explicit symmetry-breaking term

$$\Delta S_{O(4)} = \int d^4x\, \epsilon\varphi^0 \tag{48}$$

added to give the NG bosons (the pions) a mass ($\propto \sqrt{\epsilon}$), and ϕ is a matrix field constructed out of the scalar fields,

$$\phi = \varphi^0 \mathbb{1} + i\sum_{k=1}^{3}\varphi^k\sigma_k. \tag{49}$$

The σ_k are the three Pauli matrices, which act on the p and n components of N and G is the pion–nucleon coupling constant.

The action is invariant under $SU(2)\times SU(2)$ transformations

$$N \to UN, \quad \bar N \to \bar N\bar V, \quad \phi \to U_{\rm L}\phi U_{\rm R}^\dagger, \quad U_{L,R} \in SU(2). \tag{50}$$

This follows from the fact that the transformation on the matrix scalar field ϕ is equivalent to an $SO(4)$ rotation on the φ^α. Hint: check that $\phi^\dagger\phi = \varphi^2\mathbb{1}$, $\det\phi = \varphi^2$; and hence that ϕ may be written as $\phi = \sqrt{\varphi^2}\,V$, $V \in SU(2)$.

This chiral invariance of the sigma-model action is a nice expression of the symmetry properties of the underlying quark–gluon theory. When the symmetry is spontaneously broken, such that the ground-state value of the scalar field is $\phi_{\rm g} = f\mathbb{1}$, $f = \varphi^0_{\rm g}$, the action acquires a mass term $Gf\bar{N}N$: the nucleon gets its mass from spontaneous breaking of chiral symmetry, $m_N = Gf$. This relation is in fair agreement with experiment. On introducing the weak interactions into the model one finds that f equals the pion decay constant, $f = f_\pi \approx 93$ MeV, while $G = g_{\pi NN} \approx 13$ is the pion-nucleon coupling constant that is known from pion-nucleon-scattering experiments, so with $m_N = 940$ MeV we have to compare $m_N/f \approx 10$ with 13.

The field φ^0 is often denoted by σ, and φ^k by π^k, the sigma and pion fields. The pions are stable within the strong interactions but the σ is a very unstable particle with mass m_σ in the range 600–1200 MeV. Given $m_\pi = 140$ MeV and $m_\sigma = 900$ MeV, the other parameters in the action can be determined.

In this model the Goldberger-Treiman relation (19.102) is simply the nucleon mass relation mentioned above, since $g_A = 1$:

$$g_A = \frac{f_\pi}{m_N}\, g_{\pi NN}, \tag{51}$$

satisfied to about 30%. As mentioned in P & S, the actual Goldberger-Treiman relation, which reflects the physics of QCD and does not make the simplications implied by the sigma model, is satisfied to the much higher accuracy of about 5%.

References

[1] G. 't Hooft, Phys. Rep. 142 (1986) 357.

[2] E. Witten, Nucl. Phys. B156 (1979) 269.

[3] G. Veneziano, Nucl. Phys. B159 (1979) 213.