Small sets in pluripotential theory

(Joint work with Armen Edigarian)

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Potential theory in $\mathbb{C}$

Harmonic function h on an open set in $\mathbb{C}$:

- Locally: real part of holomorphic function f;
- Solution of $\Delta h \ (= (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})h) \ = 0$;
- Mean Value Equality
  $h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + re^{it})\, dt.$

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Subharmonic function u: upper semicontinuous and

- $\Delta u \geq 0$ (as distribution)
- Mean Value Inequality
  $u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})\, dt.$

Properties

$\log|f| \in \mathrm{SH}$.

Potential of μ: $P(\mu) = \int \log|z-\zeta| d\mu(\zeta)$ in SH for $\mu > 0$ a reasonable measure.

Riesz: Every $u \in \mathrm{SH}$ is (locally) the sum of a harmonic function and a potential.

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POTENTIAL THEORY!

Potential leads to:

Energy of compactly supported μ:

$I(\mu) = \int\int \log|z-\zeta|d\mu(\zeta)d\mu(z)$.

Capacity of compact E:

$$\mathrm{Cap}\, E = \sup_{|\mu|=1, \mathrm{Supp}\,\mu \subset E} e^{I(\mu)}.$$

Small sets in potential theory

Sets of capacity 0 are the small sets.
$E \subset D$. Equivalent are

- $\mathrm{Cap}\,(E) = 0$;
- $\exists h \in \mathrm{SH}$ s.t. $h|_E = -\infty$, i.e. E is polar;
- Every bounded from harmonic function on $D \setminus E$ extends harmonically to D.

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E complete polar iff E polar and a G_δ (countable intersection of opens)

Dirichlet problem

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$\mathcal{F} = \{u \in \mathrm{SH}\,(D) : \ u < g$ on $\partial D\}$
Perron family for g. Perron solution:

$$\tilde{h}(z) = \sup_{u \in \mathcal{F}} u(z).$$

$\tilde{h}$ solves the Dirichlet problem, except that the bdry values may be incorrect at an exceptional $E \subset \partial D$.

- E is polar;
- ∂D is thin at points of E

S thin at ζ if $\zeta \notin \overline{S}$ or $\exists$ a nbhd U_ζ and $u \in \mathrm{SH}\,(U_\zeta)$

$$\limsup_{\substack{z \to \zeta \\ z \in S \setminus \zeta}} u(z) < u(\zeta).$$

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A closed in $\overline{D}$, $z \in D \setminus A$. Harmonic measure

$$\begin{aligned}\omega(z, A, D) &= -\sup\{u(z) : \ u \in \mathrm{SH}\,(D), \\ &\qquad u \leq 0, u \leq -1 \text{ “at” } A\}.\end{aligned}$$

pluripotential theory

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h plurisubharmonic ($h \in \mathrm{PSH}\,(D)$) if $h \in \mathrm{SH}\,(D)$ and $\forall L$ complex line passing through D, $h|_L \in \mathrm{SH}\,(L \cap D)$.
$f, f_n \in H(D)$, and $u, v, u_n \in \mathrm{PSH}\,(D)$, $c_n \downarrow 0$ fast.

- $\log|f| \in \mathrm{PSH}\,(D)$, $\sum c_n \log|f_n| \in \mathrm{PSH}\,(D)$;
- $u + v$ and $\max\{u, v\}$ are in $\mathrm{PSH}\,(D)$;
- $(\sup c_n u_n)^* \in \mathrm{PSH}\,(D)$;
- Invariant under holomorphic change of variables
- $\left(\frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}\right) > 0.$

small sets

$A \subset D \subset \mathbb{C}^n$ is called pluripolar in the domain D if $\exists h \in \mathrm{PSH}$ with $h|_A = -\infty$.

$A \subset D \subset \mathbb{C}^n$ is called complete pluripolar in D if $\exists h \in \mathrm{PSH}\,(D)$ s.t. $A = \{z \in D : \ h(z) = -\infty\}$.

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Thm.[Josefson, 1978] Suppose A is locally pluripolar in $\mathbb{C}^n$. Then $\exists h \in \text{PSH}\,(\mathbb{C}^n)$ s.t. $h|_A = -\infty$.

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How about complete pluripolarity?

Basic notions

In $\mathbb{C}^2$ global phenomena:
$A = \{|z| < 1, w = 0\}$, $B = \{w = 0\}$; then
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Thm. [Zeriahi, 1989] Let E be a pluripolar in a (pseudo)convex domain $D \subset \mathbb{C}^n$. If $E_D^* = E$ and E is G_δ and F_σ, then E is complete pluripolar in D.

pluri-thinness

Let $S \subset \mathbb{C}^n$, $\zeta \in \mathbb{C}^n$. Call S pluri-thin at ζ if $\zeta \notin \overline{S}$ or $\exists u \in \mathrm{PSH}\,(U_\zeta)$

$$\limsup_{\substack{z \to \zeta \\ z \in S \setminus \zeta}} u(z) < u(\zeta).$$

If A is pluripolar and $z \notin A_D^*$, then A is pluri-thin at z.

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Sadullaev's questions, (1981):

1. Is $E_\alpha = \{w = z^\alpha, z \neq 0\}$ pluri-thin at $(0,0)$?
($\alpha \in \mathbb{R} \setminus \mathbb{Q}$).

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Question: Let $\Omega \subset \mathbb{C}$. Suppose $f \in H(\Omega)$, $\Gamma_f \subset \mathbb{C}^2$ its graph. Is Γ_f complete pluripolar in $\mathbb{C}^2$ (or in a domain containing $\Omega \times \mathbb{C}$)?

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No, if f has an analytic continuation.

Levenberg-Martin-Poletsky, (1992):

Let $f \in H(\Omega)$ on its domain of holomorphy $\Omega \subset \mathbb{C}$.

Is the graph Γ_f complete pluripolar in $\mathbb{C}^2$?

Answers

Sadullaev's question

1. Yes; $(E_\alpha)^*_{\mathbb{C}^2} = E_\alpha$. (Levenberg-Poletsky, 1999)
2. Yes; $A^*_D = A$, A is complete pluripolar in $\mathbb{C}^2$. (W., 2000)

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2. Yes; $A^*_D = A$, A is complete pluripolar in $\mathbb{C}^2$. (W., 2000)

L-M-P-question: Support from results on lacunary series (Sadullaev, L-M-P), (2) and

Thm. [W., 2000] Let D be a domain in $\mathbb{C}$ and let $A = \{a_1, a_2, \ldots\} \subset D$ have no limit points in D. If $f \in H(D \setminus A)$ has singularities in a_n that cannot be removed, then Γ_f is complete pluripolar in $D \times \mathbb{C}$.

The answer to L-M-P is NO

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Thm. 3 *Let* $D_1 \subset D_2$ *be domains in* $\mathbb{C}$, *such that* $D_2 \setminus D_1$ *has a limit point in* D_2.
Then $\exists f \in H(D_1)$ *with domain of existence* D_1 *such that* Γ_f *is not complete pluripolar in* $D_2 \times \mathbb{C}$.

The answer to L-M-P is NO

Thm. 5 *Let* $D_1 \subset D_2$ *be domains in* $\mathbb{C}$*, such that* $D_2 \setminus D_1$ *has a limit point in* D_2*.*
Then $\exists f \in H(D_1)$ *with domain of existence* D_1 *such that* Γ_f *is not complete pluripolar in* $D_2 \times \mathbb{C}$*.*

Thm. 6 $\exists \{a_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ *and* $\{c_n\}_{n=1}^{\infty}$ *s.t.*

$$f(z) = \sum_{j=1}^{\infty} \frac{c_j}{z - a_j} \quad \in H(\mathbb{D}). \tag{3}$$

is C^{∞} *on* $\overline{\mathbb{D}}$*, is nowhere extendible over* $\partial\mathbb{D}$*, while* Γ_f *is not complete pluripolar in* $\mathbb{C}^2$*.*

Thm. [Siciak] $\{c_n\}$ in Thm. 6 can be chosen s.t. $f \in H(\mathbb{C} \setminus \overline{\{a_j\}})$. Moreover

$$(\Gamma_{f|_D})^*_{\mathbb{C}^2} \supset \Gamma_{f|_{\mathbb{C}\setminus\{a_j\}}}.$$

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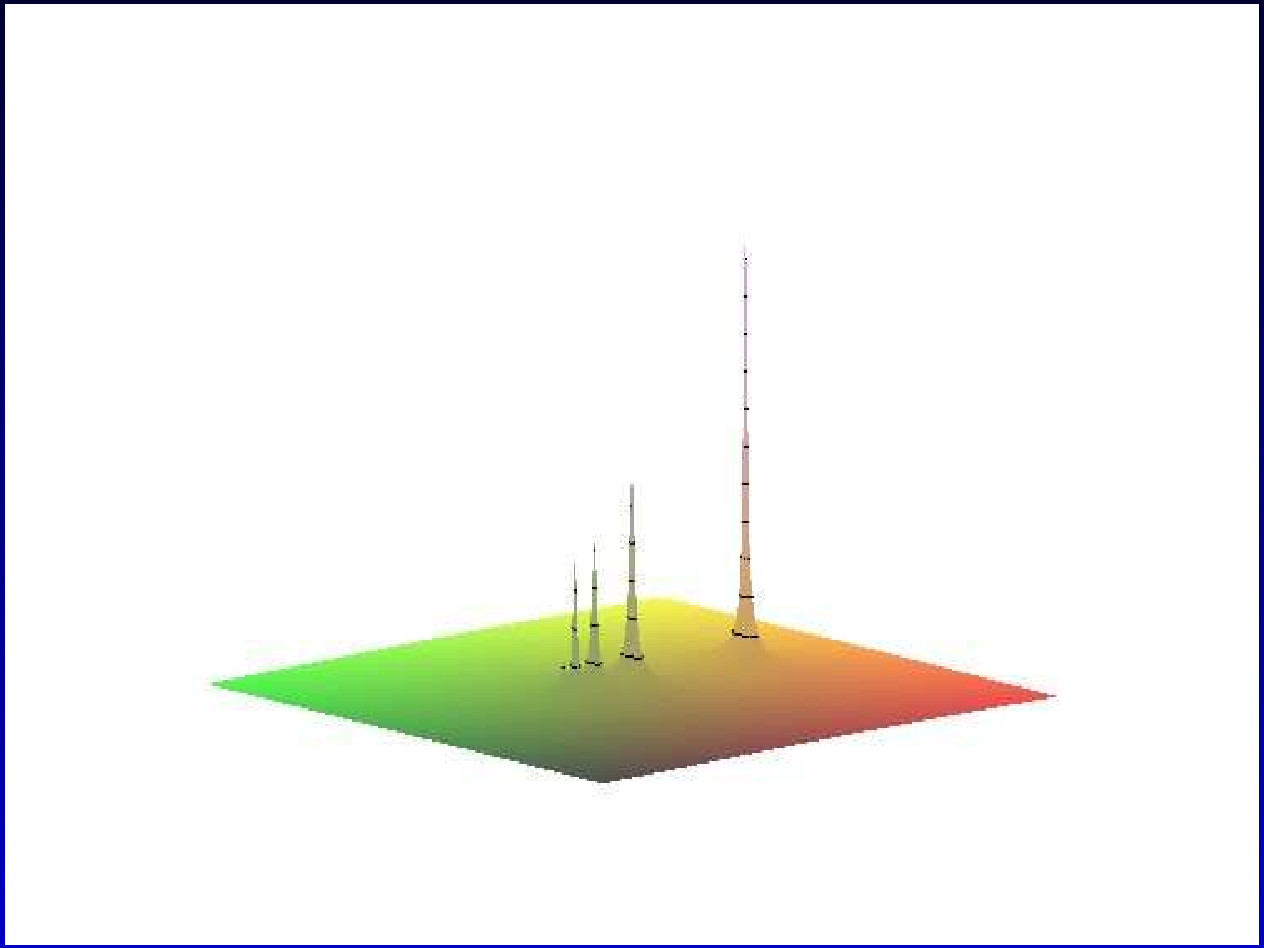
Zwonek expanded on the E-W and Siciak examples showing that there exist $f \in H^\infty(\mathbb{D})$, not extendible, but Γ_f^* contains any finite number of points over certain boundary points of $\mathbb{D}$.



Main results

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D: domain in $\mathbb{C}$, A: closed polar in D;
A': limit points of A in D.

Thm. 8 *If $f \in H(D \setminus A)$, then $\Gamma_f \cup (A \times \mathbb{C})$ is complete pluripolar in $D \times \mathbb{C}$.*
If f does not extend holomorphically over A, then $\Gamma_f \cup (A' \times \mathbb{C})$ is complete pluripolar in $D \times \mathbb{C}$.

Main results ctd

Thm. 9 *Let* $f \in H(D \setminus A)$ *not extendible over* A *and* $z_0 \in A$. *TFAE:*

1. $(\{z_0\} \times \mathbb{C}) \cap (\Gamma_f)^*_{D\times\mathbb{C}} = \varnothing$;
2. $D_{\geq R} = \{z \in D \setminus A : |f(z)| \geq R\}$ *is not thin at* z_0 *for any* $R > 0$.

If $D_{\geq R}$ *is thin at* z_0 *for some* $R > 0$, *then* $\exists w_0 \in \mathbb{C}$, *s.t.* $(\{z_0\} \times \mathbb{C}) \cap (\Gamma_f)^*_{D\times\mathbb{C}} = (z_0, w_0)$.

Main results ctd

Any graph is of G_δ- and F_σ-type, hence by Zeriahi's theorem:

Cor. 10 *Let D a domain in $\mathbb{C}$ and $A \subset D$ closed polar. Suppose $f \in H(D \setminus A)$ not extendible over A. Then Γ_f is complete pluripolar in $D \times \mathbb{C}$ iff $\forall R > 0$ the set $D_{\geq R}$ is not thin at any point of A.*

The role of w_0

Fix a disc $\mathbb{D}_\varepsilon$ in $D_{\leq R}$. If $D_{\geq R}$ is thin at z_0, then $\exists\{z_k\} \subset D_{\leq R}$ s.t. $z_k \to z_0$ and harmonic measure satisfies

$$\omega(z_n, \mathbb{D}_\varepsilon, D_{\leq R}) \geq c > 0 \quad (4)$$

Lemma The limit points w of $\{f(z_k)\}$ with z_k satisfying (4) give rise to points $(z_0, w) \in \Gamma^*$.

But

Lemma

$$\lim_{z_k \to z_0} f(z_k) = w_0$$

exists independent of the $\{z_k\}$ as long as (4) is fulfilled.

H^∞

$\mathcal{M}$: the maximal ideal space of $H^\infty(\Omega)$;
$\mathcal{M}_\lambda$: the fiber over $\lambda \in \overline{\Omega}$;
$\mathcal{M}(G) \subset \mathcal{M}$: the homomorphisms $f \mapsto f(z)$, $z \in G \subset \Omega$.

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Zalcman (1969) and Gamelin and Garnett (1970) studied distinguished homomorphisms in $H^\infty(\Omega)$.

A distinguished homomorphism: $\mu \in \mathcal{M}_\lambda$ with $\lambda \in \partial\Omega$ that is in the same Gleason part as (a component of) $\mathcal{M}(\Omega)$.

continued

There can at most be one distinguished homomorphism in $\mathcal{M}_\lambda$.

The estimate on the harmonic measure shows that the point evaluations at z_k have pseudohyperbolic distance $\leq C < 1$ to $\mathcal{M}(\mathbb{D}_\varepsilon)$, so any limit point of these must be in the same Gleason part as Ω and the limit exists and is the distinguished homomorphism.

Proof of Thm 8: Approximation.

Thm.[E-W] Assume $f \in \mathbb{C} \setminus K$, $f(\infty) = 0$, K compact polar. $\exists$ rational functions $r_n(z) = \frac{P_n(z)}{Q_n(z)}$ of degree $\leq n$ with singularities in K such that for any compact $L \subset \mathbb{C} \setminus K$

$$\|f - r_n\|_L^{1/n} \to 0, \quad \text{if } n \to \infty.$$

Here $\| \cdot \|_L$ means sup-norm in L.

Make $h_n \in$ PSH

$$h_n := \frac{1}{n} \log |(w - r_n(z))Q^n|.$$

adapt h_n

For large $n(m)$, $(m = 1, 2, \ldots)$, with

$$u_m := \max\{h_{n(m)} - \log(m+2), -m - \log(m+2)\}$$

$$\begin{aligned} u_m &< 0 \ \text{ on } |z| < m, \\ u_m(z, f(z)) &< -m \ \text{ on } 1/m < |z| < m, \\ u_m(z, w) &> -C \log m \end{aligned}$$

on $1/m < |z| < m$, $|w - f(z)| > 1/m$. Hence

$$u = \sum \frac{1}{m^2} u_m \in \text{PSH}\,(\mathbb{C}^2).$$

$$\Gamma_f \subset \{u = -\infty\} \subset \Gamma_f \cup (A \times \mathbb{C}).$$

Proof of Thm. 9

1. Use Thm. 8; $\exists h \in \text{PSH}\,(D \times \mathbb{C})$ s.t. $h = -\infty$ precisely on $\Gamma_f \cup A \times \mathbb{C}$.
2. A (new?) result in classical potential theory:

Thm. D a bdd domain in $\mathbb{C}$ and $S \subset D$ a closed disc. If $K \subset \partial D$ is compact polar, then for $z \in D$

$$\omega(z, S, D) = \inf\{\omega(z, S, D \cup U) : K \subset U \text{ open}\}.$$

If $z \in K$ is non-thin w.r.t. ∂D, then

$$\inf\{\omega(z, S, D \cup U) : K \subset U \text{ open}\} = 0.$$

Proof of Thm. 9

Fix Δ a closed disc in $D_{<R} = \{z \in D \setminus A : \ |(f(z)| < R\}$. $\forall \varepsilon > 0$ and $\forall$ non-thin $p \in A$ w.r.t. $\partial D_{<R}$ $\exists$ a nbhd U of A and $h_p \in \mathrm{SH}\,(D_{<R} \cup U)$ with $h_p < 0$, $h_p(p) = -\varepsilon$, $h|_{\Delta} = -1$.

Proof of Thm. 9.

3. View h_p as a PSH-function on a nbhd of $u = -\infty$ in $D^2_{<R-1}$. This admits extending h to a PSH-function $\tilde{h} < 0$ on $D_{<R-1} \times \mathbb{D}_{m-1}$ with $\tilde{h}|_{\Gamma_{f|_\Delta}} = -1$, $\tilde{h}(p, w) = -\varepsilon$.

4. Weighted sum of such $\tilde{h}$'s gives PSH-function v that has

5. Conclude: $v|_{\Gamma_f} = -\infty$ and $(\Gamma_f)^*_{D\times\mathbb{C}} = \Gamma_f$ is complete pluripolar.

An essential example

$a_n = \frac{1}{n}$, $A = \{a_n : n \in \mathbb{N}\}$ $D = \mathbb{C} \setminus \overline{A}$.

Let $c_n = e^{-n^2}/n^2$, $r_n = e^{-n^2}$

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - 1/n} \quad \in H(\mathbb{C} \setminus \overline{A})$$

is well-defined on $\mathbb{C} \setminus A$.

An essential example

$$f_N = \sum_{n=1}^{N} \frac{c_n}{z - 1/n} + \sum_{n=N+1}^{\infty} \frac{c_n}{-1/n}.$$

$$|f_N| < 10 \quad \text{on } \mathbb{C} \setminus \left(\bigcup_{1}^{N} B(1/n, r_n) \right);$$

$$|f| < 10 \quad \text{on } \mathbb{C} \setminus \left(\bigcup_{1}^{\infty} B(1/n, r_n) \right).$$

An essential example

With $\gamma = C(0, 2/3)$, $E_N = \mathbb{D}_{2/3} \setminus \left(\bigcup_1^N B(1/n, r_n) \right)$,

$$\omega(0, \gamma, E_N) \geq c > 0 \quad \text{independent of N(!)}$$

$|f| < 10$ is not a Dirichlet domain.

An essential example

Let $h \in \text{PSH}\,(\mathbb{D} \times \mathbb{C})$ have $h|_{\Gamma_f} = -\infty$. Let $M = \sup_{\mathbb{D}_{2/3} \times \{|w|<11\}} h(z,w)$.

Now $h(z, f_N(z)) \in \text{SH}$ on a nbhd of $\bar{E}_N$ and

$$M_N = \max_{z \in \gamma} h(z, f_N(z)) \downarrow -\infty,$$

(h USC , $f_N \to f$ unif. on γ).
By the two constants theorem:

$$\begin{aligned} h(0, f(0)) &= h(0, f_N(0)) \leq M_N \omega(0, \gamma, E_N) \\ &\qquad + M(1 - \omega(0, \gamma, E_N)). \end{aligned}$$

Conclude: $h(0, f(0)) = -\infty$; $(0, f(0)) \in (\Gamma_f)^*_{\mathbb{C}^2}$.