Inquisitive Semantics and Dialogue Management

Jeroen Groenendijk

ILLC/Department of Philosophy
University of Amsterdam
j.a.g.groenendijk@uva.nl
http://home.medewerker.uva.nl/j.a.g.groenendijk

Document for the Course on:
Alternative Logical Semantics
ESSLLI 2008, Hamburg, August 11–15

Preliminaries This is an *unfinished* document, that does not cover the material presented in the course. Important parts are still missing, but there will also be parts of this text that I will not address in the course at all, or only superficially. The text is very abstract, and lacks the necessary illustrations and natural language examples that I will present in the course.

Overview

The topic of this document is the use of inquisitive semantics in modelling the proceedings of dialogue. Certainly the way it is presented in these notes, the model is of a purely logical nature, as will be clear from the fact that it deals with dialogues in the language of propositional logic. (The extension to predicate logic is not included in this text.) Of course, I do believe that the logical model has an empirical bite, and has something to say about important features of dialogue in natural language, but there is little in the text that tries to argue for this.

In section 1 I give a global informal picture of our logical approach to dialogue, focussing on the central features characteristic for the approach. This should be followed by a section that is lacking, where I give a similar global introduction into the nature of the logical language under an inquisitive semantics. Reading (the first couple of sections of) another text I made available, *Inquisitive Semantics: Two Possibilities for Disjunction* may help a bit to fill this gap. I will present such an introduction into the inquisitive semantics for a propositional language, and its links to natural language, in the beginning of the course.

For better or worse, in Section 2, I start to unfold the model rather extensively on the most abstract level of information frames, where a language and its semantic interpretation are only there on the background. I bring them up in the central section 3, where I give the interpretation of the propositional language in inquisitive models based on the frames, and extensively discuss the semantics.

In section 4, I introduce the logical notion of compliance that characterizes the logical relatedness of subsequent moves in an inquisitive dialogue in the logical language. Here, there is again an important section still lacking. The notion of compliance can be looked upon as as a logical pragmatic notion corresponding to the Gricean Cooperation Principle, and hence also gives rise to implicatures, where the inquisitiveness of the semantics leads to a new inquisitive look on implicatures.

In section 5 I present the logical tools to manage the dialogue, to model the proceedings of a dialogue with the aid of stacks. Here, too, illustrations are largely lacking in the present text, and also the pragmatic part of the dialogue management is not included.

Up to this point we will have worked within a specific way, a rather general way, of formulating the inquisitive semantics. The main purpose of the (last) section 6 is to show that there are alternative ways of doing so. For a start, this brings us back to the so-called pair-interpretation as I use it in the paper *Inquisitive Semantics: Two Possibilities for Disjunction*, that I also made available.

It also brings us to a so-called update version of the semantics. These alternative views of the semantics all have their pros and cons, and can serve to highlight different logical aspects. The practical purpose of this section is, that several parts that are still lacking in this document, have already been dealt with in the update semantics. Once the link has been made between the semantics as it is presented here and the update semantics, we can fill the gaps in the present document.

1 The Larger Picture

What we want to get at is picture of the dynamic process of information exchange by means of raising and resolving issues, using language. Information and issues will be about something, and the nature of the language will determine what it can be about, but let us leave that unspecified for the moment. Also, when we talk about information exchange, there should be at least two agents around that are involved in a conversation, but let's leave them still behind the curtains as well. So, we are not talking about modelling information and issues of agents, we are rather thinking of information and issues as abstract entities that are brought about by the dynamics of the moves in the conversation as such.

1.1 Common Ground

I will be concerned with a model of the common ground which focusses on the role of issues in establishing a common ground. There are different uses of the notion of the common ground, different ways of looking at it.[1] The specific view

[1] In their Introduction to *Formal Semantics: The Essential Readings*, Blackwell, Oxford (2002), the editors Portner & Partee attribute the introduction of the notion 'common ground' as "the set of possible worlds compatible with what speaker and hearer can be presumed to take for granted at a given point in the conversation", to Stalnaker, See Stalnaker (1972, 1978).

I take on the common ground has been called an external view.[2] On the external view the common ground is a public entity. It is created by the discourse, by the moves of the participants in the discourse. David Lewis (1979) used the metaphor of the conversational scoreboard. Everyone, players and audience, can look at the scoreboard to see the result of what has happened sofar, what the current stage of the game is.

As with any metaphor, there is also something wrong with the scoreboard picture, because if you enter the match at halftime you can have a look at the scoreboard and see how things stand. That is not so for the common ground. You have to be there and pay attention all the time to be able to grasp the common ground.

I'm not going to use that framework, but Hans Kamp's discourse representation structures (Kamp & Reyle (1993)) and Irene Heim's file cards (Heim (1982)) are ways to deal with the common ground which are in line with my view, as long as you do not look upon them as individual mental representations, but as abstract public entities that can be shared by many, as is supposed to be the case for Fregean meanings, for senses.

From the viewpoint of discourse representation theory and its likes – the type of framework has been used widely in the logical analysis of discourse[3] – the active or passive discourse participants can shake hands and go their own way after a conversation and carry the discourse representation, or their own version of it, home with them, so to speak. That is not so for the common ground from my external viewpoint. Like Lewis' scoreboard is wiped out after the game is over – everything is set to nil, ready for the next game to start – my common ground just ceases to exist when the conversation is closed, and a brand new one is initiated as another conversation begins. (Here is a difference with Fregean senses, which live longer than anyone of us, if not eternally.)

But isn't this an idle picture of the common ground? Isn't the whole idea of information exchange in cooperative dialogue that after a successful conversation we all have gained something, that something has been learned, that by exchanging information our own information, and the information we share, has grown a bit or two?[4]

Yes, but that is the internal view of the common ground, where you, so to speak, look inside the heads of the participants in a conversation, compare their information states, before and after, and maybe even during the conversation, and you just see the miracle happen of a growing common ground. Actually,

[2] I borrow the distinction between the external and the internal view of the common ground from Gerbrandy (1999), who ascribes the distinction to Heidrach & Rieser (1998).

[3] A prime example is Segmented Discourse Representation Theory, see e.g. Asher & Lascarides (2003).

[4] Here is another difference with the game metaphor. A scoreboard usually tells you who has won and who has lost, or whether it was a draw. In the game of information exchange you can only win. Or, as Johan van Benthem says on a poster of our university: "Information is the only thing that grows without a cost". I don't know whether it is completely true, but it surely sounds good.

under this internal view, the common ground may even grow when the information exchange was fully unsuccessful. At least your information about the information of the others has grown, even when noone learned anything about the external world.

There is nothing wrong with this internal point of view of what happens during a conversation in the private hearts and minds of the participants – which is what dynamic epistemic logic is concerned with. But, even though the logical tools can be pretty much alike, you should keep the internal picture strictly separated from the external one, where the common ground is publicly monitoring how the dialogue proceeds and what it brings about.

The importance of keeping the two views apart, has been shown by Jelle Gerbrandy in his dissertation, where he proves that the two views can only happily live together if we take a rather poor view of the contents of the common ground.[5] Roughly speaking, the contents of the common ground should be restricted to information (and issues I would add) which concern the subject matter of the conversation, and not matters at a meta-level, such as what the participants get to know from the conversation about the information of the other participants, etc. And secondly, the common ground should be safeguarded against having to make repairs on it while the conversation is still on the air.

Jelle Gerbrandy brings this as a rather sad and disappointing story.[6] Well, perhaps it is from the viewpoint of dynamic epistemic logic, because higher order information and information revision are the currents in the epistemic logical porridge. But for the study of dialogue management, as it may be called, it gives important guidelines. If you want to arrive at an appropriate notion of an external common ground, that you can rely upon to correspond to the right effects on the internal common ground, if the external and internal view are to be two sides of the same coin, resist the temptation of building higher order information in it, and beware of repair, otherwise the coin will not roll very far.

1.2 Information States and Stages

The two core notions of my formal model of the common ground are those of an information state and an information stage, where an information stage is a proposal for a transition from the current state of information to a more informed state. Such a proposed transition is mediated by a proposition, where propositions are what sentences of a language express.

---

[5] See Gerbrandy (1999), where this is argued in Chapter 6: *Changing the Common Ground.*

[6] He says on the first page of the Chapter I referred to in the previous footnote: "The main result is that even in simple cases [...], the 'external' viewpoint cannot be reduced to the 'internal' one, nor vise versa. I will try to argue, and, where possible, make precise formally, that under certain minimal assumptions on information change and the way the common ground is represented, the two approaches are *in*compatible" (my emphasis). That is looking at it from the dark side. The bright side is that he also shows under what conditions the two views *are* compatible.

Propositions can be informative and inquisitive, causing the stages they give rise to to be informative and inquisitive as well. Propositions and stages can be informative and inquisitive at the same time.

To the extent that the current stage is informative it directly proposes a transition from the current state of information to a more informed state. Of course, when a participant proposes such a transition, she does so because her own information state supports the proposed transition. It can become common information, information absorbed in the common ground, if the other participant is able and willing to accept the proposed transition, in which case the proposed transition can be effectuated in the common ground.

If the other participant is not able or willing to accept a proposed transition it is essential that she signals this, and calls for cancellation of the proposed transition, in which case the proposed transition, the current stage, is to be removed from the table. This is essential in order to *maintain a common ground.* If the transition to the proposed more informed state would be effectuated in the common ground, without this being mirrored by the states of both participants, the common ground is lost.

To model cancellation of a proposed transition from the common grond, bewaring of repair, we will construct the common ground as a stack of stages.[7] The utterance of a proposition will give rise to a push of the stack, putting a new subsequent stage on top (actually, as we shall see, always two stages). If the new top of the stack is an informative stage, this calls for a reaction of cancellation or acceptance. Upon a call for cancellation of a proposed transition, removing the current stage from the table can then be effectuated by a pop of the common ground stack.

The more happy way to proceed, and hopefully that is the default, is to signal acceptance. That might be done implicitly by just coming up with a happy continuation of the dialogue. Acceptance will also have an effect on the stack, it will make it a bit more compact absorbing the information provided by the proposition a bit further down the stack. Of course, to really maintain a common ground, just signalling acceptance is not going to work, you actually have to accept the proposed information, and follow the absorption that is the effect of acceptance in your state.

We will get to the details later. But this is the general way things proceed. A new utterance leads to an expansion of the common ground stack, hypothetically updating it, which we will call the *uptake* of a proposition. When the reaction of acceptance or cancellation is absorbed, the stack wil become more compact again, leading to a real update effect, if only hypothetically.

There is still a third type of reaction (maybe there are even more to be distinguished) that is to signal *support* of a proposed informative transition. A participant may signal support if she could have proposed the transition herself, if the information provided by the proposition expressed by the sentence uttered by the other participant is not new to her, meaning that, although the participants

[7] See Kaufmann (2000) for an introduction to the notion of a stack, its logical modelling, and its use in dynamic semantics.

may not have been aware of that, they already had the information in common. (From the internal perspective, it already belonged to their common ground.)

In this case, we can percolate the effect of the proposed transition all the way down the stack until we reach the bottom of the stack. That is where the real common ground is sitting, the information of which it is fully established in the discourse that the participants have it in common. Any stages in between the bottom and the top are – roughly speaking – subsequent additions of hypothetically accepted pieces of information, that can become more fixed as the dialogue proceeds, or may turn out to be untenable as being shared.

1.3 Current Issue

Propositions and stages can not only be informative, but also inquisitive. Inquisitive stages also are proposals for a transition of the current state of information to a more informed state, but in a more indirect, auxiliary way. An inquisitive proposition will offer two or more alternative possibilities for a more informed state of the common ground. It leads to a push of the common ground stack which puts an inquisitive stage on top, the current issue. Such a stage created by the utterance of an inquisitive sentence by one of the participants invites the other participant for a follow-up with the utterance of a sentence which expresses an informative proposition that makes a choice between the proposed alternative transitions.

The main function of inquisitive propositions is to steer the process of information exchange in certain ways, to bring some structure in the exchange. This role of issues, or questions if you want, is widely acknowledged. Dialogue models usually have a component that is intended to monitor what is often called the question, or questions, under discussion. It is often a set or a stack.[8]

In our case, the question under discussion, the current issue, is not a separate component, it will just correspond to a specific type of stages that can be found in the common ground stack. Actually, we will set things up in such a way that after a transition has been proposed, and the reaction to the proposal has been absorbed, the stage on top of the common ground stack is an inquisitive stage, embodying the current issue, even if at no point in a dialogue a question has been asked.

Now you may ask, what about the very start of a dialogue, what is the current issue then? In this I follow Craige Roberts.[9] A fixed logical starting point for any dialogue is The Big Question, as she calls it, the question what the world is like. I will assume that the initial stage of the common ground is The Big Question relative to a state of information which initially is the state of complete ignorance.

And if this is how things initially are, then there is an issue to begin with, and also, whatever the contents of the initiative in the dialogue are, information

[8] See for example Asher & Lascardides (1998), and Roberts (1996).

[9] See Roberts (1996). It is not only at this point of The Big Question that my approach to modelling dialogue is akin to that of Roberts, there are many other global features that our approaches share.

it provides cannot fail to be a partial answer to The Big Question, and an issue it raises cannot fail to be a subissue of this same Big Question. So, we can always make a swinging start as long as there is any information we feel like sharing, or any particular issue that we may want to address.

The way things start points the way for any move to follow: give a partial answer to the current issue, or replace the current issue by a subissue. This, too, is widely acknowledged as the twofold steering mechanism behind any orderly dialogue. This being so, or at least taking things to be like this, sets a nice logical task: develop notions that can measure whether, and to what extent, an utterance in the dialogue follows these directions.

1.4 Compliance

We will define a logical notion of *compliance*, which judges whether of two subsequent stages in the common ground stack the one that comes after another is related to it, whether it addresses the current issue. We will also define a notion of comparative compliance which makes a quantitative comparison between possible compliant subsequent stages. The details will follow later, but let me point out one or two general things about it.

The first is that, as I hope the name 'compliance' suggests, to be compliant to the common ground, is not a law of nature, it is a regulative principle, it is a rational thing to take it to heart in the general interest of sharing information, but it can just as well occur that it can't be, or just isn't, followed by a participant in the discourse. However, if we detect this – and we can, because we can determine this from the common ground stack, which is a public thing – we will try to figure out a reason for flouting compliance. One simple reason may be that being polite overrules being compliant.[10]

And in using the word 'flouting' here, I want to indicate that being compliant to the current state of the common ground is pretty much like the Gricean Cooperation Principle (Grice (1989)). In fact, I think it is that, be it only that compliance is a fully formalized logical notion, which is helped significantly by the fact that we have a combined logical model of issues and information. This will give rise to a more precise and a more refined logical-pragmatic notion of a conversational implicature. Promises. . .

1.5 Thematizing

I said that politeness may overrule compliance, that has to do with another feature concerning issues that is central in our approach to dialogue, and arguably is demanded by maintaining a common ground, or by enhancing the

[10] In Groenendijk (1999), I defined a similar notion which I called 'licensing'. I switched to the term 'compliance', because it has a negative ring to it, and thus communicates more clearly that being non-compliant can easily be a virtue rather than a vice. I will not pay much attention here to the notion of politeness. But I agree with Leech (1983) that adding a Politeness principle to the Gricean pragmatic Cooperation Principle is important in dealing with pragmatic penomena.

common ground. This feature is – and that is probably not suggested by the term 'compliance' – that critical dialogue moves, like contradicting, or denying, or doubting what another participant in the dialogue has just proposed, can be a most compliant thing to do. But, of course, whether you are in the position to do so may depend on your social relation to the other participant, and that is where politeness enters the scene.

How does this work? Well, consider again the situation of an initiative, an opening of a conversation, and suppose the proposition expressed by this sentence is a plain assertion φ. As it happens, you, the other participant, believe that *not*-φ. So, in order to maintain a common ground, you have to publicly announce cancellation. So, you do, you say: "No." That will suffice for cancellation, but then, what is the most expected move for you to make now? Well, to continue your cancellation with "*not*-φ." Or perhaps, more politely, but with largely the same effect, to express doubt or surprise: "No! Really? φ?".

Now, if the uptake of φ in the initial common ground stack would only consist of pushing the stack with a copy of the initial stage updated with φ, and absorbing your cancellation would just be to pop the stack, thus returning to the initial Big Question on top as the now current issue, there is no way to explain this. Because relative to this, just anything goes. But that is not how it works. You may perhaps walk out of the dialogue situation altogether, but if you decide to stay, you are expected to do the sort of thing that I just sketched, respond with an utterance directly related to what was said before. That is what counts as compliant now.

Then we have to adapt our way to construct the common ground stack to this. The way we will do this is to let the uptake of a sentence consist of two subsequent steps, twice pushing the current stack, leading to two additional subsequent stages. The last of these two will just consist of a subsequent stage of the current stage which embodies the effect of the proposition expressed by the sentence that was uttered as such.

But in between the old current stage and this new stage on top we put an intermediate subsequent stage which results from an operation called *thematizing*, which adds the 'question behind' the proposition expressed by the sentence that has been uttered to the current issue to be found in the old current stage.

In the situation just sketched, where we are dealing with the initiative of a dialogue which is a plain assertion, the question behind it is the corresponding yes/no-question. The current issue is The Big Question. The way in which thematizing works in this simple case is that the current issue is replaced by the yes/no-question behind the initial assertion.

Actually, this is how things will always work if the question behind the proposition expressed by the last utterance is a subissue of the current issue. The general nature of thematizing is to take the *disjunction* of the current issue with the question behind the proposition expressed by the last sentence uttered.

Now, what we were looking at, was the situation where an initial assertion was not accepted, and cancellation was signalled. After cancellation has popped the stack, the current issue is the yes/no-question behind the assertion. And typically the negation of that assertion, and perhaps a bit less so repeating the now current issue, are moves that are in accordance with the general line: provide an answer to the current issue, or replace it by a subissue. Of course, the easy way out of this predicament, is to agree to disagree, and to let the case rest, joinedly pop the common ground stack a couple of times, by which we return to The Big Question, and can start talking about another topic, hopefully, with more success in terms of information exchange.

2 Inquisitive Information Frames

I have given a general picture of the function of the common ground in analyzing dialogue. We have seen the boring part, to beware of repair we need a stack of subsequent information stages, stages are transitions and should be able to contain information and issues, stages can be inquisitive.

We now turn to model the states and stages as such, to give a general logical picture of them. We will not be talking about language yet, but just about the structure of information and issues as such. Once we add a language to the picture, we get a view of how these structures are dynamically constructed. In the end, of course, it is not just language that is the source of information, but it is an important source, and in studying information exchange in dialogue, it is the only source we really take into consideration. (I think it does no harm to keep this in mind.)

Some of the global features of the logical model I use are not new. They are also part of other approaches to picture information and information growth, and are used, e.g., to model and study intuitionistic logic.[11] What I will to add to the logical picture of information are issues, and their role in information exchange. Well, in fact, I don't have the feeling that I really add anything, but rather that issues are already there, that it is just a matter of noticing that they are there in this general picture of information.

The overall picture is rather simple, inquisitive stages are sets of alternative possibilities, with the borderline case of an indifferent stage, in which there is a single possibility. We can study relations between these stages that play a role in comparing stages from the viewpoint of inquisitive information exchange.

The next thing that will happen is that I restrict the picture. I'll move from, what I call, general inquisitive information frames to, what I call, world-based inquisitive information frames.[12] What this move consists in is not to view pos-

[11] At the moment I have no clear view of of what the differences and correspondences precisely are. This should be the subject of further investigations. I was just recently inspired by van Benthem (2008) to take a more global look at the connections between inquisitive semantics and more general information models, conected to Kripke models for intuitionistic logic. There are also clear correspondences, but equally clear differences with Data Semantics, as developed in Veltman (1984,1985), and the information models in Landman (1986).

[12] I use the phrase 'general frames' only to distinguish the more general case from the specific case of the world-based frames. So, not in the way in which the phrase 'general frame' is used in modal logic.

sibilities as primitive entities, which is what they are in the general picture, but to define them as (non-empty) sets of possible worlds. What we arrive at then is, apart from the addition of issues, again related to an existing logical framework, that of possibility semantics, which I know from Cresswell (2004).

The reason for making this restriction is not a principled, but a practical one. It makes a big difference once we start looking at language whether we use the general or the world-based frames. Under the world-based frames the inquisitive logic that comes out is very near classical logic. As I will show, it is just one world away from classical logic. The general frames have much more in common with intuitionistic logic.

You can probably also put it like this, the restriction to world-based models precisely gives you what inquisitiveness adds to classical logic. The general models do a lot more. I think, see van Benthem (2008), that the difference can be characterized as adding the notion of evidence to the notion of information. In the world-based models information is information and you don't care about whether there is evidence for the information, in the general models you do.

The funny thing is that the way the difference comes about seems almost futile. For a propositional language it only has to do with the interpretation of atomic sentences. In both cases you can use two values for the atoms, say 1 and 0. In the general case, where possibilities are primitive, you have to do that relative to the possibilities as such. I the world-based case, where the worlds are the primitives you can do it (have to do it) relative to the possible worlds. Relative to a world the atoms are true or false.

When you consider a possibility defined as a set of worlds, you get the three cases of being true in every world in the possibility, being false everywhere, and somewhere true and somewhere false. In the latter case the information present in the possibility is not sufficient to decide whether the atom holds or not. In a primitive possibility in the general model, if an atom gets the value 1 it not just means true, but it means that it is a well-established fact. The value 0 means: not a well-established fact. Actually not excluding that it is already found true by the information. I'm running ahead of things. Back to the basics.

2.1 Information Frames

Possibilities The basic ingredients in an information frame I call *possibilities*. You can think of a possibility as a piece of information, but strictly speaking, possibilities in a frame are rather pegs to hang information on. Possibilities only become pieces of information after we have turned information frames into information models for a language, by adding an interpretation function for that language to the frame. (See section 3.1.)

Taking an optimistic point of view, the basic feature of information is that it can grow, or – which is the way I will look upon it – that it can become stricter, more restricted. The possibilities in a frame come with a relation on them that tells us whether one possibility is more restricted, more informed, than another one. In a sense, the relation is more important than the possibilities themselves. The possibilities in a frame fully get there identity from where they are relative to the other possibilities. (As is the case with numbers and the relation larger than.)

Formally, an *information frame* $\mathcal{F} = (\mathcal{P}, \leq)$, where $\mathcal{P}$ is a non-empty set, the set of possibilities in $\mathcal{F}$, and $\leq$ is a partial order on $\mathcal{P}$. When $i, j \in \mathcal{P}$, we read $i \leq j$ as: i is a possibility that *is a restriction of* the possibility j.[13] I put two additional constraints on the ordering of the set of possibilities, by which an information frame is a joined semi-lattice

Firstly, I stipulate the ordering to be such that for any finite non-empty subset I of $\mathcal{P}$ there exists a possibility $i \in \mathcal{P}$ that is *the join of* I. If j is the join of $I \subseteq \mathcal{P}$, then for every $i \in I$: $i \leq j$; and there is no $k \in \mathcal{P}$ such that $k < j$ and for every $i \in I$: $i \leq k$.

The join of $\mathcal{P}$ itself, I call the *possibility of ignorance*, and I will denote it by ω. The possibility of ignorance is the maximal element in the ordering of the possibilities, where there is no information at all.

Secondly, in line with my optimistic view on information growth, I stipulate the existence of *happy ends*. A happy end corresponds to a possibility of complete information. The set of happy ends in $\mathcal{P}$ is the set of possibilities $\imath = \{i \in \mathcal{P} \mid \forall j \in \mathcal{P}\colon\ j \leq i \Rightarrow j = i\}$.

Happy ends are the minimal elements in the ordering. For any possibility i which is not a happy end itself, there will be several happy ends among its possible restrictions, and i will be the join of all the happy ends among its restrictions. So the join if $\imath$ is ω.[14]

States You can look upon a frame $\mathcal{F}$ as such, as the frame of the initial common ground, where all possibilities are still present. The aim of information exchange is to exclude possibilities, to arrive at subframes. We can model such subframe of a frame $\mathcal{F} = (\mathcal{P}, \leq)$ as the set of pointed frames $\mathcal{F}_i = (\mathcal{P}, i, \leq)$, for $i \in \mathcal{P}$. We call these subframes, the *states in* $\mathcal{F}$.

In a state $\mathcal{F}_i$ the only possibilities that still matter are the possible restrictions of i, the ways in which the information in i can still grow, its prospects. We will denote the possible restrictions of i as $\mathcal{P}_i = \{j \in \mathcal{P} \mid j \leq i\}$. The point i in a state is the join of $\mathcal{P}_i$

The ordering on the possibilities in a frame $\mathcal{F}$ also orders its subframes, the states. The aim of information exchange is to make subsequent moves from a

[13] According to my intuitions, using the phrase 'at least as restricted' here fits the use of the notation $\leq$. And intuitively, but also because in the world-based frames $\leq$ will mean the the same as 'is a non empty subset of', I want to use $\leq$ here as the notation. It might be handier to just say 'at least as informed', but then the notation $\leq$ doesn't fit so well. A disadvantage of my choice of notation it that it runs opposite to the standard notation for the accessibility relation in Kripke frames.

[14] A footnote for sceptical philosopher, and perhaps for theologicians as well. Although I do assume that happy ends exist, I do not take it for granted that such heavenly states of divine knowledge are humanly reachable. And certainly not that they are reachable by exchanging information in dialogues, no matter for how long, and with how many other people.

state $\mathcal{F}_i$ to a state $\mathcal{F}_j$, where $j < i$. The maximal element in the ordering of the states is $\mathcal{F}_\omega$, which we call the *the initial state.* The minimal elements are states $\mathcal{F}_i$ where i is a happy end.

Although officially states are pointed subframes $\mathcal{F}_i$, we will also often refer to the possibility i as such as a state. Given a frame $\mathcal{F}$ and a possibility i it is determined what the state $\mathcal{F}_i$ is.

2.2 World-Based Information Frames

In a world-based information frame the possibilities are no longer primitive entities, but possibilities are defined as non-empty sets of possible worlds. This is the road that I will primarily take here, when I state an inquisitive semantics for a language. Once you go this way, you don't have to tell a long story about the restriction relation $\leq$, it boils down to the subset relation on the set of possibilities. All the things we stipulated above about the restriction relation are then naturally a given thing of looking at $\leq$ as the subset relation on the set of non-empty subsets of the set of possible worlds.

The happy ends correspond with singleton sets, the join of a set of possibilities corresponds to taking the union of all its elements, and the possibility of ignorance ω corresponds to the set of all possible worlds.

The essential difference between the general information frames and these world-based information frames is that in the latter it makes sense to talk about the elements of a possibility i: $\{v \in \omega \mid v \in i\}$. Which correspond one-to-one to a happy end restrictions of i, i.e. $\{\{v\} \mid v \in i\}$.

Definition 1 (World-Based Information Frame). *Let* ω*,* the set of possible worlds*, be a non-empty set. The* ω-based information frame $(\mathcal{P}_\omega, \leq)$*, is the information frame such that:*

1. $i \in \mathcal{P}$ *iff* $i \subseteq \omega$ *&* $i \neq \emptyset$
2. *For* $i, j \in \mathcal{P}$*,* $i \leq j$ *iff* $i \subseteq j$

It can easily be checked that world-based information frames are indeed information frames, and that the definitions and facts given above apply equally well to these more specific types of frames.

Since possibilities are sets of worlds, and since sets naturally come with the subset relation, it is a bit redundant to officially look upon world-based states as the set of al non-empty subsets of a possibility. Just a possibility, a set of worlds as such suffices, which is also the common way to look upon an information state.

2.3 Propositions

We can use a possibility i in a frame $\mathcal{F}$, the state $\mathcal{F}_i$ that it brings with it, to model the current state. But we still need the tools to move from a current state $\mathcal{F}_i$ to a state $\mathcal{F}_j$, such that $j < i$. Propositions are to be those tools. Once we turn information frames into information models, where we interpret a language, propositions will be what sentences express. Here, in talking about frames, they are just abstract entities. It may be helpfull to keep this in mind.

We define propositions relative to a possibility i, the point of a current state $\mathcal{F}_i$. A proposition I relative to a possibility i is a subset of $\mathcal{P}_i$, the set of possibilities in the state $\mathcal{F}_i$ that still matter, the restrictions of i. So a proposition I relative to a possibility i is a subset of the set of restrictions of i. But we add the following constraint: there are no possibilities $j, k \in I$ such that one is a proper restriction of the other, i.e., there are no two possibilities $j, k \in I$ such that $j < k$. We can read this as: any two different possibilities in a proposition at least partially *exclude each other*.

Definition 2 (Propositions). *Let $\mathcal{F}_i$ be a state.*

I is a proposition for *i iff $I \subseteq \mathcal{P}_i$ and for no $j, k \in I$: $j < k$.*

From the perspective of information exchange we can look upon the possibilities in a proposition I relative to a current state i, as *proposing one or more alternatives for i* which would further restrict the information in the current state.[15]

The definition allows for the borderline case of a proposition I for i where $I = \emptyset$. We call such a proposition an *absurd proposition for i*. An absurd proposition for the state of ignorance ω, I will just call the *absurd proposition*. From now on, when I talk about a proposition for i, I will implicitly mean a non-absurd proposition.

The distinctive feature of our propositions is that they can not only be informative but also inquisitive. A proposition I for a possibility i is *inquisitive in i* iff I contains more than one possibility, if I contains several alternative possibilities. An inquisitive proposition for ω we just call an *inquisitive proposition*.

An inquisitive proposition does not make a proposal to move from a state $\mathcal{F}_i$ to a specific other state $\mathcal{F}_j$, but rather invites a choice between a number of alternative other states $\mathcal{F}_j$, where $j < i$.

If a proposition I for i contains a single possibility j, then I is not inquisitive and can only be informative. Such a proposition is informative in case $j < i$. It

[15] As Salvador Mascarenhas pointed out to me, it needs further motivation and clarification *why* we focus on *maximal* possibilities being the elements in propositions. Why could a proposition not also contain two possibilities where one is *included* in the other? At this abstract level of propositions in frames I don't have a good answer at this moment. I can only come up with the intuition that when one possibility is fully included in another they don't form *alternatives*, that at least partial mutual exclusion is inherently connected to the notion of two alternatives. (But, then, I also thought for a long time that only *complete* exclusion was the landmark of two alternatives.) But look at it from the perspective of proposing two alternatives. You can think of a proposal of two alternatives like: going to the market, or going to the market and the shop. Here the alternative of going to the market and the shop includes the alternative of going to the market. But the way you interpret such a proposal is as: *only* going to the market, or *both* going to the market *and* to the shop. These two possibilities do exclude each other.

then proposes to make the specific move from the state $\mathcal{F}_i$ to the more restricted state $\mathcal{F}_j$.

But also an inquisitive proposition can be informative at the same time. This is the case if the *join* of the alternative possibilities in a proposition I for i is a proper restriction of i. In a world-based frame where possibilities are sets of worlds: an inquisitive proposition I for i is also informative if the union of the possibilities in I is a proper subset of i.

So this is how informativeness is generally defined. A proposition I for i is *informative in* i iff $j < i$, where j is the join of I. An informative proposition for ω we just call an *informative proposition*.

A *hybrid* proposition, a proposition I for i which is both informative and inquisitive, makes a double proposal. If j is the join of I then it first of all makes the proposal to move from the state $\mathcal{F}_i$ to the state $\mathcal{F}_j$, where $j < i$, but on top of that it invites a choice between a number of alternative other states $\mathcal{F}_k$, where $k < j$.

Another borderline case besides absurd propositions are tautological proposition. A proposition I for i is *tautological in* i iff i is the single possibility in I. If I is tautological in ω, i.e., if $I = \{\omega\}$, we just call it *the tautological proposition*.

2.4 Modern and Old-Fashioned Propositions

By way of illustration, consider a world based information frame, where possibilities are non-empty sets of worlds, then a proposition is a set of non-empty sets of worlds, such that no two different sets stand in the subset relation. So, propositions are not just a set of worlds, which is the standard logical semantical notion of a proposition. Let's call those 'old-fashioned propositions'. Then our modern propositions are sets of old-fashioned propositions.

Non-inquisitive propositions are either the empty set – both the modern and the old fashioned absurd proposition – or singleton sets consisting of a single old-fashioned proposition. So, all non-inquisitive propositions correspond one-to-one to old-fashioned propositions. What also counts as a proposition now, is any partition of the set of possible worlds, which is the old-fashioned notion of a question, where the blocks in the partition are old-fashioned propositions that correspond to complete answers to the question. Of course, as long as there is more than one block, such propositions count as inquisitive propositions now.[16]

In a proposition that forms a partition of the set of worlds, the blocks are possibilities that mutually exclude each other, and the union of all these possibilities is identical with the set of all worlds. The definition of propositions also allows for sets of possibilities that pairwise properly overlap, and of which the union is a proper subset of the set of all possible worlds.

Think of a proposition that consists just of the possibility that Alf will go to the party, and the possibility that Bea will go. There is also the possibility that

[16] Although from a standard logical semantical perspective it may seem a bit odd to call a question a proposition, it is not at odds with the use of the word 'proposition' in (slang) English, where it can mean 'task, job, problem, objective,...', which are characterizations which fit in nicely with the conversational function of questions.

both go, not an element in our proposition, but included in each of the possibilities in it. Furthermore, there is the possibility that neither of them will go, but that one is excluded by the union of the two possiblities in our proposition. You could say that relative to the set of al worlds, or the set of all possibilities if you want, this proposition is inquisitive and informative at the same time. propositions can propose information and propose an issue.

In line with the old-fashioned notion of a proposition, our new notion will also serve as a notion of semantic content, but one in which information and issues are on equal footing.

Theme and Rheme If a proposition I is informative in i, i.e., if it holds for the join j of I that $j < i$, then the proposition I excludes possibilities from $\mathcal{P}_i$, those possibilities $k \in \mathcal{P}_i$ such that $k \notin \mathcal{P}_j$. We call the join of the set of possibilities excluded by I in i *the possibility excluded by I from i*. We can also identify the possibility excluded by I from i with the join of all happy ends $h \leq i$ such that $h \not\leq j$.

If I is a proposition for i, then if we add the possibility excluded by I in i (if there is one) to I, then the result is again a proposition for i, which cannot fail to be non-informative in i, but it will be inquisitive, as soon as I is informative in, but not absurd in i. We will call this proposition *the theme of I in i*, or just *the theme of I* if $i = \omega$.

Where there is a theme, there must be a rheme. If j is the join of a proposition I, then we call the proposition $\{j\}$ *the rheme of I in i*, or just the *the rheme of I*, if $i = \omega$.

Stages We characterized propositions as proposals for transitions between states. When we model information exchange it comes handy if next to propositions, we also model stages in the exchange, of which propositions are the central part.

We define stages relative to a frame $\mathcal{F} = (\mathcal{P}, \leq)$. A *stage in $\mathcal{F}$* is a triple (i, I, j), where $i \in \mathcal{P}$ is a possibility, I is a proposition for i and j the possibility that is the join of I.

The possibility i that is the first element in a stage corresponds to the current state $\mathcal{F}_i$. The last element j, the join of the possibilities in the proposition I for i, corresponds to a new state $\mathcal{F}_j$, where if I is informative in i, i.e., if $j < i$, the proposed state $\mathcal{F}_j$ will be a proper restriction of the current state $\mathcal{F}_i$.

If the proposition I is not inquisitive in i, we just have that $I = \{j\}$. If I is not informative in i it will be the case that $i = j$. Then the proposition is not a direct proposal to move to a specific more restricted state $\mathcal{F}_j$. But if such a proposition I which is not informative in i, *is* inquisitive in i, it is an indirect proposal, an invitation to move to one of the states $\mathcal{F}_k$, where $k < i$, and k is one of the alternative possibilities $k \in I$. A hybrid proposition combines these two cases in a single informative and inquisitive move.

We let stages inherit the properties of the propositions in them, so there are informative and inquisitive, hybrid and tautological stages. Only the absurd stage does not exist, since only non-empty sets of possibilities have a join.

Stages are defined the way they are to make an easy characterization possible of two subsequent states. If $s = (i, I, j)$ and $s' = (i', I', j')$ are two stages, we will say that s' *is a subsequent stage of* s iff $i' = j$. Later, after we have introduced information models, and the semantics for a language, and propositions can be expressed by sentences, we will define the common ground as a stack of subsequent stages.

In the definition of when a stage s' is subsequent to a stage s, the nature of the propositions in s and s' plays no role, except for the informative content of the proposition in s, which determines the starting point of a subsequent stage s'.

But if we think of s being an inquisitive stage that is the result of a dialogue move made by one participant, we may expect that if the subsequent stage s' is the result of a response to stage s by another participant, that something has to be said about whether and how s' relates to s. And we will do that. We will define a logical relation of *compliance* that deals with this.

In principle, though, if a stage s is the current stage, is the stage on top of the stack of stages that models the development of the common ground, we will set things up in such a way that anything goes, compliant or not.

However, a current stage s as contributed by one participant, also determines the starting point of a subsequent stage s' which will be constructed by the next contribution of another participant. We do not expect this to happen just like that. And, indeed, it will not.

We have characterized propositions as *proposed* transitions. As such they invite for a reaction, accepting or rejecting such a proposal to make the transition from the current state to a more restricted state. It is only after such a reaction of the one participant to the proposed transition by the other particpant has been publicly made, and consequently has been *absorbed* in the common ground stack, that we move on to the next stage. This will make sure that the current state in a stage s is one that is an appropriate starting point for a subsequent stage s'.

Among the inquisitive non-informative stages in a frame $\mathcal{F} = (\mathcal{P}, \leq)$, there is an extreme case, the stage $(\omega, \imath, \omega)$, where the current state is the state of ignorance ω, the join of all possibilities in $\mathcal{P}$, and where the proposition in ω is $\imath$, the set of all happy ends in $\mathcal{P}$, *The Big Question* what the real happy end will be. We call this stage *the initial stage*.

2.5 Why bother?

You may ask yourself: why bother about these general information frames if we are going to work with world-based information frames? The answer is that in the end, you may want to look at the general case. If you look at the specific case from the perspective of the general case, you can make clear what the specific assumptions are that you are making, and what the effects are if you lift these. You are directed in this way to further investigatons, extensions of the model.

For this reason, I will always try to state things in a way that applies to the general case as well. More specifically that means that I will refer to worlds as little as possible and will stick to the level of possibilities as much as I can, because what I state then applies in the general case as well. There is the potential danger of unnecessarily complicating things, but my own experience is rather in the opposite direction. I get a better view of the specific case in the light of the more general perspective.

In fact, as I hope to show, even the specific case I concentrate on can be simplified a lot. Concretely, I will show that nothing gets lost if we look at pairs of worlds rather than sets of them. Again, you may ask: if that is so, then why bother me with sets if pairs suffice? The answer from my own experience: certain things were not that easy to motivate and explain – which I take to mean: to understand – at the level of pairs of worlds. I found that explaining things became easier at the 'unecessarily complicated' level of sets of worlds.

In the developmen of my own ideas about the subject matter over the past two years or so, I moved up and down between more specific and more general versions of telling the story. All the time 'the logic' remained the same, only my way of understanding things grew, or so I think. Only just now I feel that I have reached a level of understanding things where I precisely see the borderline between all simplifications where the logic remains the same, and the more general cases where the logic becomes really different.

3 Inquisitive Semantics

3.1 Information Models

To interpret a language $\mathcal{L}_{\mathcal{E}}$, a language $\mathcal{L}$ with basic expressions $\mathcal{E}$, we combine a world-based information frame $\mathcal{F} = (\mathcal{P}_\omega, \leq)$ with an interpretation function $\mathcal{I}$ that assigns meanings to the basic expressions of the language.

Like in intensional semantics, $\mathcal{I}$ is a function that has as its domain the possible worlds in ω, and for $v \in \omega$, $\mathcal{I}_v$ is a function that assigns an appropriate semantic value to the basic expressions in $\mathcal{E}$. So, for $\alpha \in \mathcal{E}$, the outcome of $\mathcal{I}_v(\alpha)$ will be a semantic object that is the denotation of the expression α in the possible world v.

Let $\mathcal{L}$ be a standard propositional language, with the usual repertoire of the connectives $\wedge$ for conjunction, $\vee$ for disjunction, $\rightarrow$ for implication, and the negation sign $\neg$. We build the syntax in the usual way, starting from a finite set of propositional variables $\mathcal{E} = \{p_1, \ldots, p_n\}$. We take $\{0, 1\}$ to be the semantic values that can be assigned to the propositional variables in $\mathcal{E}$.

So, a world-based information model for our propositional language will be a quadruple $\mathcal{M} = (\mathcal{P}_\omega, \leq, \mathcal{I}, \{0, 1\})$. For each atomic sentence $p \in \mathcal{E}$, and each possible world $v \in \omega$, the value of $\mathcal{I}_v(p)$ will either be 1 or 0.

First Excursion to General Models I use quite a few words in the four paragraphs above to define these rather simple things. There are easier and quicker ways to make a start for a propositional language. For example, I could just say that what the interpretaton function $\mathcal{I}$ does, is to assign a possibility to each atomic sentence: $\mathcal{I}(p) \in \mathcal{P}_\omega$.

There are two reasons for not doing this. The first is that this strategy would no longer work when we move beyond the level of a propositional language. The general way in which I state things now in the first paragraph applies to other languages as well, such as a language of first order predicate logic. Fill $\mathcal{E}$ with a set of n-place predicates, replace $\{0,1\}$ by a domain of individuals $\mathcal{D}$, and say that for each n-place predicate P^n, $\mathcal{I}_v(P^n) \subseteq \mathcal{D}^n$, and your done with the basics.

The second reason is that the way I do it now makes most clear that the worlds, and not the possibilities, are basic in the meaning assignment in a world-based information model. Whatever comes out for the possibilities should be determined by what comes out for the worlds.

If I would make the move to general, not world-based, information models, starting from a general frame $(\mathcal{P}, \leq)$, the only thing I would have to change in the first paragraph is to say that: $\mathcal{I}$ is a function that has as its domain the possibilities in $\mathcal{P}$, and for each $i \in \mathcal{P}$, $\mathcal{I}_i$ is a function that assigns an appropriate semantic value to the basic expressions in $\mathcal{E}$. So, for $\alpha \in \mathcal{E}$, the outcome of $\mathcal{I}_i(\alpha)$ will be a semantic object that is the denotation of the expression α in possibility i.

For our propositional language, the story then continues as follows: We take $\{0,1\}$ to be the semantic values that can be assigned to propositional variables in $\mathcal{E}$. So, a general information model for our propositional language is a quadruple $(\mathcal{P}, \leq, \mathcal{I}, \{0,1\})$. For each atomic sentence $p \in \mathcal{E}$, and each possibility $i \in \mathcal{P}$, the value of $\mathcal{I}_i(p)$ will either be 1 or 0.

However, now we can't leave things like this, we have to build some stability in the model:[17]

Stability Restriction If $i \leq j$, then $\mathcal{I}(i)(p) \geq \mathcal{I}(j)(p)$

What the stability restriction says is that if we move from a possibility j to a restriction i of j, then it cannot be the case that the value of an atomic sentence p is 1 in j, and 0 in i. In moving from less to more informed possibilities a 0 can become a 1 (or remain 0 forever) but not the other way around. This tells us how to read that $\mathcal{I}_i(p) = 1$: it is an established fact in i that p. And $\mathcal{I}_i(p) = 0$ is the negation of that: it is not an established fact that p, which, except at happy end possibilities, with complete information, does not mean that it is an established fact that *not-p*. What is not yet an established fact can still become one, but once something is an established fact it remains like that.[18]

[17] This property is often caled 'persistence', I borrow the term 'stability' from Veltman (1985).

[18] Philosophical footnote. Happy ends are not identified with situations where for every atomic proposition it is definitely decided whether it is 0 or 1. We can reach a possibility i where for all atomic sentences p and for all $j < i$ the value of p remains stable. So, we are dead sure about all the facts. Still, there can be many proper restrictions $j < i$, more informed situations than i. Apparently, there is more to know than just what the facts of the matter are. Of course, what we may think of

Information States With respect to a information frame $\mathcal{F} = (\mathcal{P}, \leq)$, we introduced the notion of its subframes, the pointed frames $\mathcal{F}_i = (\mathcal{P}, i, \leq)$, which we called states. Likewise, we introduce the notion of a submodel $\mathcal{M}_i$ which contains a subframe of $\mathcal{F}_i$ instead of $\mathcal{F}$ itself, and call these *information states.*

So, an *information state in* $\mathcal{M}$ is a quintuple $\mathcal{M}_i = (\mathcal{P}_\omega, i, \leq, \mathcal{I}, \{0, 1\})$. We will often refer to the point i in a an information state $\mathcal{M}_i$ as the current state. In world-based models, possibilities are non-empty sets of worlds, and hence the current state is a non-empty set of worlds.

The difference between an information state $\mathcal{M}_i$ and a state $\mathcal{F}_i$ is that whereas a state is just a register for information, in an information state there is actual information registered, partial information about the denotation of the basic expressions $\mathcal{E}$ of the language $\mathcal{L}_\mathcal{E}$. In the case of the models for our propositional language the information concerns the possible denotations of the atomic sentences of the language.

The possibilities that still matter in a state $\mathcal{M}_i$ are the possibilities in $\mathcal{P}_i$, the set of restrictions of i: $\mathcal{P}_i = \{j \in \mathcal{P} \mid j \leq i\}$, the prospects of the current state i. In a world-based model, where the current state is a non-empty set of worlds, the prospects are its non-empty subsets. The ultimate prospects from a current state i, are happy ends, the states of complete information $\{v\}$ for $v \in i$.

The initial information state, the initial state of the common ground, is $\mathcal{M}_\omega$, which is indistinguishable from the model $\mathcal{M}$ as such. Starting from the initial state, the aim of information exchange is to make subsequent moves from a state $\mathcal{M}_i$, to a more informed state of the common ground $\mathcal{M}_j$, where $j < i$.

Propositions As we saw in the case of frames, the transition from one information state $\mathcal{M}_i$ to a more restricted state $\mathcal{M}_j$, $j < i$, is mediated by a proposition, which in the process of information exchange, we look upon as a proposal to move in a certain direction.

The main function of the semantics is to tell for all sentences $\varphi \in \mathcal{L}_\mathcal{E}$ what the proposition is that φ expresses, and thereby what the proposition is that φ expresses relative to an information state $\mathcal{M}_i$, which we will denote by $i[\varphi]_\mathcal{M}$. In our inquisitive semantics, these propositions will be modern propositions, sets of old-fashioned propositions, where the elements of the set offer alternative possibilities to proceed to one or more more informed states $\mathcal{M}_j$, $j < i$.

The whole battery of notions that we met above of informative propositions, inquisitive propositions, etc, will of course also apply to propositions expressed by sentences.

Relative to a state $\mathcal{F}_i$, we defined a proposition for i to be a subset $I \subseteq \mathcal{P}_i$ such that for no $j, k \in I$: $j < k$. Likewise, relative to an information state $\mathcal{M}_i$, the proposition $i[\varphi]_\mathcal{M}$ expressed by a sentence φ in i will be such a set of possibilities that at least partially exclude each other. In our world-based models, such alternative possibilities in the proposition expressed by a sentence are sets of possible worlds, old-fashioned propositions, such that of no two such sets one

here is that what there is more to know is *why* the facts are as they are. Perhaps the real happy end is where we have answers to all questions 'Why p?'.

is properly included in the other, of no two of such old-fashioned propositions one properly entails the other.

Support We will define the proposition $i[\varphi]_{\mathcal{M}}$ expressed by a sentence φ relative to an information state $\mathcal{M}_i$ as the maximal elements among the set of possibilities $i \in \mathcal{P}_i$ such that i *supports* φ *in* $\mathcal{M}$. The notion of support of a sentence in a possibility in a model $\mathcal{M}$, which we denote as $\mathcal{M}, i \models \varphi$, is the basic recursive notion in the semantics.

The distinguishing feature of the semantics is that a sentence φ cannot only be informative, in which case φ expresses an informative proposition, but φ can also be inquisitive, in which case φ expresses an inquisitive proposition. The notion of support has to deal with these two cases of a sentence providing information and/or raising an issue.

We will define support in such a way that for a sentence φ to be supported by a possibility i means that the information φ may provide is present in i, and that an issue that φ may raise is resolved in i.

If we view support from the perspective of an information state $\mathcal{M}_i$, with its prospective possibilities $\mathcal{P}_i = \{j \in \mathcal{P} \mid j \leq i\}$, then what the notion of support of a sentence in a possibility delivers, is that if we consider for which $j \leq i$: $\mathcal{M}, j \models \varphi$, we get the selection of those restrictions $j \leq i$ such that the information φ may provide is present in j, and the issue φ may raise is resolved in j.

With respect to an issue φ may raise, there will in general be different possibilities in $\mathcal{P}_i$ that resolve the issue in different ways. So, by selecting all possibilities that resolve an issue, we may end up with possibilities that give different answers to the issue. The maximal ones among them correspond to the alternatives in inquisitive propositions.

Note on Notation for Propositions. The notation $i[\varphi]_{\mathcal{M}}$ is the same as the notation used in update semantics to denote update functions.[19] Then the outcome of an update of a current state i with φ, $i[\varphi]_{\mathcal{M}}$ would be another state, i.e., in our present terms, another possibility j. This is not what the notion means here, it is not an update function on possibilities, the outcome of $i[\varphi]_{\mathcal{M}}$ is not a possibility, but a proposition, a set of possibilities.[20]

Then, why use this notation? Well, in the end, we are interested in updating information, in updating the current state of the common ground, we do want to move from a possibility i, a current information state $\mathcal{M}_i$, to a possibility $j < i$, to a more informed state $\mathcal{M}_j$ of the common ground. Our propositions expressed by sentences serve to mediate such transitions, and to steer the exchange of information in certain directions.

[19] Update semantics originates from Veltman (1996).

[20] In section 6, we will give an equivalent reformulation of the semantics which is a 'proper' update semantics. Unlike our notion of a state here, which only concerns information, in the update version states themselves can be inquisitive, not only propositions can be, as in the present version.

Also, there is a form of updating information states present in the notion of a proposition $i[\varphi]_{\mathcal{M}}$. The information provided by the proposition $i[\varphi]_{\mathcal{M}}$ in a current state i can be identified with the join (the union in world-based models) of the possibilities in $i[\varphi]_{\mathcal{M}}$. That, in general, will be a possibility $j \leq i$, and $j < i$ if φ is informative in i.

You can look upon a proposition $i[\varphi]_{\mathcal{M}}$ for a possibility i as a proposed transition from a current state i to a state j, where j is the join of $i[\varphi]_{\mathcal{M}}$. Relative to frames, we have already introduced such transitions as stages. Lifted to models we can define *information stages* in a model $\mathcal{M}$ as triples $(i, i[\varphi]_{\mathcal{M}}, j)$.

Clearly, stages can be chained. We can chain an information stage $(i, i[\varphi]_{\mathcal{M}}, j)$ to a subsequent stage $(j, j[\psi]_{\mathcal{M}}, k)$. The second stage starts where the first ends, in state j. Such chains of subsequent stages will be what a common ground stack consists of.

Now, this is not the full picture, of course, because the fact that propositions can be sets of alternative possibilities, plays no role yet in the story as I just told it. And that is as it should be. Moving from less to more informed states is what information exchange is all about. The role of alternative possibilities in a proposition, the role of inquisitiveness, is secondary, is auxiliary to the main purpose of getting more informed.

If a proposition in a stage $(i, i[\varphi]_{\mathcal{M}}, j)$ is inquisitive, it does not want just any arbitrary subsequent stage $(j, j[\psi]_{\mathcal{M}}, k)$, but one where ψ is such that the information it provides, or the issue it embodies, is related to the issue raised by φ. We will turn to these matters of modelling inquisitive dialogues after we have looked in detail at the inquisitive interpretation of single sentences.

3.2 The Basis of Inquisitive Semantics

We start with the basic recursive definition of support relative to a world-based model. The recursive definition looks similar to other statements of the semantics for a propositional language in information oriented models. Most characteristic is the atomic clause.

Definition 3 (Inquisitive Semantics).
Let $\mathcal{F} = (\mathcal{P}_\omega, \leq)$ be a world-based frame, and $\mathcal{M} = (\mathcal{F}, \mathcal{I}, \{0,1\})$ an information model for $\mathcal{L}_\mathcal{E}$ based on $\mathcal{F}$. Let $i \in \mathcal{P}$, $p \in \mathcal{E}$.[21]

1. $\mathcal{M}, i \models p$ *iff for all* $v \in i$: $\mathcal{I}_v(p) = 1$
2. $\mathcal{M}, i \models \neg\varphi$ *iff for no* $j \leq i$: $\mathcal{M}, j \models \varphi$
3. $\mathcal{M}, i \models \varphi \to \psi$ *iff for all* $j \leq i$: *if* $\mathcal{M}, j \models \varphi$, *then* $\mathcal{M}, j \models \psi$
4. $\mathcal{M}, i \models \varphi \wedge \psi$ *iff* $\mathcal{M}, i \models \varphi$ *and* $\mathcal{M}, i \models \psi$
5. $\mathcal{M}, i \models \varphi \vee \psi$ *iff* $\mathcal{M}, i \models \varphi$ *or* $\mathcal{M}, i \models \psi$

As we have announced, on top of the recursive semantics, we will define the notion that we will be most concerned with, the notion $i[\varphi]_\mathcal{M}$ of *the proposition expressed by a sentence φ in a current state i.*

Although we could define propositions directly in terms of the notion of support, we do so in terms of a more standard notion of meaning as the set of possibilities that support φ. Like we consider propositions in current state i, we do so for meanings as well. The meaning of a sentence φ in i is the set of restrictions of i that support φ. We obtain a proposition from a meaning by selecting the maximal elements in a meaning. Finally, we also define a notion which expands a proposition into a corresponding meaning.[22]

---

[21] In accordance with the official guidelines, we state the recursive semantics relative to the model, i.e., we define $\mathcal{M}, i \models \varphi$. Relative to world-based models this overdresses the definition a bit. It would not do much harm not to make reference to the model explicitly in the clauses, and define $i \models \varphi$, and to only mention the model in the heading of the definition. In the clauses for negation and implication, we do quantify over possibilities which are restrictions of the possibility i relative to which we evaluate support. Of course, these are to be possibilities in $\mathcal{P}_\omega$ (where $i \in \mathcal{P}_\omega$ means the same as $i \leq \omega$). But even there it is not important to explicitly mention the model, because which possibilities are restrictions of i, for which j: $j \leq i$, can be 'read from' i as such. Such j are just the non-empty subsets of i. However, the situation is different when we state the semantics relative to general models. There the possibilities are primitive, and not sets of worlds. To see there which possibilities are restrictions of i, we really have to consult the model as such, where the frame $(\mathcal{P}, \leq)$ on which it is based determines what the restrictions j of i are. So, in case of general models it is essential to explicitly consider $\mathcal{M}, i \models \varphi$, and not just $i \models \varphi$. Therefore, in line with our general strategy to formulate things in such a way that we have to restate things as little as possible, were we to move to more general cases, we recursively define $\mathcal{M}, i \models \varphi$ rather than more economically $i \models \varphi$.

[22] The reason for doing things in such a roundabout way is that although the notion of a proposition is the notion we primarily want to work with, occasionally it comes

Definition 4 (Propositions and Meanings). *Let $\mathcal{M}_i$ be an information state in a model $\mathcal{M}$ for a language $\mathcal{L}_\mathcal{E}$, and $\varphi \in \mathcal{L}_\mathcal{E}$*

1. $\|\varphi\|_{\mathcal{M},i} = \{j \leq i \mid \mathcal{M}, j \models \varphi\}$
2. $i[\varphi]_\mathcal{M} = \text{MAX}(\|\varphi\|_{\mathcal{M},i}) = \{j \in \|\varphi\|_{\mathcal{M},i} \mid$ *there is no* k: $j < k$ *&* $k \in \|\varphi\|_{\mathcal{M},i}\}$
3. $\text{EXP}(i[\varphi]_\mathcal{M}) = \{j \leq i \mid$ *there is some* k: $j \leq k$ *&* $k \in i[\varphi]_\mathcal{M}\}$

We can look upon the specific instance of $i[\varphi]_\mathcal{M}$ where $i = \omega$, which we will write as $[\varphi]_\mathcal{M}$, as the meaning assigned to a sentence φ in the model $\mathcal{M}$. Likewise, the specific instance of $\omega[\varphi]_\mathcal{M}$, which we will write as $[\varphi]_\mathcal{M}$, is the proposition expressed by the sentence φ.[23]

For the language at hand, nothing can get gained or lost by switching from meanings to propositions and back again, i.e., $\text{EXP}(i[\varphi]_\mathcal{M}) = \|\varphi\|_{\mathcal{M},i}$. The reason for this is that all sentences of our language are stable:[24]

Proposition 1 (Stability).
For all sentences $\varphi \in \mathcal{L}_\mathcal{E}$ and information models $\mathcal{M}$ for $\mathcal{L}_\mathcal{E}$:

if $\mathcal{M}, i \models \varphi$, then for all $j \leq i$: $\mathcal{M}, j \models \varphi$.

This fact is stated as Theorem 2 in Cresswell (2004). The proof is by induction on the construction of the sentences.

We defined a battery of properties of propositions I in a state $\mathcal{F}_i$. They also apply to the proposition $i[\varphi]_M$ expressed by a sentence φ in an information state $\mathcal{M}_i$.

handy to shift from propositions to meanings, perform some operation on them, and then move back again to propositions. More particularly, we need this shift in defining two operations needed to construct common ground stacks, the notion of *thematizing* a sentence, and the notion of *restriction*, we need to absorb the *acceptance* of an informative stage in a dialogue. In section 6, we wll meet a third notion of meaning relative to world-based models, where meanings will be seen to also correspond to a *relation of indifference* on the set of worlds. We could use these as well to make the 'shifts' referred to here, which are needed to perform certain operations.

[23] It might make sense to make a distinction between the meaning of a sentence φ, viz., $\|\varphi\|_\mathcal{M}$ and denotation of φ in i, viz., $\|\varphi\|_{\mathcal{M},i}$, and similarly call $[\varphi]_\mathcal{M}$ the proposition expressed by φ, and $[\varphi]_{\mathcal{M},i}$ the proposition denoted by φ in i. As with the standard distinction between meaning and denotation, where meaning determines denotation, it holds here too, that the meaning fully determines the denotations in all possibilities. You could put it like this, given that a model $\mathcal{M}$ for a language $\mathcal{L}_\mathcal{E}$ corresponds to the state of ignorance $\mathcal{M}_\omega$, and given that the meanings, the propositions expressed by all sentences are determined by the model, even in the state of ignorance there is complete information about the meanings of all expressions, all sentences of the language.

[24] The property of stability is language dependent. E.g., if the language contained a *might*-operator, stability would not hold for all sentences of the language. For the language at hand, it makes no difference whether we consider our world-based models or general models. Given the Stability Restriction we have to make there for the atomic sentences, stability of all sentences holds in general models as well.

If the proposition expressed by a sentence has such properties, we will ascribe those properties to the sentence as such as well. By way of example, the notion of inquisitiveness of a sentence is defined as follows:.

Definition 5 (Inquisitive Sentences). *Let $\mathcal{M}$ be an information model for a language $\mathcal{L}_{\mathcal{E}}$, $\mathcal{M}_i$ an information state in $\mathcal{M}_i$, and $\varphi \in \mathcal{L}_{\mathcal{E}}$.*

1. *A sentence φ* is inquisitive in *an information state $\mathcal{M}_i$ iff $i[\varphi]_{\mathcal{M}}$ is an inquisitive proposition in i.*
2. *A sentence φ* is inquisitive *iff there is an information state $\mathcal{M}_i$ in a model $\mathcal{M}$, such that φ is inquisitive in $\mathcal{M}_i$.*

Since inquisitive propositions were defined as propositions which contain at least two possibilities, this means that a sentence is inquisitive if there is a model such that the proposition expressed by the sentence in that model contains at least two possibilities.

Atomic Sentences According to the atomic clause in the definition of the semantics relative to a world-based model $\mathcal{M}$, a possibility i supports an atomic sentence p iff the interpretation function in the model assigns the value 1 to p in all worlds $v \in i$.

So, from the interpretation of atomic sentence it is immediately clear that if $i \models p$, and $j \leq i$, then $j \models p$, since $j \leq i$ in world-based models means that the set of worlds in j is a non-empty subset of the set of worlds in i. Hence, atomic sentences are stable.[25] The other clauses in the semantics make sure that the property of stability is inherited by all sentences of our language.

The proposition expressed by an atomic sentence p in an information state $\mathcal{M}_i$, will consist of at most a single possibility, which consists of all the worlds $v \in i$ where p is assigned value 1.

$$i[p]_{\mathcal{M}} = \{\{v \in i \mid \mathcal{I}_v(p) = 1\}\}$$

If we are in the initial state, and $i = \omega$, the proposition $[p]_{\mathcal{M}}$ expressed by p consists of the single possibility which consists of all the worlds in the model where p is 1. That possibility is the old-fashioned proposition that is usually associated with atomic sentences.

As long as p is not absurd in a current state i, as long as there is some world $v \in i$ such that $\mathcal{I}_v(p) = 1$, the proposition expressed by p in i will consist of a single possibility, otherwise, when p is absurd in i, $i[p]_{\mathcal{M}} = \emptyset$. So, the proposition expressed by an atomic sentence p in a world-based model will at most consist of a single possibility, which means that atomic sentences are not inquisitive.

Since there will certainly be models $\mathcal{M}$ such that for some world v: $\mathcal{I}_v(p) = 1$ and for some world v: $\mathcal{I}_v(p) = 0$, there will be information states $\mathcal{M}_i$ such that $i[p]_{\mathcal{M}}$ is a singleton set $\{j\}$, where the possibility j is a proper restriction of i, which is how we defined informativeness of a proposition. Hence, atomic sentences are informative sentences.

[25] Compare the Stability Restriction in Excursion 1.

Second Excursion to General Models Except for the heading of the definition of the semantics, which now refers to world-based models, the only thing we would have to change to formulate the semantics for the propositional language relative to general models is the atomic clause in the definition, since it makes explict reference to worlds. We could have avoided that by formulating the atomic clause as:

$\mathcal{M}, i \models p$ iff for all happy end restrictions h of i: $\mathcal{I}_h(p) = 1$.

That would have made no difference for the interpretation of atomic sentences in world-based models, but it would only make sense to go about this way, if the atomic clause could read like this as well for the interpretation of atomic sentences in general models. However, the more likely option there is:

$\mathcal{M}, i \models p$ iff $\mathcal{I}_i(p) = 1$.

This clause makes a stronger requirement than that p is assigned value 1 in all the happy end restrictions of i. The general semantics does not exclude that p is assigned value 1 in all the happy end restrictions of i, but that the value is 0 in i.

Both in the world-based and in the general semantics, stability requires that for all $j \leq i$: $\mathcal{I}_j(p) = 1$ as well, but it means something different in these two cases. In the world-based semantics it only means that the information that p is present in i. In the general semantics it means not only that in i the information that p is present, but also that it is an established fact that p, that the evidence for p is available in i.

This has rather drastic, and (at least to me) unexpected consequences. This becomes clear if you consider what the proposition expressed by p in an information state $\mathcal{M}_i$ in a general model $\mathcal{M}$ is. There may be such information states $\mathcal{M}_i$, that there are two different $j, k \in \mathcal{P}$: $j \leq i$ and $k \leq i$, and $\mathcal{M}, j \models p$ and $\mathcal{M}, k \models p$, and for both j and k it holds that there is no $l \leq i$: $j < l$ or $k < l$ and $\mathcal{M}, l \models p$. Meaning what? Meaning that atomic sentences are inquisitive.

What does the inquisitiveness consist in? How can two such possibilities j and k be two alternative possibilities in the proposition $i[p]_{\mathcal{M}}$? They are alternative possible developments of the current state i, where in both it has become an established fact that p, but in a different way, on the basis of different pieces of evidence. It can easily happen that the information that p is present in the current state i – that is the situation where p is supported by all happy end restrictions of i, but not necessarily by i as such – but that it is still an open issue *Why p?* That is how atomic sentences are inquisitive. The inquisitiveness comes only to a halt in a current state i if the issue *Why p?* is resolved. And happy ends are possibilities where for every atomic sentence this question has been answered.

Classical Adequacy and Respectability Like all sentence of the language at hand, atomic sentences are stable, but under their interpretation in world-based models, atomic sentences also have a property that is not shared by all other sentences. Whether or not an atomic sentence p is supported by a possibility i is fully determined by whether p is supported by all the happy end restrictions of i. In our world based models: $\mathcal{M}, i \models p$ iff for all $v \in i$: $\mathcal{M}, \{v\} \models p$.

Allowing myself to write $v \models \varphi$ instead of $\{v\} \models \varphi$, this property of sentences can be characterized as follows:

[A] $\mathcal{M}, i \models \varphi$ iff $\forall v \in i$: $\mathcal{M}, v \models \varphi$

Note that: if $\mathcal{M}, i \models \varphi$, then $\forall v \in i$: $\mathcal{M}, v \models \varphi$ follows from the general property of stability that all sentences have. What the property [A] adds to this is that the implication also holds in the other direction, that: if $\forall v \in i$: $\mathcal{M}, v \models \varphi$, then $\mathcal{M}, i \models \varphi$.

This property of sentences is called 'classical adequacy' in Cresswell (2004). Cresswell studies a semantics like the one stated above – which he calls possibility semantics – from the perspective of intuitionistic logic. He calls [A], the 'classical adequacy requirement', and uses it to judge whether the operators in the language are interpreted in a way that they are – what he calls – 'classically respectable'. An operator is classically respectable iff when it is applied to arguments that satisfy [A], the result of the application also satisfies [A]. He argues in particular that the interpretation of disjunction as stated in the semantics given above is not classically respectable, and discusses other ways to interpret disjunction in possibility semantics which are respectable.

However, as we shall see, it is precisely the fact that our interpretation of disjunction *lacks* this property that turns the world-based semantics into an inquisitive semantics.

Disjunction Consider a disjunction $p \vee q$ of two atomic sentences. As we have seen, in world based models atomic sentences satisfy [A]. We will show that $p \vee q$ does not satisfy [A], and hence disjunction is not 'classically respectable'.

Consider a possibility i in a world-based model $\mathcal{M}$, consisting of two possible worlds: $i = \{v, u\}$, and let $\mathcal{I}_v(p) = 1$ & $\mathcal{I}_v(q) = 0$, and $\mathcal{I}_u(q) = 1$ & $\mathcal{I}_u(p) = 0$.

It is not difficult to see that $\mathcal{M}, v \models p \vee q$ and $\mathcal{M}, u \models p \vee q$, but that $\mathcal{M}, \{v, u\} \not\models p \vee q$.

According to the clause for disjunction $\mathcal{M}, v \models p \vee q$ iff $\mathcal{M}, v \models p$ or $\mathcal{M}, v \models q$. This is the case since $\mathcal{M}, v \models p$, because $\mathcal{I}_v(p) = 1$. Similarly, $\mathcal{M}, u \models p \vee q$ because $\mathcal{I}_u(q) = 1$.

Now consider $\mathcal{M}, \{v, u\} \models p \vee q$. The clause for disjunction requires that $\mathcal{M}, \{v, u\} \models p$ or $\mathcal{M}, \{v, u\} \models q$. But neither is the case. $\mathcal{M}, \{v, u\} \not\models p$ because not for all $z \in \{v, u\}$: $\mathcal{I}_z(p) = 1$, since $\mathcal{I}_u(p) = 0$. And, similarly, $\mathcal{M}, \{v, u\} \not\models q$ since $\mathcal{I}_v(q) = 0$.

Hence, disjunction does not satisfy [A].

The same example can be used to show that $p \vee q$ is inquisitive, if we let our possibility $i = \{v, u\}$ be the current state. The proposition expressed by $p \vee q$ in i consists of two possibilities: $i[p \vee q]_{\mathcal{M}} = \{\{v\}, \{u\}\}$. As we have seen above, the possibilities $\{v\}$ and $\{u\}$ are the only possibilities $j \leq i$ such that $j \models p \vee q$. Obviously, $\{v\}$ and $\{u\}$ being happy end possibilities, neither $v < u$ nor $u < v$.

So according to the definition of the proposition expressed by a sentence, these, and just these two possibilities count.

Hence the disjunction $p \vee q$ is an inquisitive sentence.

If we look at the general picture of the meaning of $p \vee q$, if we consider the proposition expressed by $p \vee q$ in the state of ignorance ω in a model $\mathcal{M}$, then assuming that at least in some $v \in \omega$: $\mathcal{I}_v(p) = 1$, and in some $v \in \omega$: $\mathcal{I}_v(q) = 1$, we also obtain two possibilities, the possibility that p and the possibility that q.[26]

$$[p \vee q]_{\mathcal{M}} = \{\{v \in \omega \mid \mathcal{I}_v(p) = 1\}, \{v \in \omega \mid \mathcal{I}_v(q) = 1\}\}$$

There can be many other possibilities $i \leq \omega$ such that $\mathcal{M}, i \models p \vee q$, but for any of such i it will hold that either $i < \{v \in \omega \mid \mathcal{I}_v(p) = 1\}$ or $i < \{v \in \omega \mid \mathcal{I}_v(q) = 1\}$. Hence, according to the definition of the proposition expressed by a sentence, which only selects the maximal possibilities that support a sentence, they will not turn up in $[p \vee q]_{\mathcal{M}}$. Only the largest possibilities that either support p or support q will end up in the proposition expressed by $p \vee q$ in $\mathcal{M}$.

The disjunction $p \vee q$ is not only an inquisitive sentence, but also an informative sentence. Add a world w to our current state $\{v, u\}$, where v and u are as above and w is such that $\mathcal{I}_w(p) = 0$ and $\mathcal{I}_w(q) = 0$. With $i = \{w, v, u\}$ as current state, the proposition expressed by $p \vee q$ in i remains the same as it was above with current state $i = \{v, u\}$. In both cases $i[p \vee q]_{\mathcal{M}} = \{\{v\}, \{u\}\}$. The world $w \in i$ will be in none of the possibilities in the proposition $i[p \vee q]_{\mathcal{M}}$.

Hence, the disjunction $p \vee q$ is a hybrid sentence, $p \vee q$ is both an inquisitive and an informative sentence.[27]

Note that if we add a world z to our current state $\{w, v, u\}$, where w, v and u are as above and z is such that $\mathcal{I}_z(p) = 1$ and $\mathcal{I}_z(q) = 1$. With current state $i = \{z, w, v, u\}$, the proposition expressed by $p \vee q$ in i becomes $i[p \vee q]_{\mathcal{M}} = \{\{v, z\}, \{u, z\}\}$. The possibilities $\{v, z\}$ and $\{u, z\}$ are the largest possibilities $j \leq i$ such that $j \models p \vee q$. The two possibilities in the proposition overlap. The proposition is indifferent with respect to z, z cannot be paired with any other

[26] What comes out as the proposition expressed by a disjunction in inquisitive semantics, is like what comes out in so-called Hamblin semantics, based on Hamblin (1973), which he originally developed in the framework of Montague Grammar for a semantics of questions. More and less recently, Hamblin semantics, or alternative semantics, has been applied in the analysis of several linguistic phenomena, in particular to focus in Rooth (1985, 1992), and to a wide range of other semantic and pragmatic phenomena, e.g., in Aloni (2007), Alonso-Ovalle (2006), Simons (2000, 2001). At a global level, the main difference between inquisitive and alternative semantics is that in inquisitive semantics the alternatives come out of the basic interpretation of, e.g., disjunction, whereas in alternative semantics sets of alternative denotations are constructed besides the ordinary interpretation. My hope is that inquisitive semantics, in combining informativeness and inquisitiveness in a single notion of meaning, offers a more principled way to deal with the kinds of phenomena that alternative semantics has succesfully been applied to.

[27] Interpreted relative to general models, atomic sentences are also hybrid.

world in one of the alternatives in the proposition to form a pair that makes a difference. The only pair that does is the pair consisting of v and u.

Inquisitiveness and Classical Inadequacy We have seen by way of the example $p \vee q$ that there are sentences which do not satisfy [A] and are inquisitive. But there is a necessary connection between having the one property, being inquisitive, and lacking the other, satisfying [A].

Proposition 2. *For all* $\varphi \in \mathcal{L}$: φ *is inquisitive iff* φ *does not satisfy* [A].

In one direction, that if φ satisfies [A], then φ is not inquisitive, this is trivial. If φ satisfies [A], then in any possibility i in a world-based model $\mathcal{M}$, the proposition expressed by φ in i consists of at most of a single possibility: $i[\varphi]_{\mathcal{M}} = \{\{v \in i \mid v \models \varphi\}\}$.

The other direction, that if φ is inquisitive, then φ does not satisfy [A], is not much more complicated. First note: that φ satisfies [A], implies that in any world-based model $\mathcal{M}$, if $\mathcal{M}, i \models \varphi$ & $\mathcal{M}, j \models \varphi$, then so does the join of i and j, i.e., then $\mathcal{M}, i \cup j \models \varphi$.

If φ is inquisitive, then for some information state $\mathcal{M}_i$ in a model $\mathcal{M}$, the proposition expressed by φ in i contains at least two possibilities $j \neq k$ such that $j \models \varphi$ & $k \models \varphi$. For any two such possibilities $j, k \in i[\varphi]_{\mathcal{M}}$, it cannot be the case that $j \cup k \in i[\varphi]_{\mathcal{M}}$. Since, if $j \neq k$, then $j < j \cup k$. By the definition of what a proposition is, it cannot be the case that $j \in i[\varphi]_{\mathcal{M}}$ and $j \cup k \in i[\varphi]_{\mathcal{M}}$, because the definition requires for any two $j, k \in i[\varphi]_{\mathcal{M}}$ that $j \not< k$. So, if φ is inquisitive then there is a model $\mathcal{M}$ such that $\mathcal{M}, j \models \varphi$ & $\mathcal{M}, k \models \varphi$ and $\mathcal{M}, j \cup k \not\models \varphi$.

But that contradicts that φ satisfies [A], because we have seen that that implies that in any model $\mathcal{M}$, if $\mathcal{M}, j \models \varphi$ & $\mathcal{M}, k \models \varphi$, then $\mathcal{M}, j \cup k \models \varphi$.

We have seen by way of the example $p \vee q$ that disjunctions do not satisfy [A] and can be inquisitive. We have also seen that lacking to satisfy [A] and being inquisitive are necessarily connected. It can also be shown that relative to the interpretation of the language in world-based models, disjunction is the only source of inquisitiveness in the language.

In order to show this, we have to look globally and in a quick pace at the properties of negation and implication, which we will consider more slowly and in more detail in sections to follow, leading to the same conclusions as I will derive in a compact way in the next section. That is why I mark it as a preview. You can safely skip it and move on to the discussion of negation and implication.

Preview: Disjunction as *the* Source of Inquisitiveness We show that if it were not for disjunction, any sentence of our propositional language would satisfy [A].[28]

[28] We formulated the property [A] relative to world-based models. So, this proposition is also restricted to that case. When we consider general models, the proposition ceases to hold from the very start, since then atomic sentences do not satisfy [A].

Proposition 3. *Let $\mathcal{L}_{\mathcal{E}}^{\setminus\vee}$ be the disjunction free fragment of $\mathcal{L}_{\mathcal{E}}$.*

For all $\varphi \in \mathcal{L}_{\mathcal{E}}^{\setminus\vee}$: φ satisfies [A].

For all atomic sentences $p \in \mathcal{E}$, it follows immediately from the atomic clause in the world-based semantics that they satisfy [A]. What is also obvious from the semantics of conjunction is that if both φ and ψ satisfy [A], then $\varphi \wedge \psi$ satisfies [A]. Next, we prove two more things:[29,30]

(i) For any sentence $\varphi \in \mathcal{L}_{\mathcal{E}}$, its negation $\neg\varphi$ satisfies [A], irrespective of whether φ as such satisfies [A].
(ii) For any sentence $\varphi \to \psi \in \mathcal{L}_{\mathcal{E}}$, if ψ satisfies [A], then $\varphi \to \psi$ satisfies [A], irrespective of whether φ satisfies [A].

If we have shown these two things, then we have surely shown that all sentences in the disjunction free fragment $\mathcal{L}_{\mathcal{E}}^{\setminus\vee}$ of $\mathcal{L}_{\mathcal{E}}$ satisfy [A]. We wil actually have shown that as long as disjunctions occur inside the scope of a negation, or in the antecedent of an implication in a sentence φ in the full language $\mathcal{L}_{\mathcal{E}}$, then φ will still satisfy [A].

(i) For any sentence $\varphi \in \mathcal{L}_{\mathcal{E}}$, its negation $\neg\varphi$ satisfies [A], irrespective of whether φ as such satisfies [A]. Given that any sentence is stable, we only have to consider the case where for all $v \in i$: $v \models \neg\varphi$, and see whether that implies that $i \models \neg\varphi$. Suppose that: for no $v \in i$: $v \models \varphi$. Stability of φ guarantees that: for all i, j: if $i \leq j$ & $i \not\models \varphi$, then $j \not\models \varphi$. That means that if for all $v \in i$: $v \not\models \varphi$, then also for $j \leq i$: $j \not\models \varphi$, which means that $i \models \neg\varphi$.

(ii) For any sentence $\varphi \to \psi \in \mathcal{L}_{\mathcal{E}}$, if ψ satisfies [A], then $\varphi \to \psi$ satisfies [A], independently of whether φ satisfies [A]. Suppose ψ satisfies [A], then $i \models \varphi \to \psi$

[29] Actually, just the second of the two already suffices, if we had defined $\neg\varphi$ as $\varphi \to \bot$, where we define: $\mathcal{M}, i \models \bot$ iff $i = \emptyset$. Since $\bot$ obviously satisfies [A], if we prove the fact about implication, we have taken care of negation at the same time.

[30] Unless I misread Cresswell (2004), he shows that negation and implication are 'classically respectable', meaning that if their arguments satisfy [A], so does the negation, c.q., implication as a whole. What I show here is stronger: negation is classically adequate, it meets [A] no matter what, and an implication is classically adequate, i.e., satisfies [A], as soon as its consequent satisfies [A]. We can also give a reformulation of [A], which is neutral between world-based and general models stating classical adequancy as:

[A'] $\mathcal{M}, i \models \varphi$ iff for all happy ends h in $\mathcal{M}$: $h \leq i \;\Rightarrow\; \mathcal{M}, h \models \varphi$.

The facts (i) and (ii) shown here for negation and implication for [A], hold for [A'] as well. Meaning that also relative to general models negation and implication are (more than) classically respectable. To the extent that anything non-classical is going on in general models, there are two causes for this: atomic sentences and disjunction. In particular, that in general models $\neg\neg p$ and p are not equivalent, is not to blame on non-classical behavior of negation, but on the non-classical behavior of atomic sentences. In world-based models we tame atomic sentences into classical behavior. The only remaining source for non-classical, i.e., inquisitive behavior is disjunction.

iff for all $j \leq i$: if $j \models \varphi$, then $\forall v \in j$: $v \models \psi$. But then, given that φ is stable, we can just as well restrict quantification to all $v \in i$: $v \models \varphi$, instead of to all $j \leq i$. I.e., if ψ satisfies [A], then $i \models \varphi \to \psi$ iff for all $v \in i$: if $v \models \varphi$, then $v \models \psi$, and hence, $\varphi \to \psi$ satisfies [A] as well.

Many things follow from this logical fact. For example that it is impossible to define disjunction in terms of negation and conjunction or implication. It also tells us that $\neg\neg\varphi$ will not generally be equivalent to φ. The example $p \vee q$ is a case in point. We have seen above that $p \vee q$ does not satisfy [A]. But we have shown above that $\neg\varphi$ satisfies [A], for any sentence φ, hence, $\neg\neg(p \vee q)$ satisfies [A]. But that means that $\neg\neg(p \vee q)$ behaves classically and unlike $p \vee q$ is not inquisitive. End of the quick pace preview, back to a slow pace.

3.3 Negation

The semantic clause for negation says that a possibility supports $\neg\varphi$ iff no restriction of it supports φ:

$$\mathcal{M}, i \models \neg\varphi \text{ iff } \neg\exists j \leq i\colon\ \mathcal{M}, j \models \varphi$$

If no restriction of a possibility i supports φ, then no happy end restriction of i supports φ. Stability tells us that if a possibility does not support a sentence φ, if $i \not\models \varphi$, then it cannot be the case for any j such that $i \leq j$ that $j \models \varphi$. So, to see whether $i \models \neg\varphi$ it suffices to inspect that no happy end restriction of i supports φ. In a world based model the happy end restrictions of a possibility i are the possibilities $\{v\}$ such that $v \in i$. So, relative to world based models negation boils down to the following:

$$\mathcal{M}, i \models \neg\varphi \text{ iff } \neg\exists v \in i\colon\ \mathcal{M}, v \models \varphi$$

In other words, a negation $\neg\varphi$ satisfies [A], and is not inquisitive.

For the proposition expressed by $\neg\varphi$ in an information state $\mathcal{M}_i$ in a world-based model $\mathcal{M}$, this means that $i[\neg\varphi]_{\mathcal{M}}$ can at most consist of a single possibility. This single possibility is the join of all happy end restrictions of i that do not support φ, which in a world-based model is a subset of i.

$$i[\neg\varphi]_{\mathcal{M}} = \{\{v \in i \mid v \not\models \varphi\}\}.$$

Negation behaves completely classically. For any sentences φ the proposition $[\neg\varphi]_{\mathcal{M}}$ that is expressed by $\neg\varphi$ in the state of ignorance in a model $\mathcal{M}$, contains as its only possibility the old-fashioned proposition that $\neg\varphi$ classically expresses.[31]

In discussing propositions in frames, we defined for propositions which are informative in a possibility i, the notion of the possibility excluded by a proposition from $\mathcal{P}_i$, i.e., the set of restrictions of i, as the join of those happy end

[31] Coincidence or not, in dynamic predicate logic (Groenendijk & Stokhof (1991)), negation has a similar 'classical effect' of blocking the dynamics of an existential quantifier.

restrictions of i, which are not restrictions of the join of the possibilities in the proposition.

For a proposition $i[\varphi]_{\mathcal{M}}$ in a world based model this means that the possibility that is excluded by $i[\varphi]_{\mathcal{M}}$ from $\mathcal{P}_i$ is the possibility that consists of the set of worlds $\{v \in i \mid \neg\exists j \in i[\varphi]_{\mathcal{M}}\colon\ v \in j\}$. Not surprisingly, if φ is informative in i, then the possibility excluded by φ in a possibility i is the single possibility in $i[\neg\varphi]_{\mathcal{M}}$.

Consider the example of the proposition expressed by $p \vee q$ in an information state $\mathcal{M}_i$, which above we have seen to be:

$$i[p \vee q]_{\mathcal{M}} = \{\{v \in i \mid \mathcal{I}_v(p) = 1\}, \{v \in i \mid \mathcal{I}_v(q) = 1\}\}$$

For the proposition $i[p \vee q]_{\mathcal{M}}$ we find that the possibility that it excludes is: $\{v \in i \mid \mathcal{I}_v(p) = 0\ \&\ \mathcal{I}_v(q) = 0\}$, which is indeed the sole possibility in $i[\neg(p \vee q)]_{\mathcal{M}}$, as long as $p \vee q$ is informative in i, otherwise $p \vee q$ does not exclude any possibility in i and $i[\neg(p \vee q)]_{\mathcal{M}} = \emptyset$.

Assertions We have seen that in general $\neg\varphi$ satisfies [A] and is not inquisitive, which means that the same holds for $\neg\neg\varphi$. For our example $p \vee q$, which we have seen to be an inquisitive sentence, this means that it cannot be equivalent with $\neg\neg(p \vee q)$, which is not inquisitive. Equivalence of two sentences is defined in the obvious way:

Definition 6 (Equivalence).

$\varphi \Leftrightarrow \psi$ *iff for every model* $\mathcal{M}$ *and possibility* i: $i[\varphi]_{\mathcal{M}} = i[\psi]_{\mathcal{M}}$.

Let us consider what $\neg\neg\varphi$ means. The semantic clause for negation delivers the following:

$\mathcal{M}, i \models \neg\neg\varphi$ iff $\neg\exists j \leq i\colon\ \mathcal{M}, j \models \neg\varphi$, i.e.,
$\mathcal{M}, i \models \neg\neg\varphi$ iff $\neg\exists j \leq i\colon\ \neg\exists k \leq j\colon\ \mathcal{M}, k \models \varphi$, i.e.,
$\mathcal{M}, i \models \neg\neg\varphi$ iff $\forall j \leq i\colon\ \exists k \leq j\colon\ \mathcal{M}, k \models \varphi$

So, what we end up with is that a possibility i supports $\neg\neg\varphi$ iff for every restriction j of i we can find a restriction k that supports φ. Among the restrictions j of i we will find happy end restrictions h, of which the only restriction is h itself. So, what is required is that every happy end restriction of i supports φ. But then stability tells us that this is not only necessary, but also sufficient. In our world based models, the happy end restrictions of a possibility i are the possibilities $\{v\}$ such that $v \in i$. So, what we have is:

$$\mathcal{M}, i \models \neg\neg\varphi \text{ iff } \forall v \in i\colon\ \mathcal{M}, v \models \varphi$$

Not surprisingly, this tells us that $\neg\neg\varphi$ satisfies [A] and is not inquisitive.

If we consider our example $p \vee q$ again, then what we get for the proposition expressed by $\neg\neg(p \vee q)$ in an information state $\mathcal{M}_i$ in a model $\mathcal{M}$ is:

$$i[\neg\neg(p \vee q)]_{\mathcal{M}} = \{\{v \in i \mid \mathcal{I}_v(p) = 1 \text{ or } \mathcal{I}_v(q) = 1\}\}$$

As soon as there are worlds $v, u \in i$ such that $\mathcal{I}_v(p) = 1$ & $\mathcal{I}_v(q) = 0$, and $\mathcal{I}_u(q) = 1$ & $\mathcal{I}_u(p) = 0$, i.e., as soon as $p \vee q$ is inquisitive in a possibility i in a model $\mathcal{M}$, $i[p \vee q]_{\mathcal{M}}$ and $i[\neg\neg(p \vee q)]_{\mathcal{M}}$ are not the same. Which suffices to show that $p \vee q$ and $\neg\neg(p \vee q)$ are not equivalent.

Unlike the proposition $[p \vee q]_{\mathcal{M}}$ expressed by $p \vee q$ in a model $\mathcal{M}$, the proposition $[\neg\neg(p \vee q)]_{\mathcal{M}}$ expressed by $\neg\neg(p \vee q)$ in $\mathcal{M}$ contains as its single possibility the old-fashioned proposition classically expressed by $p \vee q$. The difference between the two is that although $p \vee q$ and $\neg\neg(p \vee q)$ contain the same information, i.e., the possibilities they do and do not exclude are the same, whereas $p \vee q$ is also inquisitive in that it embodies the issue whether p or q, $\neg\neg(p \vee q)$ is indifferent.[32]

Generally, $\neg\neg\varphi$ only concerns information and not issues. This can also be seen from the fact that the single possibility in the proposition expressed by $\neg\neg\varphi$, if $\neg\neg\varphi$ is not absurd, is the join of the possibilities in the proposition expressed by φ.

In discussing propositions in frames, we called the proposition that consists of the single possibility that is the join of the possibilities of a proposition, the *rheme* of that proposition. So, we can also look upon $\neg\neg\varphi$ as the proposition that expresses the rheme of a proposition φ. We introduce the exclamation mark as a separate operator that we add to the language and assign it the meaning we found double negation to have.

Definition 7 (Assertive Closure).

If $\varphi \in \mathcal{L}_{\mathcal{E}}$*, then* $!\varphi \in \mathcal{L}_{\mathcal{E}}$

$\mathcal{M}, i \models\, !\varphi$ *iff* $\forall j \leq i$: $\exists k \leq j$: $\mathcal{M}, k \models \varphi$

We call a sentence of the form $!\varphi$ the assertive closure of φ. I could also have introduced $!\varphi$ as an abbreviation for $\neg\neg\varphi$, but by introducing the operator separately I want to make clear that there need not be a necessary connection between an operator $!\varphi$ that results in a sentence that just expresses the informative content of φ, its rheme, and double negation of that sentence. Now, it is just a fact about the present semantics that $\neg\neg\varphi \Leftrightarrow\, !\varphi$.

On the basis of the following fact, we may also call any sentence that is equivalent to its assertive closure, an assertion. As it so happens assertions are the sentences that satisfy [A].

Proposition 4 (Assertions). $\varphi \Leftrightarrow\, !\varphi$ *iff* φ *is not inquisitive.*

We will mainly make use of the operator to single out the rheme of a sentence.s

Questions We have seen when discussing frames, that if a proposition I is informative, i.e., if I excludes some possibility, then if we add the possibility that I excludes as an alternative to the possibilities already in I, then the result

[32] Relative to general models, you could perhaps put it as follows: $p \vee q$ shows interest in why $p \vee q$, is it because p or because q?

is a non-informative but inquisitive proposition. We called this proposition the *theme of* the proposition I.

That means for a sentence φ, that if φ is informative, then if we add the single possibility in $[\neg\varphi]_{\mathcal{M}}$ to the possibilities in $[\varphi]_{\mathcal{M}}$, then the result is a non-informative inquisitive proposition. This inquisitive proposition is the proposition expressed by $\varphi \vee \neg\varphi$. So, $\varphi \vee \neg\varphi$ expresses the theme of φ.[33]

As we did for the rheme of a sentence, to get at the theme of a sentence we introduce the question mark as a separate operator that we add to the language ans assign it the same interpretation as $\varphi \vee \neg\varphi$ has.

Definition 8 (Inquisitive Closure).

If $\varphi \in \mathcal{L}_{\mathcal{E}}$*, then* $?\varphi \in \mathcal{L}_{\mathcal{E}}$

$\mathcal{M}, i \models ?\varphi$ *iff* $\mathcal{M}, i \models \varphi$ *or* $\neg\exists j \leq j\colon\ \mathcal{M}, i \models \varphi$

We call a sentence of the form $?\varphi$ the inquisitive closure of φ. I could also have introduced $?\varphi$ as an abbreviation for $\varphi \vee \neg\varphi$, but by introducing the operator separately I want to make clear that there need not be a necessary connection between an operator $?\varphi$ that results in a sentence that expresses the theme of φ, and the disjunction of that sentence with its negation. Now, it is just a fact about the present semantics that $\varphi \vee \neg\varphi \Leftrightarrow ?\varphi$.

In the case of $?\varphi$, we will like to read it as corresponding to a question, an interrogative sentence, in natural language. For that reason it is good that we 'officially disconnected' $?\varphi$ from $\varphi \vee \neg\varphi$. Although I do think that such a non-informative disjunction can easily fulfill the same role as the corresponding question does in natural language, and some languages also use such disjunctions for that purpose, we can have some worries about whether the semantic content of $\varphi \vee \neg\varphi$, and hence whether the interpretation we assigned to $?\varphi$ above, tells the whole story.

For example, if we consider what $\neg(\varphi \vee \neg\varphi)$, and hence what $\neg?\varphi$ means given the interpretaton we assigned to it above, the semantics tells us that it is an absurd proposition, a contradiction. We have not discussed implication yet, but it

[33] What we can do in the case of $\varphi \vee \neg\varphi$ to get at $i[\varphi \vee \neg\varphi]_{\mathcal{M}}$ is to simply take: $i[\varphi]_{\mathcal{M}} \cup i[\neg\varphi]_{\mathcal{M}}$. This does not hold in general, it is not always guaranteed that the result of $i[\varphi]_{\mathcal{M}} \cup i[\psi]_{\mathcal{M}}$ is a proposition. What can go wrong is that for some possibility $j \in i[\varphi]_{\mathcal{M}}$ it holds that $j < k$ for some possibility $k \in i[\varphi]_{\mathcal{M}}$, or vice versa. Then both j and k would end up in the union, whereby it does not count as a proposition. However, in the particular case of φ and $\neg\varphi$ there is no risk that this can happen. The unique possibility in $i[\neg\varphi]_{\mathcal{M}}$ is excluded by any possibility in $i[\varphi]_{\mathcal{M}}$.

Taking the union does work at the level of not taking just the largest, but the full set of possibilities that support φ. That is why we also defined the notion $\|\varphi\|_{\mathcal{M},i} = \{j \leq i \mid \mathcal{M}, i \models \varphi\}$. At this level, we get:

$\{j \leq i \mid \mathcal{M}, i \models \varphi \vee \psi\} = \{j \leq i \mid \mathcal{M}, j \models \varphi\} \cup \{j \leq\mid \mathcal{M}, i \models \psi\}$.

We can take $\|\varphi \vee \psi\|_{\mathcal{M},i} = \|\varphi\|_{\mathcal{M},i} \cup \|\psi\|_{\mathcal{M},i}$. Then we can get at the proposition expressed by $\varphi \vee \psi$ by collecting those $j \in \|\varphi \vee \psi\|_{\mathcal{M},i}$ such that there is no $k \in \|\varphi \vee \psi\|_{\mathcal{M},i}$ $j < k$, which is what the operation MAX does: $\text{MAX}(\|\varphi \vee \psi\|_{\mathcal{M},i}) = i[\varphi \vee \psi]_{\mathcal{M}}$.

also holds that $?\varphi \rightarrow !\varphi$ is equivalent with $!\varphi$. It is a fact about natural language, or at least I take it to be that way, that questions, interrogatives, cannot occur under the scope of negation, or as the antecedent of an conditional sentence. (There might be a longer story for conditionals, though.) Restricting myself to the case of negation, I don't think that the fact that $\neg\varphi$ is a contradiction is a sufficient explanation for that.

For such reasons we might search for a variation of the interpretation of $?\varphi$ that just makes a difference between $\varphi \vee \neg\varphi$ and $?\varphi$ such that the one can, and the other cannot, sensibly occur under negation, or as the antecedent of an implication.

I don't want to go into the details of that now, but one option I see is to add a presuppositional element to $?\varphi$ that deems it uninterpretable when evaluated in a happy end. It is a specific feature of the interpretation of negation (and the same holds for the antecedent of an implication) that in the evaluation of $\mathcal{M}, i \models \neg\varphi$ you are bound to also have to inspect happy end restrictions of i, hence you are bound to consider cases where the presupposition I proposed above for $?\varphi$ is not fulfilled, which would cause the uninterpretability of $\neg\varphi$ (and similarly for sentences with questions as the antecedent of an implication). That, I would consider a sufficient explanation for why questions cannot occur under negation (as antecedent of an implication) in natural language.

But I will restrict my use of $?\varphi$ here mainly as a means to obtain the theme of a sentence φ, just as I will mainly use $!\varphi$ to get at the rheme of φ.

On the basis of the following fact, we may also call any sentence that is equivalent to its inquisitive closure, a question. As it so happens questions are the sentences that do not satisfy [A].

Proposition 5 (Questions). $\varphi \Leftrightarrow ?\varphi$ *iff* φ *is not informative.*

One thing this tells us that $??\varphi$ is equivalent with φ, iteration of the question mark is superfluous. The theme of a question is just that question. Note that it is not sufficient to arrive at an *inquisitive* question $?\varphi$ that φ is not informative. For $?\varphi$ to be inquisitive, φ should not be tautological or absurd.

Questions and Partitions The simplest example of a question is the sentence $p \vee \neg p$. The proposition expressed by p in an information state $\mathcal{M}_i$ is $i[p]_{\mathcal{M}} = \{\{v \in i \mid \mathcal{I}_v(p) = 1\}\}$, which will not be the empty set as long as p is not absurd in i. If p is informative in i, then the possibility it excludes is $\{v \in i \mid \mathcal{I}_v(p) = 0\}$, which indeed is the sole possibility in $i[\neg p]_{\mathcal{M}}$, and:

$$i[p \vee \neg p]_{\mathcal{M}} = \{\{v \in i \mid \mathcal{I}_v(p) = 1\}, \{v \in i \mid \mathcal{I}_v(p) = 0\}\}$$

Clearly, this proposition is not informative, the join of the possibilities in $i[p \vee \neg p]_{\mathcal{M}}$ will always equal i. But that does not mean that $p \vee \neg p$ is tautological, i.e., that $i[p \vee \neg p]_{\mathcal{M}} = \{i\}$. When p is informative in i, there will be two possibilities in $i[p \vee \neg p]_{\mathcal{M}}$, and $p \vee \neg p$ is inquisitive in i. Since there is an information state $\mathcal{M}_i$ in a model $\mathcal{M}$ such that $p \vee \neg p$ is inquisitive in i, $p \vee \neg p$ is an inquisitive sentence.

What results as the proposition expressed by $p \vee \neg p$ in a world-based model, the proposition $[p \vee \neg p]_{\mathcal{M}}$, is a bi-partition of the set of possible worlds ω, an old-fashioned yes/no-question.

In a standard partition semantics for questions, interrogative sentences are usually obtained on the basis of a standard assertoric language to which a question operator is added.[34] So a standard representation for the yes/no-question whether p would be $?p$. The interpretation is then obtained by collecting worlds where the denotation of p is the same, which for $?p$ gives you two sets of worlds, the set of worlds where p holds, and the set of worlds where p does not hold. So, the interpretation in a standard partition semantics is based on an equivalence relation on the set of worlds. Equivalence relations give rise to partitions.[35]

You could write down the following clause in our possibility semantics to mirror the partition interpretation of questions:[36]

$$\mathcal{M}, i \models ?^2\varphi \text{ iff } \forall j \leq i\text{: } \exists k \leq j\text{: } \mathcal{M}, k \models \varphi \text{ or } \neg\exists j \leq i\text{: } \mathcal{M}, j \models \varphi$$

In the case of $?^2p$, the interpretation in a partition semantics is the same as we obtain for $p \vee \neg p$ in our world based semantics (and hence the same as for $?p$ under the interpretation we assigned to $?\varphi$ above).[37] Generally, under this interpretation of $?^2\varphi$ we get the same result as for $\varphi \vee \neg\varphi$ as long as φ satisfies [A]. The reason is obvious when we compare the clause for $?^2\varphi$ with the interpretation of $!\varphi \vee \neg\varphi$, which is completely the same, and hence also the same as we obtain for $?!\varphi$. As long as φ is not absurd and not tautological, $?!\varphi$ is a polar questions that has the two answers ‘Yes’ and ‘No’.[38]

[34] This is how things are, e.g., in Groenendijk (1999), building on Groenendijk & Stokhof (1982, 1984), where a partition semantics for (embedded) questions is worked out and motivated in detail.

[35] As we shall see in section 6, the meanings our semantics assigns to the sentences of the language can also be characterized in terms of a relation on the set of worlds, but it is not an equivalence relation. An equivalence relations is reflexive, symmetric and transitive. The relation our semantics gives rise to lacks the property of transitivity.

[36] This is more complicated than needed if we just consider the world-based semantics. What would suffice there is what we wrote down above in the definition of $?\varphi$. That clause would not do for the general semantics, in the sense that there the interpretation of $?\varphi$ would not always correspond to a bi-partition, whereas the clause as I gave it in the text does guarantee that. The case is clear if you just consider $?p$. In the general semantics, p as such is already inquisitive, so $?p$ may lead to a proposition which contains more than two possibilities. If a yes/no-question is a purely informative question, where the answer ‘yes’ just provides the information that, and the answer ‘no’ the information that not, then the clause for $?^2\varphi$, is the one that covers this. What does hold is that you may still say that our interpretation of $?\varphi$ relative the general semantics gives you the theme of φ.

[37] Following up on the previous footnote, in case of the general semantics this should read: ‘is the same as we obtain for $!\varphi \vee \neg\varphi$.’

[38] As has been proved by Christopher Brumwell, if we only have negation, conjunction and the ?-operator in the language, then under the interpretations assigned to these operators here, in the world-based semantics what results is a partition semantics.

Alternative Questions If we consider the thematization of the hybrid disjunction $p \vee q$ we get the following proposition with three possibilities, the possibility hat p, the possibility that q and the possibility that neither p nor q:

$$i[(p \vee q) \vee \neg(p \vee q))]_{\mathcal{M}} = i[?(p \vee q)]_{\mathcal{M}} = \\ \{\{v \in i \mid \mathcal{I}_v(p) = 1\}, \{v \in i \mid \mathcal{I}_v(q) = 1\}, \{v \in i \mid \mathcal{I}_v(p) = 0 \ \& \ \mathcal{I}_v(q) = 0\}\}$$

If we start from $\neg(p \vee q)$, or from $!(p \vee q)$, we get only two possibilities, the possibility that p or q, and the possibility that not p or q:

$$i[\neg(p \vee q) \vee !(p \vee q)]_{\mathcal{M}} = i[?!(p \vee q)]_{\mathcal{M}} \\ \{\{v \in i \mid \mathcal{I}_v(p) = 1 \text{ or } \mathcal{I}_v(q) = 1\}, \{v \in i \mid \mathcal{I}_v(p) = 0 \ \& \ \mathcal{I}_v(q) = 0\}\}$$

The latter proposition has in common with the proposition expressed by $p \vee \neg p$ in a possibility i that the two alternative possibilities are alternatives in a strong sense of the word, they completely exclude each other. That is not necessarily so for the three possibilities for $(p \vee q) \vee \neg(p \vee q)$, the two possiblities that were already present in the proposition expressed by $p \vee q$ may overlap. They will overlap in the proposition $p \vee q$ expresses in a possibility i, if there is a world $v \in i$ such that $\mathcal{I}_v(p) = \mathcal{I}_v(q) = 1$, i.e., if $p \wedge q$ is not absurd in i. The possibility that p and the possibility that q do not exclude each other. They do both exclude that neither p nor q.

Though not all alternative possibilities for $(p \vee q) \vee \neg(p \vee q)$ are alternatives in the strongest sense of the word, they are in the weaker sense that for any alternative possibility $j \in i[(p \vee q) \vee \neg(p \vee q)]_{\mathcal{M}}$ there is a world $v \in j$ such that for no possibility $k \neq j$ and $k \in i[(p \vee q) \vee \neg(p \vee q)]_{\mathcal{M}}$ it holds that $v \in k$. Every alternative possibility for $(p \vee q) \vee \neg(p \vee q)$ has a part that is unique for it. For the alternative that p it is the possibility that p and not q, and for the the alternative that q it is the possibility that q and not p.

But this is not the end of it, alternatives can be alternatives in an even weaker sense. Consider $(p \vee \neg p) \vee (q \vee \neg q)$, i.e., $?p \vee ?q$. There can be four possibilities for this sentence in a possibility i, the possibility that p, that $\neg p$, that q and that $\neg q$. In this case, if all four alternatives are present in the proposition expressed by $(p \vee \neg p) \vee (q \vee \neg q)$ in a possibility i, none of them has a part that it does not share with any of the other alternatives. Take the possibility that p. For any $v \in i$ where $\mathcal{I}_v(p) = 1$, it will either be the case that $\mathcal{I}_v(q) = 1$, or that $\mathcal{I}_v(q) = 0$.

3.4 Implication

In the clause for implication in the definition of the semantics, in evaluating whether a possibility i supports a conditional sentence $\varphi \to \psi$, we quantify over all restrictions j of i:

$$\mathcal{M}, i \models \varphi \to \psi \text{ iff } \forall j \leq i\colon \ \mathcal{M}, j \models \varphi \ \Rightarrow \ \mathcal{M}, j \models \psi$$

In words, a possibility i supports $\varphi \to \psi$ iff all restrictions of i that support the antecedent φ support the consequent ψ as well.

Athough we quantify over all restrictions of i that support the antecedent φ, since like any sentence in our language, the consequent ψ is stable, it suffices to consider *the maximal* possibilities that are restrictions of i that support the antecedent φ, and see whether *they* support the consequent ψ.

The maximal possibilities that are restrictions of i that support φ are the possibilities in the proposition expressed by φ in the information state $\mathcal{M}_i$, i.e., the possibilities in $i[\varphi]_{\mathcal{M}}$. So, we can also write the interpretation of conditional sentences as follows:

$$\mathcal{M}, i \models \varphi \to \psi \text{ iff } \forall j \in i[\varphi]_{\mathcal{M}}\colon\ \mathcal{M}, j \models \psi$$

What holds in general is that:

$$\mathcal{M}, i \models \varphi \text{ iff there is some } j \in [\varphi]_{\mathcal{M}}\colon\ i \leq j$$

In words, a possibility i in $\mathcal{M}$ supports φ iff i is a restriction of a possibility in the proposition expressed by φ in $\mathcal{M}$. This follows immediately from the way in which propositions are defined.

This gives us yet another reformulation of the interpretation of conditional sentences:

$$\mathcal{M}, i \models \varphi \to \psi \text{ iff } \forall j \in i[\varphi]_{\mathcal{M}}\colon\ \exists k \in [\psi]_{\mathcal{M}}\colon\ j \leq k$$

In words, a possibility i in $\mathcal{M}$ supports $\varphi \to \psi$ iff every possibility in the proposition expressed by the antecedent φ in the information state $\mathcal{M}_i$, is a restriction of some possibility in the proposition expressed by the consequent ψ.

Now, the existential quantification over the possibilities in the proposition expressed by ψ only has work to do, if there can be more than one, i.e., if ψ is inquisitive. If the consequent ψ is not inquisitive, and is not absurd, the single possibility we have in $[\psi]_{\mathcal{M}}$ is $\{v \in \omega \mid \mathcal{M}, v \models \psi\}$. If we then require that $j \leq \{v \in \omega \mid \mathcal{M}, v \models \psi\}$, we require that: $\forall v \in j\colon\ \mathcal{M}, v \models \psi$.

But then, irrespective of whether φ is inquisitive or not, quantification over all possibilities in $i[\varphi]_{\mathcal{M}}$, can be reduced to quantification over all worlds $v \in i$ such that $\mathcal{M}, v \models \varphi$. Because the possibilities in $i[\varphi]_{\mathcal{M}}$ are the maximal possibilities, the largest sets of worlds $v \in i$, such that v supports φ, checking for all possibilities in $i[\varphi]_{\mathcal{M}}$ whether all worlds in them support ψ, is the same as checking for all worlds $v \in i$ that support φ, whether v supports ψ as well:

If ψ is not inquisitive: $\mathcal{M}, i \models \varphi \to \psi$ iff $\forall v \in i\colon\ \mathcal{M}, v \models \varphi \Rightarrow \mathcal{M}, v \models \psi$.

If we have relative to a world v that: if $\mathcal{M}, v \models \varphi$, then $\mathcal{M}, v \models \psi$, that means that $\mathcal{M}, v \models \varphi \to \psi$. In other words, the following holds:

If ψ is not inquisitive: $\mathcal{M}, i \models \varphi \to \psi$ iff $\forall v \in i\colon\ \mathcal{M}, v \models \varphi \to \psi$.

But this says that if ψ is not inquisitive, then $\varphi \to \psi$ satisfies [A], and given that to satisfy [A] means not to be inquisitive, we arrive at the conclusion that:

If ψ is not inquisitive, then $\varphi \to \psi$ is not inquisitive.

This means that as long as the consequent of an implication is not inquisitive, the implication as a whole behaves as classical implication, irrespective of the inquisitiveness or non-inquisitiveness of the antecedent.[39]

It is also not difficult to see that the following holds as well.

If ψ is not informative, then $\varphi \to \psi$ is not informative.

If ψ is not informative, then $[\psi]_{\mathcal{M}}$ does not exclude any possibilities. Suppose $\varphi \to \psi$ is informative, then it should be the case that there is a possiblity i such that $i[\neg(\varphi \to \psi)]_{\mathcal{M}} \neq \emptyset$. Then, since negation behaves classically, it should hold that $\exists v \in i\colon \mathcal{M}, v \models \varphi$ & $\mathcal{M}, v \not\models \psi$, and hence that there is some $v \in \omega$: $\mathcal{M}, v \not\models \psi$. But that contradicts that ψ is not informative and does not exclude any possibilities.

This is not surprising, since in classical logic, if ψ is a tautology, then so is $\varphi \to \psi$. But here, being a tautological sentence is not the same as being not informative, but means: neither informative nor inquisitive. The two facts about non-inquisitive consequents and non-informative consequents of an implication, together tell us that in inquisitive logic as well, if the consequent ψ of an implication $\varphi \to \psi$ is tautological, then so is the implication as a whole. But it can very well be the case that though the consequent ψ of an implication $\varphi \to \psi$ is not informative, ψ is inquisitive, and that then $\varphi \to \psi$ is inquisitive as well.

Implications with Inquisitive Consequents Let's return to our formulation of the interpretation of implication in terms of propositions.

$$\mathcal{M}, i \models \varphi \to \psi \text{ iff } \forall j \in i[\varphi]_{\mathcal{M}}\colon \exists k \in i[\psi]_{\mathcal{M}}\colon j \leq k$$

First note that I made a small change in the formulation, I replaced $\exists k \in [\psi]_{\mathcal{M}}\colon j \leq k$ by $\exists k \in i[\psi]_{\mathcal{M}}\colon j \leq k$. This makes no difference. The possibilities k in $i[\psi]_{\mathcal{M}}$ kan be proper restrictions of those in $i[\psi]_{\mathcal{M}}$, but that does not 'make it more difficult' for a possibility $j \in i[\varphi]_{\mathcal{M}}$ 'to find' a possibility $j \leq k\colon k \in i[\psi]_{\mathcal{M}}$, than it was to find such a $k\colon j \leq k$ and $k \in [\psi]_{\mathcal{M}}$, since j must be a restriction of i anyway.

Let us now consider the case where something more interesting may happen, the case where the consequent ψ is inquisitive, simple examples of which are the sentences $p \to (q \vee \neg q)$ and $p \to (q \vee r)$. The first of these we could also write as $p \to ?q$, it corresponds to a conditional question.

In case the antecedent φ of an implication $\varphi \to \psi$ is not inquisitive, as in the two examples we just noted, there will be (at most) a single possibility $j \in i[\varphi]_{\mathcal{M}}$. Assuming that ψ is inquisitive in i, there will be at least two (two for the two illustrating examples) possibilities $k \in i[\psi]_{\mathcal{M}}$. What the clause for implication requires is that j is a restriction of at least one of these alternative possibilities k. So, $j \leq k_1$ or ... or $j \leq k_n$.

[39] In fact this was just a lengthy, but hopefully illuminating remake of something we proved already in the Preview in just a couple of lines.

For the example $p \to (q \vee \neg q)$, or $p \to ?q$, this means that:

$$\{v \in i \mid \mathcal{M}, v \models p\} \subseteq \{\ v \in i \mid \mathcal{M}, v \models q\} \text{ or}$$

$$\{v \in i \mid \mathcal{M}, v \models p\} \subseteq \{\ v \in i \mid \mathcal{M}, v \models \neg q\}$$

I.e., either $\forall v \in i$: $\mathcal{M}, v \models p \Rightarrow \mathcal{M}, v \models q$, or $\forall v \in i$: $\mathcal{M}, v \models p \Rightarrow \mathcal{M}, v \models \neg q$. This says that either $\mathcal{M}, i \models p \to q$, or $\mathcal{M}, i \models p \to \neg q$. which in turn means that $\mathcal{M}, i \models (p \to q) \vee (p \to \neg q)$.

So, what we have in general is that the proposition expressed by the conditional sentence $p \to (q \vee \neg q)$ in an information state $\mathcal{M}_i$ is:[40]

$$i[p \to (q \vee \neg q)]_{\mathcal{M}} = \{\{v \in i \mid \mathcal{M}, v \models p \to q\}, \{v \in i \mid \mathcal{M}, v \models p \to \neg q\}\}$$

In other words $p \to (q \vee \neg q) \Leftrightarrow (p \to q) \vee (p \to \neg q)$. In a sense, this is not surprising, because the equivalence holds in classical logic as well, but the difference is, of course, that unlike in classical logic we have that in inquisitive logic these formulas are not tautological, they are contingent. Not in the sense that they are informative, they are not, but because they are inquisitive.

Similarly, in case of the other example $p \to (q \vee r)$ we get that this means the same as $(p \to q) \vee (p \to r)$. Here, too, it holds that the two are classically equivalent as well, but unlike in the previous example they are not classical tautologies, which means in our semantics that they are informative, they exclude the possibility that $p \wedge \neg q \wedge \neg r$. But they are not only informative, they are inquisitive as well. Both $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$ are hybrid sentences.

The difference with classical semantics shows itself by the fact that whereas classically $p \to (q \vee r)$ and $p \to \neg\neg(q \vee r)$, or $p \to\ !(q \vee r)$ are equivalent, in inquisitive semantics they are not. Whereas $p \to (q \vee r)$ is, $p \to \neg\neg(q \vee r)$ is not inquisitive.

Inquisitive Antecedent and Inquisitive Consequent Consider the example $(p \vee q) \to (r \vee \neg r)$, or $(p \vee q) \to ?r$. In this case, since the antecedent is inquisitief, if we consider the alternative possibilities $j \in i[p \vee q]_{\mathcal{M}}$, there are two if both p and q are not absurd in i and $p \wedge q$ is informative in i:

$j_1 = \{v \in i \mid \mathcal{M}, v \models p\}$;
$j_2 = \{v \in i \mid \mathcal{M}, v \models q\}$.

If it is to hold that $\mathcal{M}, i \models (p \vee q) \to (r \vee \neg r)$, then it has to hold that both j_1 and j_2 are restrictions of a possibility $k \in i[r \vee \neg r]_{\mathcal{M}}$, of which there are two if $r \vee \neg r$ is inquisitive in i:

[40] This result is obtained for the semantics of conditinal questions in Velissaratou (2000). Inquisitive semantics came about as the result of trying to combine her semantics for conditional questions with the partition semantics of Groenendijk (1999). Proposals for a semantics of conditional questions in a partition semantics can be found in Hulstijn (1997) and Isaacs & Rawlins (2008).

$k_1 = \{v \in \omega \mid \mathcal{M}, v \models r\}$;
$k_2 = \{v \in \omega \mid \mathcal{M}, v \models \neg r\}$.

So, if $\mathcal{M}, i \models (p \vee q) \to (r \vee \neg r)$, then both j_1 and j_2 are to be restrictions of either k_1 or in k_2. So we get four possibilities:

$\mathcal{M}, i \models (p \vee q) \to (r \vee \neg r)$ iff

$j_1 \leq k_1$ and $j_2 \leq k_2$; or
$j_1 \leq k_1$ and $j_2 \leq k_2$; or
$j_1 \leq k_2$ and $j_2 \leq k_1$; or
$j_1 \leq k_2$ and $j_2 \leq k_2$.

Each of these four cases corresponds to a situation where i supports two implications:

$\mathcal{M}, i \models (p \vee q) \to (r \vee \neg r)$ iff

$\mathcal{M}, i \models p \to r$ and $\mathcal{M}, i \models q \to r$; or
$\mathcal{M}, i \models p \to r$ and $\mathcal{M}, i \models q \to \neg r$; or
$\mathcal{M}, i \models p \to \neg r$ and $\mathcal{M}, i \models q \to r$; or
$\mathcal{M}, i \models p \to \neg r$ and $\mathcal{M}, i \models q \to \neg r$.

Given the interpretation of conjunction:

$\mathcal{M}, i \models (p \vee q) \to (r \vee \neg r)$ iff

$\mathcal{M}, i \models (p \to r) \wedge (q \to r)$; or
$\mathcal{M}, i \models (p \to r) \wedge (q \to \neg r)$; or
$\mathcal{M}, i \models (p \to \neg r) \wedge (q \to r)$; or
$\mathcal{M}, i \models p \to \neg r) \wedge (q \to \neg r)$.

Given the interpretation of disjunction:

$\mathcal{M}, i \models (p \vee q) \to (r \vee \neg r)$ iff $\mathcal{M}, i \models$

$(p \to r) \wedge (q \to r) \vee$
$(p \to r) \wedge (q \to \neg r) \vee$
$(p \to \neg r) \wedge (q \to r) \vee$
$(p \to \neg r) \wedge (q \to \neg r)$.

What this says is that there can be four possibilities in $i[(p \vee q) \to (r \vee \neg r)]_{\mathcal{M}}$. The sentence is inquisitive, but not informative.

There is more to say on the semantics, but we have to leave it to this now, and move on to the logical notion of compliance.

4 Compliance

We have discussed the inquisitive semantics for our propositional language. But our larger aim is to model the dynamic process of information exchange. So, we have to make the move from single sentences to sequences of them that constitute a dialogue, where we want to model how the moves in a dialogue build a common ground.

We have already announced that we will model the common ground as a stack of information stages. In discussing propositions in frames we introduced the notion of an stage as a triple (i, I, j), where i is the possibility that is the current state, I is a proposition for the current state i, and j is the join of the alternative possibilities in the proposition I. In discussing propositions in models, stages became information stages $(i, i[\varphi]_{\mathcal{M}}, j)$, where the proposition in a stage is a proposition as expressed by a sentence φ of the language in the current state.

Note that for an information stage $(i, i[\varphi]_{\mathcal{M}}, j)$, we can also 'express the join j' in the language, j is the single possibility in the proposition $i[!\varphi]_{\mathcal{M}}$, the rheme of φ, the information that φ provides in i.

A subsequent stage for (i, I, j) takes j as the new current state, and hence is a stage (j, J, k). Filling this with a proposition expressed by a sentence as well, two subsequent information stages may look like this: $(i, i[\varphi]_{\mathcal{M}}, j)$ followed by $(j, j[\psi]_{\mathcal{M}}, k)$. Don't take this to mean that if an utterance of φ is followed by an utterance of ψ that these two stages as such will be subsequent stages in the corresponding common ground stack. Things are more intricate. But that is the topic of the next section.

The topic of the present section is to introduce the logical notions that compare two stages, compare two propositions, from the perspective of whether one can be seen as a move that is compliant to the other or not.

4.1 Difference and Inquisitiveness

It is the notion of a proposition as the set of maximal possibilities that support a sentence, that gives rise to the simple characterization of an inquisitive proposition as a proposition that contains more than one possibility. But this is not a very fine grained characterization of inquisitiveness. It does not give you a handle to compare two propositions and see whether one is more inquisitive than another. Just comparing the number of possibilities in two propositions would not get you very far.

We are in need of such a notion of comparative inquisitiveness, e.g., if we want to be able to judge whether a move in a dialogue is one which replaces the current issue by a subissue which is easier to answer.

We can arrive at another characterization of inquisitiveness which does give the means to compare inquisitiveness. We can characterize an inquisitive proposition as a proposition such that there are pairs of possibilities that make a difference. Like other notions in this section, I define the notion of difference relative to stages in frames, but it can easily be seen to apply to propositions expressed by sentences, and to sentences as such.

Definition 9 (Difference). *Let (i, I, i') be a stage, and let i and j be two possibilities such that $j \leq i'$ and $k \leq i'$.*

> j and k make a difference in (i, I, i') *iff there is a pair of possibilities $j', k' \in I$ such that $j \leq j'$ & $k \leq k'$ and there is no possibility $l \in I$ such that $j \leq l$ & $k \leq l$.*

We will also say that i and *j* make a difference in a proposition *I* for *i, when i and j make a difference in (i, I, i'). And, similarly, for i* and *j* make a difference in *$i[\varphi]_{\mathcal{M}}$* or in *$[\varphi]_{\mathcal{M}}$.*

Note that if a proposition I for i contains several alternatives, then any such pair $j, k \in I$ counts as a pair of possibilities that make a difference in I. Conversely, when I is not inquisitive in i, you will not be able to find two possibilities i, j that make a difference. But what gives us the fine grainedness we were looking for is that many more pairs of possibilities than just the alternatives in I count as pairs of possibilities that make a difference in I.

For example any pair of happy ends where one only contains a world v such that $\mathcal{I}_v(p) = 1$ and the other only contains a world u such that $\mathcal{I}_u(p) = 0$, is a pair of possibilities that makes a diference in $[p \vee \neg p]_{\mathcal{M}}$. And any pair of happy ends where one only contains a world v such that $\mathcal{I}_v(p) = 1$ and $\mathcal{I}_v(q) = 0$, and the other only contains a world u such that $\mathcal{I}_u(p) = 0$ and $\mathcal{I}_u(q) = 1$, is a pair of possibilities that makes a diference in $[p \vee q]_{\mathcal{M}}$. On the other hand, a happy end that only contains a world z such that $\mathcal{I}_z(p) = 1$ and $\mathcal{I}_z(q) = 1$ will not be an element of any pair of possibilities that makes a difference in $[p \vee q]_{\mathcal{M}}$. This possibility is in the overlap of both alternatives in $[p \vee q]_{\mathcal{M}}$.

I illustrated the fine grainedness of the notion of inquisitiveness in terms of pairs of possibilities making a difference in a proposition, but you only have to consider pairs of happy ends to determine whether a proposition is inquisitive.

Proposition 6. *I is an inquisitive proposition iff there is a pair of happy ends that make a difference in I.*

If there are two happy ends h and e that make a difference in I, then there must be more than one alternative in I. Both h and e should be restrictions of some, but not of the same alternative in I. Then there must be more than one.

The other direction: If I is inquisitive, then there is a pair of happy ends that makes a difference. Suppose this were not so. Then there could be an inquisitive proposition I, where for any two happy ends h and e that are restrictions of some alternative possibilities $i, j \in I$, there is an alternative possibility $k \in I$: $h \leq k$ & $e \leq k$. Suppose there was such an alternative $k \in I$. Then every happy end that would be a restriction of some alternative in I would be a restriction of k. But that would mean that k is the join of I. But that contradicts hat k is an alternative in I.

This may seem a rather obvious fact, hardly worth of spending half a page on it, but we will see later (in section 6) that is has important consequences. The notion of pairs of possibilities making a difference in a proposition, that gives us the tools to characterize the notion of inquisitiveness that is the crucial notion in the semantics, looks upon the meaning of sentences 'in a relational way'. What this logical fact says is that you do not have to consider the full relation of pairs of possibilities, but that you have enough to go by if you restrict yourself to the relation on happy ends. For the possible worlds semantics it suggests that just pairs of worlds, rather than our possibilities, non empty sets of all sizes, might suffice as points of evaluation in the semantics. We will show in section 6, on the basis of the fact discussed here, that this is indeed the case.

4.2 Inquisitiveness and Relatedness

After this little digression, let us return to the issue at hand. We introduced the notion of difference to arrive at a characterization of inquisitiveness to make it possible to compare inquisitiveness of two propositions, more in particular to be able to judge whether a proposition embodies an easier to answer subissue of the current issue. The notion of comparative inquisitiveness that the notion of two possibilities making a difference in a proposition gives rise to is that 'less inquisitive' means 'less pairs that make a difference'. We will incorporate that into our logical notion of compliance that is to judge whether an utterance is compliant to the current issue.

But we need another notion next to less inquisitiveness in order to determine compliance to the current issue. If only because less inquisitiveness demands nothing of purely informative non-inquisitive propositions, but as we shall see, we also need it to determine of inquisitive propositions, next to less inquisitiveness, whether they embody a less inquisitive *subissue* of the current issue.

The second notion that plays a role is *relatedness*, which requires of an informative proposition that the information it provides really addresses the current issue, whether it is an answer, at least a partial answer to the current issue. If a proposition is informative and not inquisitive, and hence contains a single possibility, it provides a complete answer to the current issue if that possibility equals one of the possibilities in the current issue. It provides a partial answer if that possibility is the join of some of the possibilities in the current issue. The general picture is then, for propositions that are informative and/or inquisitive, that each possibility in the proposition equals the join of a subset of the set of propositions in the current issue.

As I announced, we need relatedness also to determine subissues of the current issue. Consider the example where the current issue is $[p \vee \neg p]_{\mathcal{M}}$ (or $[?p]_{\mathcal{M}}$), and the proposition of which we are to evaluate whether it is a less inquisitive subissue is $[(p \vee \neg p) \vee (q \vee \neg q)]_{\mathcal{M}}$ (or $[?p \vee ?q]_{\mathcal{M}}$). The latter is less inquisitive than the former. In $[p \vee \neg p]_{\mathcal{M}}$ every two happy ends h and e such that p has a different value in the world in that happy end forms a pair of possibilities that makes a difference. But if the value of q in the worlds in h and e is the same, h and e, then they do not form a pair that makes a difference in $[(p \vee \neg p) \vee (q \vee \neg q)]_{\mathcal{M}}$.

They are both in the possibility that q (or both in the possibility that not q) in the proposition $[(p \vee \neg p) \vee (q \vee \neg q)]_{\mathcal{M}}$.

However, although $[(p \vee \neg p) \vee (q \vee \neg q)]_{\mathcal{M}}$ may be less inquisitive than $[p \vee \neg p]_{\mathcal{M}}$, we don't want to consider the former to be a *subissue* of the latter. Well, relatedness takes care of that. The possibilities that q and that not q, which are contained in $[(p \vee \neg p) \vee (q \vee \neg q)]_{\mathcal{M}}$, do not cover possibilities in $[p \vee \neg p]_{\mathcal{M}}$.

Relatedness on its own is also not capable of characterizing *easier to answer* subissues. If the current issue is $[(p \vee \neg p) \vee (q \vee \neg q)]_{\mathcal{M}}$, then $[p \vee \neg p]_{\mathcal{M}}$ is related to that. The two possibilities in the latter are also possibilities in the former. But with the aid of the additional requirement of less inquisitiveness $[p \vee \neg p]_{\mathcal{M}}$ does not count as proposing an easier to answer subissue, since as we have seen, $[p \vee \neg p]_{\mathcal{M}}$ is more inquisitive than $[(p \vee \neg p) \vee (q \vee \neg q)]_{\mathcal{M}}$.

We combine the notions of less inquisitiveness and relatedness together in the notion of compliance. We formulate the definition for subsequent stages, but since what these are is fully determind by the state where we start and the propositions in the stages they apply just as well to propositions, or for that matter, to sentences expressing these propositions.

Definition 10 (Relatedness, Inquisitiveness and Compliance).
Let $s = (i, I, j)$ and $r = (j, J, k)$ be two subsequent stages.

1. r is related to s *iff for all $k \in J$: k is the join of some $K \subseteq I$*
2. r is at most as inquisitive as s *iff every pair of possibilities that makes a difference in r makes a difference in s as well.*
3. r is compliant to s *iff r is related to s and r is at most as inquisitive as s.*

Compliance is a very strict notion, as it should be as a logical notion. In particular, compliance characterizes propositions which provide more information than the current issue asks for, as non-compliant, $p \wedge q$ is not compliant to $p \vee \neg\neg p$, nor, for that matter, to $p \vee q$. There is nothing particularly bad about that, we will take them up in the common ground stack just like that. At the same time, if the logical notion also is to have empirical linguistic relevance, it should be the case, as I believe it is, that such over-informative reactions to the current issue count as marked cases. We shall return to the issue later.

Compliance gives an absolute logical criterion to judge whether a proposition addresses the current issue, but not all propositions are equally compliant. E.g., an informative compliant proposition is to be preferred over a purely inquisitive one which counters an issue with a subissue. A proposition that more fully resolves the current issue is preferred over one that does less so. Within the bounds of related information, the more information the better.

I also take it to be the case that less inquisitiveness, within the bounds of relatedness, is preferred over more. It may certainly not be obvious from an empirical point of view whether dialogues in natural language really care about this. I think so, though. From a 'logical conversational' point of view it makes a lot of sense, if you take it, as is our intention, that the current issue is (basically) brought about by the utterance of one participant, and the proposition of which we are judging comparative compliance originates from an utterance of the other participant. An inquisitive, certainly an inquisitive and non-informative proposition in response to the current issue only makes sense if you believe that this subissue of the current issue is one that might be answerable by the person who created the current issue. This *can* be the case, *partially.* The less inquisitive a counter question is, the more likely it is that this particular *part* of the current issue might be answerable by the person who created the current issue.

For such logical reasons, the notion of comparative compliance is defined the way it is.

Definition 11 (Comparative Compliance). *Let s be a stage, and r and r' two stages subsequent to s.*

r is at least as compliant to s as r' iff

1. *r is compliant to s and r' is compliant to s;*
2. *r is at least as informative as r';*
3. *r is at most as inquisitive as r'.*

We leave the empirical issues for what they are, and move on to what our logical notions apply to, building up the common ground stack.

5 Dynamic Dialogue Management

5.1 Stages in Stacks

In modelling the proceedings of a dialogue, the basic building blocks are *stages*, and their basic ingredients are propositions, the semantic content of a sentence uttered in the dialogue, in the context of the current state of information. In a stage we also explicitly register what new current state of information the proposition proposes, information that is already implicity there in the proposition as such, it is given by the join of the alternative possibilities in the proposition. These alternative possiblities in the proposition, if such there are, if the proposition is inquisitive in the context, propose a new current issue to direct the dialogue.

We met the notion of stages before, our findings are repeated in the definition below, which is stated relative to frames.

Definition 12 (Stages). *Let $\mathcal{F}$ be an information frame.*

1. *A* stage *in $\mathcal{F}$ is a triple (i, I, j) such that i is a possibility in $\mathcal{F}$, I is a proposition in $\mathcal{F}_i$, and j is the join of I.*
2. *Two stages $s = (i, I, j)$ and $s' = (i', I', j')$ in $\mathcal{F}$ are* subsequent *iff $i' = j$.*
3. The initial stage *in $\mathcal{F}$ is $(\omega, \imath, \omega)$.*
4. *A stage $s = (i, I, j)$ in $\mathcal{F}$ is a* ground stage *iff $I \subseteq \imath$.*

5. *A stage $s = (i, I, j)$ in $\mathcal{F}$ is a* final stage *iff s is a ground stage and j is a happy end.*

The first element in a stage (i, I, j) is the possibility that represents the current state of information, as established by what went before. If we add a language and a model to the picture, the second element will be a proposition $i[\varphi]_{\mathcal{M}}$, the proposition expressed by a sentence φ in the current state of information i. The last element j, being the join of the possibilities in the proposition $i[\varphi]_{\mathcal{M}}$, gives the new current state of information, relative to which we can determine the proposition expressed by the next sentence.

In the initial stage we begin with The Big Question, the proposition which has all the happy ends in the frame as its alternative possibilities. The initial current state of information is the state of ignorance ω. In case a dialogue proceeds happily, we may joinedly succeed in eliminating certain possiblities, which will lead to a better informed ground stage, where The Big Question, is just a big question. If all goes extremely well, we might end up in a final stage, where we have reached a state of complete information, where nu further issues remain.

If you were to represent a sequence of stages, starting from a ground stage, then there is a lot of superfluous information that you will want to suppress. Since every ground stage starts with ω, why mention it? And since the last element in any stage (i, I, j) is j the join of I, which can be read from I as such, why mention it? Then, in general, since what is to be the first element of a stage can be read from the proposition in the preceeding state, it would suffice to just represent the sequence of the propositions in the stages to be able to obtain all the information that determines the triples in each stage. However, from the viewpoint of defining things easily, the official triples in stages are helpful.

We will model the common ground as a stack of subsequent stages. The bottom of the stack consists of a language and a model that interprets the language: $(\mathcal{M}, \mathcal{L}_{\mathcal{E}})$. Directly on top of that we find a ground stage that represents the current state of established common information, the 'real' common ground at the present stage of the dialogue, and the big question that still remains. Further subsequent stages are propositions that are still under consideration, that holds in particular for the stage on top, for the proposition in the stage on top of the stack.

Definition 13 (Common Ground Stacks). *Let $\mathcal{M}$ be a model for a language $\mathcal{L}_{\mathcal{E}}$ based on a frame $\mathcal{F}$, and s a ground stage in $\mathcal{F}$. The set of* common ground stacks for $\mathcal{M}$ and $\mathcal{L}_{\mathcal{E}}$ *is the set such that:*

1. *$\langle(\mathcal{M}, \mathcal{L}_{\mathcal{E}}), s\rangle$ is a common ground stack.*
2. *if $\langle\sigma, s\rangle$ is a common ground stack, and r is a subsequent stage of s in $\mathcal{F}$, then $\langle\langle\sigma, s\rangle, r\rangle$ is a common ground stack.*

We assume that a dialogue always starts from an initial common ground stack $\langle(\mathcal{M}, \mathcal{L}_{\mathcal{E}}), s\rangle$, where s is the initial stage. You may take the presence of a language and its interpretation at the bottom of the stack to embody the assumption that the participants in the dialogue fully share a language and its interpretation. That is always assumed to be part of their common grond.

But the presence of the language and the model at the bottom of the stack also allows us to let the rest of the stack, the stages, to just consist of elements from the frame, pegs to hang information on, the model at the bottom can hang the information on them, takes care of the interpretation of the elements in the stages.

Once we are beyond the initial stage, the current stage s on top of a stack $\langle\sigma, s\rangle$ will be a stage $(i, i[\varphi], j)$, where j is the current state of information as established by σ, $i[\varphi]_{\mathcal{M}}$. As I indicated, when representing the proposition in the stack, we can omit reference to the model $\mathcal{M}$, since we can retrieve that from the basis of the stack. Finally, j is the join of the possibilities in $i[\varphi]$, and if we compare i and j we can see whether the proposition $i[\varphi]$ is informative in i, which will be the case if $j < i$. (We could also say $j \neq i$, since it will always be the case that $j \leq i$.)

5.2 Operations on Stacks

We will define two types of operations on a stack σ, that always come in tandem after each other. The first type of operation is that when a sentence φ is uttered by one participant as a move in the dialogue, there will be an operation on σ that performs the *uptake* of the proposition expressed by φ in the current stack σ. The next step is that the other participant has to react, explicitly or implicitly, to this move. We will distinguish three possibilities for that: *cancellation*, *acceptance*, and *support*. The second type of operations concerns the *absorption of the reaction* in the stack.

The reactions, in particular the option of cancellation, are essential in making sure that the common ground stack really remains a common ground. In particular when the current proposition made by one of the participants is informative it is important to know whether the other participant can, or is willing to, accept the information provided. If not, this should be signalled by calling for cancellation, which will basically lead to a pop of the stack, removing the last stage that resulted from the uptake of the sentence uttered by the other participant. More about this later, we now first turn to the uptake of a sentence.

Uptake The uptake of a sentence φ in a common ground stack with top s, consists of two subsequent pushes of the stack. The first is called *thematizing* φ and puts a new stage on top of the stack where we add the theme $?\varphi$ of φ to the current issue in s. (There will always be one.) The second is called *assuming* φ and consists in another push of the stack where we perform a hypothetical update of the current state in s with φ as such. Note, also inquisitive sentences, questions, can be 'assumed'.

Definition 14 (Uptake). *Let $\langle\sigma, s\rangle$ be a common ground stack for $(\mathcal{M}, \mathcal{L}_{\mathcal{E}})$, $\varphi \in \mathcal{L}_{\mathcal{E}}$, and $s = (i, I, j)$.*

$\langle\sigma, s\rangle[\varphi]^{\Uparrow} = \langle\langle\langle\sigma, s\rangle, s[\varphi]^{?}\rangle, s[\varphi]^{\uparrow}\rangle$, *where*
$s[\varphi]^{?} = (j, \text{MAX}(\text{EXP}(I) \cup \|?\varphi\|_{\mathcal{M},j}, j)$
$s[\varphi]^{\uparrow} = (j, j[\varphi]_{\mathcal{M}}, k)$, *where k is the join of $j[\varphi]_{\mathcal{M}}$.*

The last part, to assume φ is rather straightforward. The new stage that is put on top takes the current state j we find in the old top s as its starting point, we add the proposition expressed by φ relative to that possibility, and register the new current state, being the join of the possibilities in the proposition.

Thematizing looks a bit more complicated. What it wants to achieve, is that if the current issue can be represented by the question $?\psi$, thematization of φ leads to the issue that can then be expressed by the *disjunction* of questions $?\psi \vee ?\varphi$. I explain the definition first, and then motivate this a bit. But note already that the effect of thematization will always meet one aspect of compliance: a disjunction of two questions cannot fail to be at most as inquisitive as each of its disjuncts. The only thing that can go 'wrong' in terms of compliance is that the stage resulting from thematization is not related to the current issue.

In discussing questions and disjunction in the semantics, we noted that at the level of propositions expressed by sentences, it does not hold in general that $i[\varphi \vee \psi]_{\mathcal{M}} = i[\varphi]_{\mathcal{M}} \cup i[\psi]_{\mathcal{M}}$, since it is not guaranteed that the outcome has the properties of a proposition. What does hold at the level of meanings is that $\|\varphi \vee \psi\|_{\mathcal{M},i} = \|\varphi\|_{\mathcal{M},i} \cup \|\psi\|_{\mathcal{M},i}$. From this we can arrive at the proposition expreesed by $\varphi \vee \psi$ by taking the minimal elements in $\|\varphi\|_{\mathcal{M},i} \cup \|\psi\|_{\mathcal{M},i}$, which is what the operation $\text{MAX}(\|\varphi\|_{\mathcal{M},i} \cup \|\psi\|_{\mathcal{M},i})$ delivers.

So, basically, what is happening in thematizing φ is that we construct a proposition which is the disjunction of the proposition I from the stage on the old top of the stack and the theme $?\varphi$ of the new sentence φ that has been uttered. If the old stage s looked liked this: $(i, i[\psi], j)$, then the effect of thematizing φ will be $(j, j[\psi \vee ?\varphi], j)$.[41]

Note that we have to take the current state j created by the proposition in s as starting point for the stage that is to result from thematizing to make a proper subsequent stage. We can be assured that the join of the proposition $j[\psi \vee ?\varphi]$ will equal j. If i differs from j in the old stage s, then the proposition expressed by ψ in i is informative, but relative to j it is not, because then j already contains the information that ψ provided relative to i. The question $?\varphi$ that we add also makes sure that no information can be added by the stage of thematizing φ. From this we see, that we can also be assured, as the definition takes for granted, that in the next stage where φ is assumed we can still start from the current state as created in s. If φ as such is informative, the end result may be a different more informed current state.

[41] You cannot put the definition of thematizing in this form, assuming that the previous proposition was $i[\psi]$ for some sentence ψ. In the definition you have to put things in purely semantic terms. Just think of the proposition in the initial stage. That does not come about as the result of the utterance of any sentence. It is not even obvious whether it is in general something that can be expressed by a sentence.

Thematization Let me try to give a bit of motivation for thematization, where first I want to say that the way I deal with it now is not the end of the story but just the beginning of it. Where there is a lot of room for further investigations, empirical and logical, is what precisely determines the theme of an utterance. I just take it to be the contents of $?\varphi$ for a sentence φ of our propositional language, but I do not really believe that things are as simple as that. Just to mention the most obvious thing, sentences in natural language can have focus, or other such features, that essentially have to do with determining the theme of the sentence and/or the issue presupposed by the sentence.

Having said his, let us turn to the way in which thematization is presently dealt with. It is tempting to think of the stage that results from thematization as creating the new current issue after the utterance of a sentece φ, relative to the previous current issue, but hat is not really so. The result of thematizing φ will not be on top after the uptake of φ, on top is the result of to assume φ, and the top of the stack determines the current issue.

The result of thematization is more like a 'back up' issue that comes in force if the top of the stack were to be popped. We still have to see how this works, but this happens if the reaction upon the utterance of φ by the one participant is a call for cancellation by the other. Most standardly so, because the other participant cannot accept the information that φ provides, but cancelling a question is possible as well. Only a call for cancellation after the utterance of φ, leading to the removal of the result of to assume φ from the stack, will turn the result of thematization into the current issue.

One of the basic effects of this is that critical dialogue moves are characterized by the model as compliant, as cooperative dialogue moves. Suppose the dialogue has started with a simple assertion p. Thematization of p in the initial stage leads to the stage $(\omega, \omega[p \vee \neg p], \omega)$, assumption leads further to the subsequent stage $(\omega, \omega[p], \omega)$. (Everything compliant by the way.)

If p is inconsistent with the information of the other participant, or if for other reasons she cannot or does not want to update her own state with p, it is essential that she publicly announces cancellation, otherwise the common ground as it is constructed by the dialogue fails to be a 'real' common ground. From the common interest in maintaining a common ground, cooperativity requires the call for cancellation.

This puts the issue $(\omega, \omega[p \vee \neg p], \omega)$ on top. A *most* compliant move to make now is to utter $\neg p$, which is what the other participant can and should do according to the rules, in the case where p was indeed inconsistent with her information because it supports $\neg p$.

This as such gives good reasons to have some operation of thematizing a sentence that goes ahead of assuming its content, but not for the specific 'disjunctive' nature that we assigned to it. Well, to some extent it does. If you go along with the idea that The Big Question is the initial issue (which not everyone finds equally easy to buy), and you go along with the idea that there should be some operation that works on the current issue and the theme of the sentence to create a new issue, then I see no other option but to choose for a disjunction of questions. (For a start *conjunction* would be the worst idea.) More generally, if a sentence constitutes a less inquisitive subissue of the current issue, a compliant move, it is the disjunction of the two that precisely gives the result that just the subissue as such is the result of thematization.

My original motivation for adding $?\varphi$ to the current issue the way thematization does, and not to just take $?\varphi$ as such as new current theme, comes from conditional questions. Suppose the dialogue has opened with a conditional question $p \to ?q$. Then $p \to q$ counts as one of the two answers. So, suppose the question is followed by this answer. The theme of $p \to q$ is the questioned conditional $?(p \to q)$. Suppose we take this to be the new current issue. That would mean that a critical response to the answer $p \to q$ would address $?(p \to q)$. But the most natural critical response to $p \to q$ is not its negation "No, $\neg(p \to q)$", but rather the other answer to the conditional question $p \to ?q$, i.e.: "No, $p \to \neg q$". This is precisely what adding the theme $?(p \to q)$ of $p \to q$ to the original issue $p \to ?q$ delivers. The disjunction of these two questions is equivalent with the original conditional question. I.e., if the current issue is $p \to ?q$, then thematization of $p \to q$ relative to this, leaves the current issue as it was.[42]

A second example that motivated me in opting for the definition of thematization as it is, concerns a response with $?q$ to an initial question $?p$. (Or after an initial assertion q followed by a reaction of cancellation.) This is a clear case of a non-compliant response, at least, according to the logical rules. But that does not exclude that it is a cooperative move in the dialogue. If interpreted as such, the counterquestion implicates that although the responder has no direct answer to the initial question $?p$, if she had an answer to her question $?q$ she might be able to come up with something of an answer to the initial question $?p$ after all. She might for example have the information that $q \to p$.

Where the way thematization is defined starts playing a role is when the participant who asked the initial question ?, and got no direct answer, in turn has no direct answer to the counterquestion $?q$. Cancelling that question ("Well,...", "I don't know but,...") brings her back to the result of thematizing $?q$ after $?p$, which according to the definition is $?p \vee ?q$. Now, you may think that does not get her very far. Given that she asked $?p$ herself, and cancelled $?q$, she has none

[42] Though this was my initial motivation for defining thematizaton the way I do, I am not so convinced by this example anymore. The reason is that I believe that the theme of $\varphi \to \psi$ *as such*, in general, irrespective of the nature of its antecedent and consequent, should by default already come out as $\varphi \to ?\psi$. I have shown in Groenendijk (2008), that it does hold in general that $\varphi \to \psi$ is equivalent with its division in the conjunction of the assertion $\varphi \to !\psi$ and the question $\varphi \to ?\psi$. That may be interesting, but I don't see yet how that fact can be used to adapt the general nature of division in theme and rheme of a sentence to this. Anyway, this does not go against the nature of thematization as I have it, it rather invites to investigate what is the best general procedure is to define what the theme of a sentence is. Just taking the disjunction with its negation as theme seems to crude. The intuition is that one way or the other among the propositions that are options for serving as the background question, the least inquisitive one is to be chosen. But I don't know yet how to go about this.

of the four complete answers to $?p \vee ?q$. Right, but this disjunction of questions has quite a few partial answers like $p \vee q$, and $q \rightarrow p$. Some of these might help the other person to come up with an answer "Then p" to the original question $?p$.[43] Which might, or might not, be acceptable to the stimulator. But if it is, her original question $?p$ is resolved.

It is the disjunction of the two questions $?p \vee ?q$ that results from thematization that precisely gives room for compliant responses with propositions that express information about dependencies between p and q of the sort that were implicated by the counter question $?q$ to $?p$ to be potentially helpful.

But let me end these comments on thematization with what I started with. There is a lot more room here for investigations in the nature of the theme of a sentence and hence in the nature of thematization as well.

Absorption We now turn to the operations on stacks that absorp the reactions of cancellation, acceptance and support. For the latter two, we define an auxiliary notion of restricting a stage s to the information present in a state r.

Definition 15 (Restriction). *Let $s = (i, I, j)$ and $r = (j, J, k)$ be two subsequent stages. The* restriction of s to r *is:*

$$s\lceil r\rceil = (i, \textsc{max}(\textsc{exp}(I) \cap \textsc{exp}(\{k\}), k))$$

In the definition of restriction we find the same kind of pattern as we met in the definition of thematizing, in the sense that we have to shift from the level of propositions to the level of meanings, perform the operation of intersection there, which corresponds to conjunction, and in the end shift back again to the propositional level.

Here, too, it may help to see what is going on, by considering an example of a proposition as expressed by a sentence. So, suppose $r = (j, j[\varphi], k)$. The resulting state k is the join of the possibilities in $j[\varphi]$. Then $\{k\}$ can be nothing but $j[!\varphi]$, since the join of $j[\varphi]$ equals the sole possibility in $j[!\varphi]$. The possibility k represents the informative content of φ relative to j. If $\{k\} = j[!\varphi]$, then $\textsc{exp}(\{k\}) = \|!\varphi\|_{\mathcal{M},j}$.

Suppose that the proposition I in s is $i[\psi]_{\mathcal{M}}$, then $\textsc{exp}(I) = \|\psi\|_{\mathcal{M},i}$. And $\|\psi\|_{\mathcal{M},i} \cap \|!\varphi\|_{\mathcal{M},j} = \|\psi \wedge !\varphi\|_{\mathcal{M},i}$. Note that although i and j may differ we can consider the conjunction relative to i, since it can only be the case that $j < i$. Finally, what $\textsc{max}(\|\psi \wedge !\varphi\|_{\mathcal{M},i})$ will deliver is the proposition $i[\psi \wedge !\varphi]_{\mathcal{M}}$.

[43] This is another story, but the *Then* in front of the response is rather essential. It signals that the responder draws the conclusion that p on the basis of combining her own information and information provided by the stimulator. Combining information from different sources is a tricky affair. Natural language is aware of that, and forces to signal a warning that this has happened. That is the discourse function of *Then* in the response. (See Groenendijk, Stokhof & Veltman (1997)). To model this formally, we could add to our stack approach a dialogue version of the *conclude*-operation, as it is given in Kaufmann (2000), in dealing with the use of stacks to model modal subordination.

For the notion of support we also define the auxiliary notion of percolating an operation on stages all the way down the stack. It will be used in the definition of support in combination with the operation of restriction we just defined. So, in case of support we will percolate information all the way down, which will inevitably lead to a stronger ground stage of the common ground stack.

Definition 16 (Percolation). *Let* $[\cdot]$ *be an operation on stages.*

1. $\langle(\mathcal{M}, \mathcal{L}_{\mathcal{E}})\rangle^{\lhd}[\cdot] = \langle(\mathcal{M}, \mathcal{L}_{\mathcal{E}})\rangle$
2. $\langle\sigma, s\rangle^{\lhd}[\cdot] = \langle\sigma^{\lhd}[\cdot], s[\cdot]\rangle$

We are now ready for the defnition of the three types of reactions we distinguish to an utterance made by the other participant.

Definition 17 (Cancellation, Acceptance, and Support).

1. $\langle\langle\sigma, s\rangle, t\rangle\ [\bot] = \begin{cases} \langle\sigma, s\rangle & \textit{if } s \textit{ is inquisitive} \\ \langle\sigma, s\rangle[\bot] & \textit{otherwise} \end{cases}$
2. $\langle\langle\sigma, s\rangle, t\rangle\ [\Diamond] = \begin{cases} \langle\langle\sigma, s\rangle, t\rangle & \textit{if } s = s\lceil t\rceil \\ \langle\sigma, s\lceil t\rceil\rangle & \textit{if } s \neq s\lceil t\rceil \textit{ and } s\lceil t\rceil \textit{ is inquisitive} \\ \langle\sigma, s\lceil t\rceil\rangle[\Diamond] & \textit{otherwise} \end{cases}$
3. $\langle\langle\sigma, s\rangle, t\rangle\ [\top] = \begin{cases} \langle\langle\sigma, s\rangle, t\rangle & \textit{if } s = s\lceil t\rceil \\ \langle\sigma^{\lhd}\lceil t\rceil, s\lceil t\rceil\rangle & \textit{if } s \neq s\lceil t\rceil \textit{ and } s\lceil t\rceil \textit{ is inquisitive} \\ \langle\sigma, s\lceil t\rceil\rangle[\top] & \textit{otherwise} \end{cases}$

All three operations deconstruct a stack, popping a stage from the stack in case of cancellation, and pulling information down the stack in case of acceptance and support, and the operations keep on doing so until they meet a stage in which there remains an issue. Even after that, support keeps percolating information all the way down the stack.

In case of acceptance and support, there is also a 'check' at the beginning (the first clause in their definition), whether the reaction concerns a sentence which was informative in the stage of the common ground relative to which the sentence was uttered. Restriction of s to the informative content of t will typically have no effect in case compared to s, t embodies no new information, i.e., if t just embodies a change in current issue as compared to t.

So, after one of these operation has been performed, there is always an issue to relate to for the next move in the dialogue, until we meet the unlikely situation where all possible issues have been resolved.

Of course we should llok at examples, and how the model relates to natural language, but we do not include this in the present document.

6 One World Makes The Difference

For the semantics in terms of world-based models, the difference between classical and inquisitive logic was marked by the fact that not all sentences satisfy [A]. In fact, precisely the inquisitive sentences of the language do not satisfy [A]. This means that there is no way to state the semantics relative to single worlds, you need to evaluate sentences relative to sets of worlds, relative to possibilities, only at that level inquisitiveness can be detected, so to speak.

However, what I want to show here, is that we do not need sets of worlds of arbitrary size, just possibilities with no more than two worlds suffice. So, I want to prove the following fact:[44]

[A2] $\mathcal{M}, i \models \varphi$ iff $\forall v, u \in i\colon \mathcal{M}, vu \models \varphi$.

I think that if we can show this to hold, it makes clear that what we are dealing with is a semantics, and a logic, that differs minimally from classical logic: just one world makes the difference. Inquisitive logic is very near classical logic.

In one direction, for the language at hand where all sentences are stable, [A2] is just as easy as [A]. The remaining thing to show is:

Proposition 7. *If* $\forall v, u \in i\colon \mathcal{M}, vu \models \varphi$, *then* $\mathcal{M}, i \models \varphi$.

What this says is, look at all single worlds in a possibility (because I didn't say that $v \neq u$), and compare all the pairs of two different worlds in a possibility, see whether all of these support φ, and if they do, you can be assured that the possibility as such supports φ as well.

I'll prove the contraposition: if $\mathcal{M}, i \not\models \varphi$, then $\exists v, u \in i\colon \mathcal{M}, vu \not\models \varphi$.

If $\mathcal{M}, i \not\models \varphi$, then there are two cases to consider: (i) the case where φ is informative in a possibility i; and (ii) the case where φ is inquisitive in i.

Case (i) of informativeness is easy. If φ is informative in i, then there must be a world $v \in i$: $\mathcal{M}, v \not\models \varphi$, and by stability for any pair v, u: $vu \not\models \varphi$.

Remains case (ii) of inquisitiveness. If φ is inquisitive in i, then $i[\varphi]_{\mathcal{M}}$ is an inquisitive proposition in a possibility i.

What we have shown already, when we discussed the notion of *difference*, is the crucial fact that if $i[\varphi]_{\mathcal{M}}$ is an inquisitive proposition, then there are worlds (happy ends) $v, u \in i$ such that the pair v, u makes a difference in $i[\varphi]_{\mathcal{M}}$.

What that means according to the definion of two possibilities making a difference for a proposition, is that each of the two worlds belongs to some alternative possibility in the proposition but that there is no possibility in the

[44] This may mean something different to you than what it means to me. Until rather recently I always stated inquisitive semantics relative to pairs of worlds. So, I don't have to convince myself that if you consider pairs of worlds you can make an inquisitive semantics. And probably, you don't find that too difficult to believe either. So, the main point to me, and perhaps to you also, is that if I can show this to hold, then I know that whatever results I obtained before in my pair-semantics carries over to the more general case, that it does not really make a difference as far as the logic is concerned, at least not for this particular language, and this particular semantics.

proposition to which they both belong. So, for our pair of worlds v, u, of which we know that they must exist, this means that there are possibilities $j, k \in i[\varphi]_{\mathcal{M}}$, where $v \in j$ & $u \in k$, but there is no possibility $l \in i[\varphi]_{\mathcal{M}}$ such that $v \in l$ & $u \in l$.

Given that a proposition expressed by a sentence φ in a possibility i, is the set of largest possibilities $j \leq i$ such that $\mathcal{M}, i \models \varphi$, we get that both $\mathcal{M}, v \models \varphi$ and $\mathcal{M}, u \models \varphi$, but $\mathcal{M}, vu \not\models \varphi$. And we are done. [A2] holds.

6.1 Inquisitive Pair Semantics

That [A2] holds for our language $\mathcal{L}_{\mathcal{E}}$ and world-based models, tells us that we can set up the semantics at a lower level, relative to points consisting of two worlds, and that we can then 'collect the points' and arrive at an equivalent way to represent the meaning of a sentences of the language.

We leave the models completely as they were, The only thing we change is that we define support relative to pairs of worlds instead of possibilities, non-empty sets of worlds. The semantics then reads as follows.[45]

Definition 18 (Inquisitive Pair Semantics). *Let $\mathcal{F} = (\mathcal{P}_\omega, \leq)$ be a world-based frame, and $\mathcal{M} = (\mathcal{F}, \mathcal{I}, \{0,1\})$ an information model for $\mathcal{L}_{\mathcal{E}}$ based on $\mathcal{F}$. Let $(u, v) \in \omega^2$, $p \in \mathcal{E}$.*

1. $\mathcal{M}, (v, u) \models p$ *iff* $\mathcal{I}_v(p) = 1$ *&* $\mathcal{I}_u(p) = 1$
2. $\mathcal{M}, (v, u) \models \neg\varphi$ *iff* $\mathcal{M}, (v, v) \not\models \varphi$ *&* $\mathcal{M}, (u, u) \not\models \varphi$
3. $\mathcal{M}, (v, u) \models \varphi \to \psi$ *iff for all* $\pi \in \{v, u\}^2$ *if* $\mathcal{M}, \pi \models \varphi$*, then* $\mathcal{M}, \pi \models \psi$
4. $\mathcal{M}, (v, u) \models \varphi \wedge \psi$ *iff* $\mathcal{M}, (v, u) \models \varphi$ *and* $\mathcal{M}, (v, u) \models \psi$
5. $\mathcal{M}, (v, u) \models \varphi \vee \psi$ *iff* $\mathcal{M}, (v, u) \models \varphi$ *or* $\mathcal{M}, (v, u) \models \psi$

In the clause for implication, we quantify over the four pairs you can make from v and u, the pairs $(v, u), (u, v), (v, v), (u, u)$. The identity pairs sort of play the role of single worlds. It you inspect the clauses, then it is clear that if $(v, u) \models \varphi$, then $(u, v) \models \varphi$. We also have that if $(v, u) \models \varphi$, then $(v, v) \models \varphi$ and $(u, u) \models \varphi$. The reverse, of course does not hold.

If we literally collect the pairs (u, v) such that $(u, v) \models \varphi$, we get a set of pairs of worlds that we may call *the relational meaning of* φ. I define this notion in line with our practice, relative to a possibility, a non empty set of worlds i. We get the full relational meaning if we choose ω for i. The relational meaning of φ in i is defined as follows:

[45] See Groenendijk (2008), where much the same features of the semantics as have been illustrated above, are presented relative to the pair semantics. There is a cryptic remark in Ten Cate and Shan (2007: p. 69) that: "To test a LoI entailment [Logic of Interrogation, see Groenendijk (1999)], it suffices to consider structures with only two possible worlds." It was Balder ten Cate — in his capacity as an anonymous referee for the submitted abstract of the talk on which Groenendijk (2008) is based — who suggested the semantics relative to pairs of worlds as it is defined here, as a logically more simple alternative for the update semantics I gave for the language in the abstract. (See below.)

$$\langle\varphi\rangle_{\mathcal{M},i} = \{(u,v) \in i^2 \mid (u,v) \models \varphi\}$$

The relation expressed by a sentence φ in i, $\langle\varphi\rangle_{\mathcal{M},i}$ can be proved to be a symmetric and reflexive relation on i.[46] If an identity pair $(v,v) \notin \langle\varphi\rangle_{\mathcal{M},i}$, that means that v is excluded by the information φ provides.

If $(v,v) \in \langle\varphi\rangle_{\mathcal{M},i}$ and $(u,u) \in \langle\varphi\rangle_{\mathcal{M},i}$, but $(v,u) \notin \langle\varphi\rangle_{\mathcal{M},i}$, it means that φ is inquisitive in i, and that (v,u) is a pair that makes a difference. You can take the relation $\langle\varphi\rangle_{\mathcal{M},i}$ to be a *relation of indifference*. When v and u are related 'φ is indifferent' with respect to ways in which the worlds v and u in i may differ.

We can make the connection with the possibility semantics by defining the notion of the meaning of $\|\varphi\|_{\mathcal{M},i}$ in terms of the relational meaning:

$$\|\varphi\|_{\mathcal{M},i} = \{j \leq i \mid \text{for all } v,u \in j\text{: } (v,u) \in \langle\varphi\rangle_{\mathcal{M},i}\}$$

From here we know how to arrive at the proposition $i[\varphi]_{\mathcal{M}}$, by selecting the maximal elements from $\|\varphi\|_{\mathcal{M},i}$: $i[\varphi]_{\mathcal{M}} = \textsc{max}(\|\varphi\|_{\mathcal{M},i})$.

The direct route from relational meanings to proposition is by taking the maximal possibilities, i.e., largest sets of worlds, such that all worlds in the possiblity are related to each other in the relational meaning.

$$i[\varphi]_{\mathcal{M}} = \textsc{max}(\{j \leq i \mid \text{for all } v,u \in j\colon (v,u) \in \langle\varphi\rangle_{\mathcal{M},i}\})$$

We can also easily move in the other direction, from propositions to relational meanings, by collecting all pairs of worlds that are together in one of the alternative possibilities in a proposition.

$$\langle\varphi\rangle_{\mathcal{M},i} = \{(v,u) \mid \text{for some } j \in i[\varphi]_{\mathcal{M}}\text{: } v \in i \;\&\; u \in i\}$$

So, we basically have three ways of looking upon inquisitive meanings in world-based models: the set of possibilities that support a sentence; a relation of indifference on the set of worlds; and propositions. The most basic way to state the semantics is the pair semantics that I gave above.

I find the representation of inquisitive meanings as propositions the most perspicuous. Also, sofar I have found no way to go around propositions in stating the logical notion of relatedness. Nor do I see how compliance could be defined without the notion of relatedness.

The relational view on meaning also has something to say for it, because you can use it in stating a 'proper' udate semantics for the language.

6.2 Inquisitive Update Semantics

Given the relational semantics, it is also not difficult to see how you can formulate an equivalent 'true' update semantics. In fact, in this format I, and others, started to develop inquisitive semantics.[47] The update semantics defines the

[46] The proof is reasonably straightforward, but perhaps should be given here, because the fact that the relational semantics has this property also determines the nature of propositions.

[47] See Jäger (1996), Hulstijn (1997,2000), Groenendijk (1999).

interpretation of the sentences of our propositional language in terms of update functions on *states*, as functions from states to states.

Beware, this is a new use of the term 'state' as compared to how I used it sofar, where states only concerned information. States in the update semantics can be inquisitive, i.e., they are more like the propositions. In fact, they correspond one-to-one to them. To make things worse, I used the symbols s, t, r to denote stages, but I will use them now also to denote states in the update semantics. Why this is handy will become clear later. In a sense, it is not *so* bad, since propositions were the essential central part of stages, and our new states are like propositions.

States have the same nature as relational meanings, i.e., a state is a reflexive and symmetric relation on a subset of the set of possible worlds, i.e., on what we have called a possibility i. A state s is a set of pairs of worlds such that if $(v, u) \in s$, then $(u, v) \in s$ and $(v, v) \in s$ and $(u, u) \in s$.

The possibility i on which a state s is a relation corresponds to the *information* present in s, and is obtained by $i = \{v \in \omega \mid \text{for some } u \in \omega\colon (v, u) \in s\}$. This set corresponds to an information state in the way I used the term before, which only concerned information and not issues. Relative to our propositions this set corresponds to the join of the alternative possibilities in a proposition.

In an state issues can be present as well. A state s is *inquisitive* iff there are two worlds v and u such that $(v, v) \in s$ and $(u, u) \in s$ and $(v, u) \notin s$. If this is not so, the state is called *indifferent*.

The updates for our propositional language are defined as follows:[48]

Definition 19 (Inquisitive Update Semantics). *Let $\mathcal{M}$ be a world-based information model, and let s be a* state *in $\mathcal{M}$, a reflexive and symmetric relation on a subset of the set of worlds ω.*

1. $s[p]_{\mathcal{M}} = \{(v, u) \in s \mid \mathcal{I}_v(p) = 1 \;\&\; \mathcal{I}_u(p) = 1\}$
2. $s[\neg\varphi]_{\mathcal{M}} = \{(v, u) \in s \mid (v, v) \notin s[\varphi]_{\mathcal{M}} \;\&\; (u, u) \notin s[\varphi]_{\mathcal{M}}\}$
3. $s[\varphi \to \psi]_{\mathcal{M}} = \{(u, v) \in s \mid \forall \pi \in \{v, u\}^2\colon\ \pi \in s[\varphi]_{\mathcal{M}} \Rightarrow \pi \in \sigma[\varphi]_{\mathcal{M}}[\psi]_{\mathcal{M}}\}$
4. $s[\varphi \vee \psi]_{\mathcal{M}} = s[\varphi]_{\mathcal{M}} \cup s[\psi]_{\mathcal{M}}$
5. $s[\varphi \wedge \psi]_{\mathcal{M}} = s[\varphi]_{\mathcal{M}}[\psi]_{\mathcal{M}}$

I use standard features of the formulation of update semantics which make certain clauses look a bit different from the corresponding clauses in our previous statements of the semantics.[49] But the truth of the matter is that these standard features do not do any real work here, because, as it is, there are no 'real dynamic effects' in the semantics.

For example, I defined conjunction standardly in terms of sequencing the updates of the conjuncts, but I could just as well have defined conjunction as:

[48] The inquisitive update semantics is presented and discussed in Groenendijk (2009).
[49] I am non-standard in that I index the update function with the model. That is standardly suppressed in update semantics. I do the indexing here explicitly for easier comparison with the other statements of the semantics.

$$s[\varphi \wedge \psi]_{\mathcal{M}} = s[\varphi]_{\mathcal{M}} \cap s[\psi]_{\mathcal{M}}$$

There is nothing in the semantics that causes that 'order matters'. Likewise, implication can just as well be written as:

$$s[\varphi \rightarrow \psi]_{\mathcal{M}} = \{(u, v) \in s \mid \forall \pi \in \{v, u\}^2\colon\ \pi \in s[\varphi]_{\mathcal{M}} \Rightarrow \pi \in s[\psi]_{\mathcal{M}}\}$$

If you then compare the relational semantics with the update semantics you probably see the correspondences more easily. But you can also make them explicit in the following ways:

$$s[\varphi]_{\mathcal{M}} = \{(v, u) \in s \mid \mathcal{M}, (v, u) \models \varphi\}$$
$$\mathcal{M}, (v, u) \models \varphi \text{ iff } \{v, u\}^2[\varphi]_{\mathcal{M}} = \{v, u\}^2$$

What is uniformly behind these equivalence facts is that the update semantics is not really dynamic.[50]

6.3 Inquisitive Update Semantics and Dialogue Management

What comes a bit more easily in the update semantics is building and performing operations on the common ground. To begin with, a common ground stack can simply be a stack of states from the update semantics, rather than the stages that we used sofar. This will make clear why I choose to use s, t, r to denote states, where they used to denote stages, I don't have to change anything in the way I denoted stacks.

First, parallel to ground stages I define ground states, where it will not be difficult to recognize in the initial state the proposition I called The Big Question – which is why a use the same notation for it – and big questions in the other ground states.

Definition 20 (Ground States).

1. *The* initial state *is* $\imath = \{(u, u) \mid u \in \omega\}$
2. s *is a* ground state *iff* $s \subseteq \imath$

Ground states are to fulfil the same functions as ground stages, a ground state is stipulated to be at the bottom of the common ground stacks.

Next we define the following operation on states, called *indifferentiation* of a state s:

$$s^\star = \{(v, u) \mid (v, v) \in s \ \&\ (u, u) \in s\}$$

What indifferentiation on a state does is that it gives you *all* the pairs you can form from the set of worlds on which s is a relation. In a state $s^\star$ there are no pairs that make a difference anymore. The information in s and $s^\star$ is the same. But in $s^\star$ the issues that may have been there in s are removed from the table.

[50] Groenendijk (1998) present a dynamic version, which combines dynamic predicate logic with an inquisitive logic.

The indifferentiation of the initial state $\imath^\star$ gives you ω^2, the universal relation on the set of worlds. This is the state of complete ignorance and indifference. Now, here I have to allow myself another aberration of notation to make sure that everything looks superficially the same in the common ground stacks: I will just write ω instead of ω^2.

Using indifferentiation, parallel to the notion of two subsequent stages, I define the notion of two subsequent states.

Definition 21 (Subsequent States). *s and r are* subsequent states *iff $r^\star \subseteq s^\star$.*

This follows the pattern of subsequent stages in that we ignore issues. We just require that a subsequent state contains at least as much information as the previous one.

Now the stacks.

Definition 22 (Common Ground Stacks). *Let $\mathcal{M}$ be a model for a language $\mathcal{L}_\mathcal{E}$, and s a ground state. The set of* common ground stacks for $\mathcal{M}$ and $\mathcal{L}_\mathcal{E}$ *is the set such that:*

1. *$\langle(\mathcal{M}, \mathcal{L}_\mathcal{E}), s\rangle$ is a common ground stack.*
2. *if $\langle\sigma, s\rangle$ is a common ground stack, and s and r are subsequent states, then $\langle\langle\sigma, s\rangle, r\rangle$ is a common ground stack.*

What really becomes easier to define is the uptake of a sentence in the common ground. The main reason is that, unlke was the case for propositions, the union of two states cannot fail to be a state.

Definition 23 (Uptake). *Let $\langle\sigma, s\rangle$ be a common ground stack for $(\mathcal{M}, \mathcal{L}_\mathcal{E})$, $\varphi \in \mathcal{L}_\mathcal{E}$.*

$\langle\sigma, s\rangle[\varphi]^\Uparrow = \langle\langle\langle\sigma, s\rangle, s[\varphi]^?\rangle, s[\varphi]^\uparrow\rangle$, *where*

$s[\varphi]^? = s \cup s^\star[?\varphi]$

$s[\varphi]^\uparrow = s^\star[\varphi]$.

Note that both in thematizing φ and assuming φ, we don't update the current state s as such with the theme $?\varphi$ and φ, but first take the indifferentiation $s^\star$ of the current state s. That follows the pattern of two propositions $i[\varphi]$ and $j[\psi]$, in subsequent stages, where the fact hat they are subsequent guarantees that j is the join of the possibilities in $i[\varphi]$. We ignore the issues in $i[\varphi]$, and just consider the information φ may provide, and take that as the starting point for interpreting ψ. The indifferentiation here as the same effect.

By the way, given the contents of the uptake operation, it will be clear that the two pushes of s will indeed lead to new states on top that are subsequent states according to the definition. When we start from the intial state, and perform a sequence of uptake operations we will always end up with a stack in accordance with how stacks are defined.

I don't have to restate the operations of cancellation, acceptance and support. They can remain literally the same. But I do have to redefine the notion of restriction. Like thematizing, this works more easily with states than with stages.

Definition 24 (Restriction). *Let s and r be two subsequent states. The* restriction of s to r *is:*

$$s\lceil r\rceil = \{(v,u) \in s \mid (v,v) \in r \;\&\; (u,u) \in r\}$$

Remember that the function of restriction was to absorb information from the top of the stack to earlier stages. The identity pairs of worlds in r represent the information in r, by removing any pair of worlds from s that contains worlds that are no t present in r we update s with the information contained in r.

6.4 Compliance and Two Views on Meaning

As was the case relative to propositions in subsequent stages, we want to judge for subsequent states in the stack whether new states that have been added to the top of the common ground stack, are compliant with the state that was on top before they were added. Of course, the same notions of relatedness (applying to the result of thematization) and being at most as inquisitive (applying to to assume) play a role.

I will not bother to explicitly restate these notions, I just want to make the observation that whereas the notion of comparative inquisitiveness takes more effort to get at in the case of propositions in stages, this holds for relatedness with respect to states.

Since states are defined in terms of a relation of indifference, it takes no effort to see whether of two subsequent states s and r, r is at most as inquisitive as s. Just see whether for any pair (u,v) such that $(v,v) \in r$ & $(u,u) \in r$ and $(v,u) \notin r$, it also holds that $(v,u) \notin s$. Of course, these are the pairs that make a difference.

In the case of propositions in stages we first had to define what in a proposition are pairs of possibilities that make a difference, before we could measure comparative inquisitiveness. Where we saw, by the way, that really only pairs of happy ends, pairs of worlds matter.

Relative to relatedness the situation is reverse. For propositions in stages, this went easy, just see whether the alternative possibilities in the one proposition each cover the union of a subset of the alternative possibilities in the other.[51]

To be able to do the same for states, we first have to define what the alternative possiblities in a state are. We have seen how to do that in discussing relational meanings: take the maximal sets of worlds such that they are all related to each other in the state.

So, if we take the propositional view, we have to move to the relational view to be able to define one of the two essential ingredients of the core logical notion of compliance, if we take the relational view as present in the states, we have to

[51] What this ignores, though, is that to arrive at the notion of a proposition, we do have to take the *maximal* possibilities that support a sentence. We got there in two steps, first defining a notion of meaning that gave us all the possibilities that support a sentence, and then selecting the maximal elements. But that is not a necessary feature, we could define the propositions directly.

move to the propositional view to be able to define the other essential ingredient of the core logical notion of compliance.

Actually, the third notion that I used, that of the set of possibilities that support a sentence, which I used as an intermediary notion to get to the propositions, by taking the maximal elements, is probably not really needed. Propositions can be defined directly in terms of the maximal possibilities that support a sentence.

My main reason for introducing the intermediary notion was that union and intersection of two propositions need not be propositions, and I needed union and intersection in defining thematization and restriction. But for relational meanings, for states, union and intersection work fine, when applied to states they deliver states. So, at least for the world-based version of these notions, we should have been able to use that as an intermediary notion to get at appropriate definitions for thematization and restriction relative to propositions. It will be a matter of moving from propositions to the corresponding relations, do the necessary operations there as we defined them relative to states, and turn the resulting relation back into a proposition.

Anyway, conclusion, any way you may want to go, updating states or using propositions – taking a relational or a propositional primary view, respectively – you need the other view at a certain point to get things fully to work. At least, that is the current state of the art.

What is not so obvious is, whether, if we turn to further investigations in inquisitive semantics and dialogue management, choosing for one of the two views may not become a more serious affair. I started out from the relational opdate perspective, and via the pair semantics ended up with the propositional approach. And I went in that direction because it seems a promising to consider an inquisitive semantics relative to general models that bring you (further) in the area of intuitionistic logic, or at least to stronger intermediate logics, where next to pure information evidence starts playing a role. I would not know at this moment whether such a move is equally easy in the case of update semantics.

Conversely, the update format of inquisitive semantics is all set and ready to be extended to a really dynamic inquisitive update semantics, not just at the dialogue level, but also already in the semantics as such.[52] Just hink of a dialogue like: (A) "Will John go to the party?" (B) "And will Mary go with him?". Or: (A) "John went out with a girl yesterday" (B) "Was it Mary?". In the uptake of such sequences of utterances in a dialogue it would not do anymore to do the uptake relative to the indifferentiation of the current state. It is rather the state as such that has to be updated with the next utterance. These are different types of moves in the dialogue than just answering the current issue or replacing it by a subissue. Work to do.

[52] I did a lot of work there already before *The Logic of Interrogation*, which is also in update format, but not dynamic, and which therefore had the subtitle *Classical Version*.

6.5 Five-Valued Flat Semantics

We already showed by means of the relational interpretation that the world-based inquisitive semantics is only one world away from classical semantics, but starting from the relational interpretation we can also show that we can use a truth table method with five values to give the meanings of the connectives.

Starting point is the observation that there are four different situations where we get that $vu \not\models \varphi$, plus one where $vu \models \varphi$. Let these correspond with five values as indicated below.

○○ $v \not\models \varphi$ & $u \not\models \varphi$
●○ $v \models \varphi$ & $u \not\models \varphi$
○● $v \not\models \varphi$ & $u \models \varphi$
●● $v \models \varphi$ & $u \models \varphi$ & $vu \not\models \varphi$
⬬ $vu \models \varphi$

This gives the tools to define the semantics in terms of a five-valued valuation function relativized to two worlds: $\mathcal{V}_{vu}(\varphi) \in$ {⬬, ●●, ●○, ○●, ○○}. We can use truth tables to explicate the meanings of the logical operations.

For the atomic case, the value of $\mathcal{V}_{vu}(p)$ is determined by $\mathcal{I}_v$ and $\mathcal{I}_u$

$\mathcal{V}_{vu}(p)$	$\mathcal{I}_v$	$\mathcal{I}_u$
⬬	1	1
●○	1	0
○●	0	1
○○	0	0

Fig. 1. Atoms

The table for implication, where the last column also corresponds with negation, reads as follows.

$(\varphi \to \psi)$		ψ: ⬬	●●	●○	○●	○○	
	⬬	⬬	●●	●○	○●	○○	
	●●	⬬	⬬	●○	○●	○○	
φ	●○	⬬	⬬	⬬	○●	○●	$\neg\varphi$
	○●	⬬	⬬	●○	⬬	●○	
	○○	⬬	⬬	⬬	⬬	⬬	

Fig. 2. Implication and Negation

The 5 values are partially ordered in the following way: ○○ < ●○ < ●● and ○○ < ○● < ●●, and ●● < ⬬. According to the table $\mathcal{V}_{vu}(\varphi \rightarrow \psi) =$ ⬬, when $\mathcal{V}_{vu}(\varphi) \leq \mathcal{V}_{vu}(\psi)$, else $\mathcal{V}_{vu}(\varphi \rightarrow \psi) = \mathcal{V}_{vu}(\psi)$, with two exceptions in the last column where $\mathcal{V}_{vu}(\psi) =$ ○○ (which corresponds to negation). Then $\mathcal{V}_{vu}(\varphi \rightarrow \psi) =$ ○● if $\mathcal{V}_{vu}(\varphi) =$ ●○, and $\mathcal{V}_{vu}(\varphi \rightarrow \psi) =$ ●○ if $\mathcal{V}_{vu}(\varphi) =$ ○●.

It isn't so exceptional if you realize that with respect to the 'classical' values, i.e., {⬬, ●○, ○●, ○○}, we are dealing with a so-called product system. You calculate the value of the pair, by looking independently at the left and right side. If you then look at the 'exceptional' case where $\mathcal{V}_{vu}(\varphi) =$ ●○ and $\mathcal{V}_{vu}(\psi) =$ ○○, and you read ● as 1, and ○ as 0, it is all but natural that the outcome is ○●. And similarly for the other 'exceptional' case.

Note that there is only one case where an implication is assigned the value ●●, i.e., where the implication raises an issue in the point. That is where the antecedent is ⬬ and the consequent ●●.

As is to be expected, we find a lot more ●● outcomes in the table for disjunction:

$(\varphi \vee \psi)$		ψ: ⬬	●●	●○	○●	○○	
	⬬	⬬	⬬	⬬	⬬	⬬	
	●●	⬬	●●	●●	●●	●●	
φ	●○	⬬	●●	●○	●●	●○	$?\varphi$
	○●	⬬	●●	●●	○●	○●	
	○○	⬬	●●	●○	○●	○○	

Fig. 3. Disjunction and Questions

In all cases where the values of the disjuncts are ordered with respect to <, the outcome is the highest value of the two.

In the two cases where the values of the disjuncts are not ordered with respect to <, i.e. where the one disjunct has value ●○ and the other ○●, we get the value that is immediately above them in the ordering, ●● as outcome. These two cases witness the fact that a disjunction can introduce an issue.

The table also tells us what the value of $?\varphi$ is, given that $?\varphi = (\varphi \vee \neg\varphi)$.

Finally, the table for conjunction, which has little surprises. In all cases where the values of the conjuncts are ordered with respect to <, the outcome is the lowest value of the two.

In the two cases where the values of the conjuncts are not ordered with respect to <, i.e. where the one conjunct has value ●○ and the other ○●, we get the value that is immediately below them in the ordering, ○○ as outcome.

Well, it's nice to have truth tables by which you can calculate certain things through. And it will certainly be possible to compare informativeness and inquisitiveness of sentences, to define entailment, but I think that the notion of

$(\varphi \wedge \psi)$		ψ				
		⬬	●●	●○	○●	○○
	⬬	⬬	●●	●○	○●	○○
	●●	●●	●●	●○	○●	○○
φ	●○	●○	●○	●○	○○	○○
	○●	○●	○●	○○	○●	○○
	○○	○○	○○	○○	○○	○○

Fig. 4. Conjunction

relatedness escapes such a low level approach and needs intensionality at a larger level.

6.6 Intensionality of the Semantics of Questions

The issue of intensionality of the semantics of questions was raised in Groenendijk & Stokhof (1997) and they prove that the semantics of questions has to be intensional. They state the argument for a query language which has an ordinary language of propositional logic as its indicative basis, and forms interrogative sentences by putting a question mark in front of an indicative sentence. The semantics they use is a partition semantics. So the language can only express polar yes/no questions.

Nelken & Francez (2001) tried to prove the opposite by giving an extensional Five-valued semantics of questions, for a much richer language, but, without going into any details, we can consider the fragment of their language which corresponds to the very simple query language as we just sketched it.

In saying that they propose an extensional semantics, I mean that, unlike in our pair semantics, there is a single evaluation point. A single world, you could say, not pairs of them, as the relational semantics does. The relational semantics *is* intensional, be it minimally so, it needs two possible worlds.

The Five values of Nelken & Francez are related to our 5 values as indicated in the table in Figure 18.

true	⬬
resolved	
unresolved	●●
unknown	●○
	○●
false	○○

Fig. 5. Five and 5 values

The FIVE values form a bilattice, the values *true* and *resolved* are the top elements of the two parts. Indicative sentences can get the values *true, unknown*, and *false*; the interrogatives the values *resolved* and *unresolved.* As you can probably guess, $?\varphi$ counts as *resolved*, when φ is *true* and when φ is *false*, and $?\varphi$ is *unresolved* when φ has the value *false.*

It goes via the entailment relation, but a sentence ψ counts as an answer to $?\varphi$ iff whenever ψ is *true*, $?\varphi$ is *resolved.* Clearly, this means, e.g., that both p and $\neg p$ count as answers to $?p$.

However, they meet a serious problem, which they note, but do not really solve.[53] In this way also $p \vee \neg p$ counts as answer to $?p$. Whenever $p \vee \neg p$ is *true* (its value can be *unknown* by the way), either p is *true*, and hence $?p$ is *resolved*, or $\neg p$ is *true*, and hence $?p$ is *resolved* as well. Hence, whenever $p \vee \neg p$ is true, $?p$ is resolved. So $p \vee \neg p$ is characterized as an answer to $?p$.

Given this rather unwelcome result, I think it is fair to say that Nelken & Francez didn't really succeed in proving that an extensional semantics for questions is possible.[54] Of course, that doesn't yet mean that it isn't possible. And, actually, our relational semantics shows that it almost is.

[53] Ironically, they outline a solution using intuitionistic logic. And that would be an *extensional* semantics for questions?

[54] See also Nelken & Shan (2006), who try to show that a minimal intensional semantics is possible, treating $?\varphi$ as $\Box\varphi \vee \Box\neg\varphi$, where the intensional operator is interpreted in such a way that one only needs to consider two possibilities, which actually boils down to a 4-valued 'flat' semantics.

References

Aloni, M. (2007), 'Free Choice, Modals and Imperatives', in *Natural Language Semantics*, **15**, 65-94.

Aloni, M., A. Butler, and P. Dekker (2007), *Questions in Dynamic Semantics*, Elsevier, Oxford.

Alonso-Ovalle L. (2006), *Disjunction in Alternative Semantics*, PhD thesis, University of Massachusetts, Amherst

Asher, N, and A. Lascarides (1998), 'Questions in Dialogue', *Linguistics and Philosophy* **21**, pp.237-309. Asher, N, and A. Lascarides (2003), *Logics of Conversation*, Cambridge University Press, 2003.

van Benthem, J. (2008), ''The Information in Intuitionistic Logic', to appear in *Synthese*, special issue on 'Philosophy of Information' edited by F. Floridi & S, Sequoiah-Grayson (Oxford).

ten Cate, B. and C. Shan (2007) 'Axiomatizing Groenendijk's Logic of Interrogation', in: M. Aloni, A. Butler, and P. Dekker, *Questions in Dynamic Semantics*, Elsevier, Oxford, pp. 63-82.

Gerbrandy, J. (1999), *Bisimulations on Planet Kripke*, Ph.D. thesis, University of Amsterdam.

Grice, H.P. (1989), *Studies in the Ways of Words*, Harvard University Press, Cambridge, Mass.

Groenendijk, J. (1998), 'Questions in update semantics', in: J.H. Hulstijn & A. Nijholt, *Formal semantics and pragmatics of dialogue: proceedings of the thirteenth Twente Workshop on language technology (Twendial '98)*, Universiteit Twente, Enschede, 25-137

Groenendijk, J. (1999), 'The Logic of Interrogation', in T. Matthews & D.L. Strolovitch (eds), *The Proceedings of the Ninth Conference on Semantics and Linguistic Theory*, CLC Publications, Ithaca, NY, pp. 109-126. Also in: M. Aloni, A. Butler, and P. Dekker (2007).

Groenendijk, J. (2008), 'Inquisitive Semantics: Two Possibilities for Disjunction', to appear in the Proceedings of Tbilisi 2007: Seventh International Tbilisi Symposium on Language, Logic and Computation.

Groenendijk, J. (2009), 'Inquisitive Semantics: Conditionals, Questions and Disjunction', to appear in the Proceedings of SALT XVIII at Amherst MA, edited by Tova Friedman & Satoshi Ito. .

Groenendijk, J., and M. Stokhof (1982), 'Semantic Analysis of Wh-Complements', *Linguistics and Philosophy*, **5**, pp. 175-233.

Groenendijk, J., and M. Stokhof (1984), '*Studies on the Semantics of Questions and the Pragmatics of Answers*, Ph.D. thesis, University of Amsterdam.

Groenendijk, J., and M. Stokhof (1991), 'Dynamic Predicate Logic', *Linguistics and Philosophy*, **14**, 39-100.

Groenendijk, J., and M. Stokhof (1997), 'Questions', in: J. van Benthem and A. ter Meulen, *Handbook of Logic and Linguistics*, Elsevier, Amsterdam, pp. 1055-1124.

Groenendijk, J., M. Stokhof, & F' Veltman: (1997), 'Coreference and Modality in the Context of Multi-Speaker Discourse', in: H. Kamp & B. Partee, *Context Dependence in the Analysis of Linguistic Meaning*, Elsevier, Amsterdam., 195-215.

Hamblin, C.L.: (1973), 'Questions in Montague English', *Foundations of Language* **10**, 41-53.

Heidrich W. and H. Rieser (eds) (1998), *Mutual Knowledge, Common Ground and Public Information, Workshop at the 10th European Summerschool in Logic, Language and Information (1998).*

Heim, I.: (1982), *The Semantics of Definite and Indefinite Noun Phrases*, University of Massachusetts. Published in 1989 by Garland, New York.

Hulstijn, J. (1997), 'Structured Information States. Raising and Resolving Issues', in: A. Benz and G. Jäger (eds), *Proceedings of MunDial97*, University of Munich.

Hulstijn J.: (2000), *Dialogue Models for Inquiry and Transaction*, Ph.D. thesis, University of Twente.

Isaacs, J., and K. Rawlins (2008), 'Conditional Questions', *Journal of Semantics*.

Jäger, G. (1996), 'Only Updates. On the Dynamics of the Focus Particle *Only*', in: M. Stokhof and P. Dekker (eds), *Proceedings of the Tenth Amsterdam Colloquium*, ILLC University of Amsterdam, Amsterdam, pp.387-405. Also in: M. Aloni, A. Butler, and P. Dekker (2007).

Kamp H. and U. Reyle (1993), *From Discourse to Logics*, Kluwer, Dordrecht.

Kaufmann S. (2000), 'Dynamic Context Management', in M. Palmer, S.Kaufman & M. Pauly, *Formalizing the Dynamics of Information*, CSLI Publications, Stanford.

Landman, F. (1986), *Towards a Theory of Information*, Ph.D. thesis, University of Amsterdam.

Leech, G. (1983), *Principles of Pragmatics*, Longman, London.

Lewis, D. (1979), 'Score-keeping in a Language Game', in: R. Bäuerle, U. Egli & A. von Stechow, *Semantics from Different points of View*, Springer.

Lewis, D. (1982), ' 'Whether' Report', in his: *Papers in Philosophical logic*, Cambridge University Press, Cambridge, 1998, pp. 45-56.

Mann, W.C.: (1988), 'Dialogue Games: Conventions of Human Interactions', *Argumentation* **2**, 511-532.

Nelken, R. and N. Francez (2001), 'Bilattices and the Semantics of Natural Language Questions', *Linguistics and Philosophy* **25**, pp.37-64.

Nelken, R, and C. Shan (2006), 'A Modal Interpretation of the Logic of Interrogation', *Journal of Logic, Language and Information*, **15**, pp. 251-271.

Ramsey, F. P. (1931), 'General Propositions and Causality', in his: *The Foundations of Mathematics and other Logical Essays*, edited by R. B. Braithwaite, London: Kegan Paul, Trench & Trubner, pp. 237-255.

Roberts, C.: (1996), 'Information Structure in Discourse', In J.H. Yoon and A. Kathol (eds), *Working Papers in Linguistics 49*, The Ohio State University, pp. 91-136.

Rooth, M. (1985), *Association with Focus*, Ph.D. thesis, University of Massachusetts, Amherst.

Rooth, M. (1992), 'A Theory of Focus interpretation', in *Naural language Semantics*, **1** 75-116.

Stalnaker, R. (1972), 'Pragmatics', in: Harman and Davidson (eds), *Semantics of Natural Language*, Reidel, Dordrecht, 380-379.

Stalnaker, R. (1978), 'Assertion', in: P. Cole (ed), *Pragmatics (Syntax and Semantics 9)*, Academic Press, New York, 315-332.

Simons, M.: (2000), *Issues in the Semantics and Pragmatics of Disjunction*, Garland, NY.

Simons, M.: (2001), 'Disjunctions and Alternatives', *Linguistics and Philosophy* **25**, pp. 597-619.

Velissaratou, S.: (2000), *Conditional Questions and Which-Interrogatives*, MA thesis, ILLC, University of Amsterdam.

Veltman, F,: (1984), 'Data Semantics', in J, Groenendijk, Th. Janssen, & M. Stokhof (eds), —it Truth, Interpretation and Information. Foris, Dordrecht, 44-63.

Veltman, F,: (1985), *Logics for Conditionals*, Ph.D. thesis, University of Amsterdam.

Veltman, F. (1996), 'Defaults in Update Semantics', *Journal of Philosophical Logic* **25**, 221-261.