Dynamically Generated Gauge Fields:
QED as an Effective Low Energy Theory

E.E. van der Kraaij

Master's Thesis Physics
Supervisor: Prof. dr. ir. F. A. Bais

Institute for Theoretical Physics
Valckenierstraat 65
1018 XE Amsterdam
The Netherlands

University of Amsterdam
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Abstract

A purely fermionic theory with current-current interaction is analysed from a renormalization group viewpoint in a $1/N$ expansion, and it is shown that by a correct fine-tuning of the free parameters the result at low energies is equivalent to that of Quantum Electrodynamics. The non-remormalizable four fermi theory generates a massless vector degree of freedom, which by its coupling with the fermion fields can be interpreted as a composite photon. In the low energy limit of the purely fermionic theory the gauge invariance is a dynamically generated symmetry, to leading non-trivial order in a $1/N$ expansion.

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Chapter 1

Introduction

At present Quantum Electrodynamics (QED) is one of the most accurately tested theories in particle physics. It provides us with a description of the interaction of the electrons, positrons and photons in the language of Quantum Field Theory. The photon corresponds to the quantized transversal electromagnetic wave, which is a massless particle that carries the electro-magnetic force between charged particles.

An important property of this theory is local gauge invariance, stating that the action is invariant under the simultaneous phase shift of the fermion field ψ and the translation of the photon field A_μ. The fields transform as follows

$$\psi(x) \quad \rightarrow \quad e^{ie\omega(x)}\psi(x) \quad ; \tag{1.1}$$

$$A_\mu(x) \quad \rightarrow \quad A_\mu(x) - \partial_\mu \omega(x) \ , \tag{1.2}$$

where $\omega(x)$ parametrizes the local gauge transformation. Gauge symmetries are the underlying symmetries of all four interactions in nature: the electromagnetic, the weak, the strong and the gravitational force, and except for the gravitational force all are well described with renormalizable field theories. QED is based on a local $U(1)$ symmetry given by eq. (1.1) and (1.2), and is a subgroup of the local $SU(2) \times U(1)$ gauge group giving a unified description of the electromagnetic and weak force. The strong force is based on a local $SU(3)$ symmetry and together with the electro-weak force is unified in the $SU(3) \times SU(2) \times U(1)$ Standard Model.

In all these theories the gauge fields can be seen as the carriers of the force between particles. Due to the tremendous success of these theories local gauge invariance is nowadays seen as a guiding principle behind the four elementary interactions, but an open question remains whether this invariance is a fundamental principle or a consequence of underlying dynamics.

The conventional interpretation is that the gauge particles are elementary particles, i.e. particles which are not made up of any other constituents. In this thesis however we will investigate the alternative possibility that the photon is a composite particle, realized as a bound state of some degrees of freedom of a more fundamental theory.

We will consider a purely fermionic theory with a four fermi interaction and show that at low energies this theory can describe a massless vector degree of freedom. This effective vector particle is a collective excitation of two fermion fields and because of its coupling to the fermion field it can be interpreted as a photon. Indeed the parameters of the non-renormalizable four fermi theory can be fine-tuned in such a way that its effective low energy limit becomes QED. In this situation local gauge invariance is a dynamically generated property: the symmetry is brought forth by the interacting fermions and not as a symmetry principle imposed a priori.

This possibility is of course of great interest and as it concerns the fundaments of local gauge invariance itself, it immediately concerns all gauge particles. The basic idea is that gauge fields, the carriers of forces between interacting particles, may arise as low energy collective excitations of some underlying theory.

1.1 The composite gauge particle

In this section we give a brief historical overview of the composite gauge particle and along the way discuss several of its aspects.

The idea of the photon being a bound state has its origin in the sixties. In 1961 Nambu and Jona-Lasinio [1] published two articles in which they showed their purely fermionic NJL model to acquire bosonic bound states, produced after chiral symmetry breaking. As the quarks were not known yet at that time, they thought these fermions to be the nucleons, that is protons and neutrons, and the bound state to be the pion.

They were not far from being right and this model nowadays represents the chiral symmetry breaking at low energies of the strong force, producing pions consisting of two quarks. Nambu and Jona-Lasinio based their model on an analogy with the Cooper-pair, a bound state of two electrons which breaks the $U(1)$ gauge symmetry, consequently yielding superconductivity.

After this enormous breakthrough there have been many attempts to apply the same dynamics of symmetry breaking to other symmetries in order to uncover possible underlying theories. In 1963 both Bialynicki-Birula [3] and Bjorken [2] independently published an article on a four fermi theory in which a neutral vector particle is shown to appear, with the same properties as the photon. In Bjorken's model this composite photon is produced by Lorentz symmetry breaking: in a fermion model with current-current interaction,

$$\mathcal{L}_{Bj} = \overline{\psi}(i\partial\!\!\!/ - m)\psi - G\left(\overline{\psi}\gamma_\mu\psi\right)\left(\overline{\psi}\gamma^\mu\psi\right) \ , \tag{1.3}$$

the spontaneous symmetry breaking implies a non-zero vacuum current density $<\overline{\psi}\gamma_\mu\psi>=\eta_\mu$. The massless excitation modes in the broken phase correspond to the photon field.

At first sight, the breaking of Lorentz symmetry is cause for great concern, but luckily this problem fixes itself: Bjorken argued that the vacuum expectation value is in fact no more than a gauge fixing in the effective gauge theory and has no implications for physical measurable phenomenon. In 1968 Y. Nambu [4] expanded on this particular aspect and confirmed Bjorken's analysis. By calculating S-matrices of interaction processes of significance, he showed that the physical outcomes are Lorentz invariant and equal to those in QED.

Bjorken found a vector condensate which under the right circumstances took the form of the photon. This is not as obvious as it may seem, for a vector particle is not by definition a gauge particle. In 1973 Cornwall *et al.* [5] made an important step in this problem by showing that at low energies any bound vector meson with a mass tending to zero must approach the form of a gauge vector particle. The requirement of non-singular behavior for tree diagrams dictates the coupling to behave as a gauge coupling.

Three years later in 1976 Eguchi [6] extended the idea of Bjorken to the non-abelian case and showed that in a wide class of fermionic theories the interactions between the fermions and bound states at low energies turn out te be renormalizable. This rendered the original purely fermionic theory renormalizable and by showing at the same time the equivalence with theories in which the would be bound states are fundamental, the possibility of composite gauge particles was established. The equivalence is first shown of the NJL theory with the σ-model, then of a current-current fermionic theory with QED and finally of a non-abelian analog of the latter with QCD.

In a sequel [7] however Eguchi extended on the subject and comes to the conclusion that the exact condition for a possible equivalence is the requirement of an ultra-violet fixed point in the renormalization flow. As this is not the case in QED, which has an infra-red fixed point, the equivalence with the current-current fermionic theory does not hold. The non-abelian theories however do comply and can be interpreted as effective theories.

The large N expansion

The large N expansion was introduced in 1974 by G. 't Hooft [31] in his attempts to understand the strong force by extending the $SU(3)$ group for the color interaction between quarks from three colors to a number N of colors. In the limit of $N \to \infty$ the expressions for the Feynman diagrams are arranged in a different way than in an expansion of the charge coupling constant, i.e. with the large N considered as a free parameter the diagrams are arranged by powers of $1/N$, leading to crucial simplifications in this limit.

For such expansions in purely fermionic theories, we consider N flavors of fermions introducing a global but not a local $SU(N)$ symmetry. After an appropriate N-dependent rescaling of the coupling constants, the Feynman diagrams can then be arranged such that calculations in the leading order of $1/N$ become much simpler. The system of one fermion field with current-current interaction used by Bjorken in eq. (1.3) for example can be extended to N fields by writing

$$\mathcal{L} = \overline{\psi}^I (i\partial\!\!\!/ - m)\psi^I - \frac{G}{N}\left(\overline{\psi}^I \gamma_\mu \psi^I\right)\left(\overline{\psi}^J \gamma^\mu \psi^J\right) , \tag{1.4}$$

with $I, J = 1, \ldots, N$ and a summation over repeated I and J is implied. In section 3.1.1 we discuss this expansion in greater detail; for a detailed analysis in a textbook, see [32].

Several authors have tried to apply this method to the equivalences between purely fermionic theories and fermion-boson theories. In 1988 Suzuki [8] showed at low energies in a $1/N$-expansion that composite light vector bosons for the right value of the coupling constant become massless and equivalent to the fundamental gauge particles. The equivalence is shown between the standard $SU(2) \times U(1)$ model for the electroweak gauge bosons and a model with composite weak mediators plus fundamental photon, with in both cases the same $SU(2)$ global symmetry breaking and the correct coupling between the composite gauge particles and the fermions. This work was commented on and further extended in the same year by Cohen, Georgi and Simmons [9].

Renormalization flows

Another perspective on this subject is provided by the renormalization group approach and the analysis of renormalization flows, see also appendix C. The basic idea, developed by Wilson [15], is that although two theories might superficially seem totally different, this need not be the case. If both systems have the same fixed point and identical renormalization flows near this point, then both theories describe the same physics. The gauge theories of the Standard Model are renormalizable while purely fermionic theories, with interactions as in eq. (1.4) are not. However, even if one theory is renormalizable and the other is not, Wilson's idea can still be applied.

A. and P. Hasenfratz [10] achieved this in the non-abelian case and established the equivalence of an N flavor fermionic theory with current-current interaction to the fundamental asymptotically free $SU(N)$ Yang-Mills theory. In two articles by Sonoda [12, 13] this is explicitly applied to QED and the equivalence between a composite vector boson and the photon is made clear. These last two papers have been the main source of inspiration for this thesis.

1.2 Thesis outline

Chapter 2 In this chapter we elaborate on the phenomenon of symmetry breaking and highlight a few important examples. From the idea of scalar condensates formed in the strong coupling limit we go to vector condensates and the possible formation of composite gauge particles. Along the way we discuss similarities and differences between the two types.

Chapter 3 Instead of using the standard analysis of QED by means of perturbation theory we will be applying the $1/N$ expansion. In this chapter we discuss its implementation and consequences. Also will we be using the gauge invariance violating cut-off regularization scheme as this results in a renormalization of the photon mass, giving us the possibility of fine-tuning this parameter.

Chapter 4 After introducing QED in the previous chapter, the renormalization procedure is now performed by fixing all counterterms and as an end result of the analysis of QED we obtain several renormalization flows with fixed points.

Chapter 5 The fermionic theory we will be analysing is a system with interactions similar to the current-current term in Bjorken's model. In this chapter we explain how and why exactly we introduce the specific interactions we will be using and how these can be redefined with our goal of an effective gauge theory in mind.

Chapter 6 The purely fermionic set up of the previous chapter still is not directly related to QED. In this chapter we determine the correct conditions we need and finally we show how the system can be fine-tuned such that its low energy limit is in effect QED.

Chapter 7 We discuss the obtained equivalence between the two theories and evaluate how the resulting dynamically generated gauge theory fits among other effective theories.

Appendices In appendix A we state the conventions we use throughout this thesis. In appendix B we show briefly how regularization by means of the dimensional scheme works and we explain why we may as well use the somewhat oldfashion cut-off scheme which is more suitable for our purpose. In appendix C we summarize the philosophy behind the renormalization group approach and introduce often used definitions.

Chapter 2

Symmetry Breaking

In this chapter we introduce the phenomenon of symmetry breaking. Symmetries can be broken in many different ways and we think it useful to give a comprehensive survey of the possibilities. We have adopted the following, what we believe to be the most common, definitions:

Explicit symmetry breaking: if the Lagrangian of a system is extended by hand with an extra term which manifestly breaks the symmetry. For example, when the QED Lagrangian of massless electrons is extended with a mass term, the chiral symmetry is explicitly broken.

Spontaneous symmetry breaking: when the mere form of the potential is such that the invariance of the Lagrangian is not present in the ground state of the system. A good example is the ferromagnet: its Hamiltonian of interacting spins is symmetric under rotations, but the groundstate in which all spins of the ferromagnet are aligned is not.

Dynamical symmetry breaking: if the spontaneous breaking is an effective description of an underlying dynamical effect. In the BCS theory of superconductivity for example, is the $U(1)$ gauge symmetry spontaneously broken by the formation of Cooper pairs. Dynamical symmetry breaking is thus a special case of spontaneous symmetry breaking.

The BCS theory was one of the first successful quantum theories in which an observed symmetry breaking was explained by a deeper underlying dynamics. After the success of the BCS theory came the Nambu-Jona-Lasinio model, which became the prototype for dynamical symmetry breaking in many quantum field theories. Even with today's successful Standard Model we see that the same explanation in terms of a deeper dynamics is attempted once again in the so far experimentally unreached Higgs sector.

In the scenario of spontaneous symmetry breaking we have a condensate of a fundamental scalar field ϕ, implying a non-zero vacuum expectation value

$$< \phi > \neq 0 \; . \tag{2.1}$$

We distinguish two important consequences: the first is Goldstone's theorem, which states that for each broken symmetry one massless Goldstone boson appears. The second is the Higgs-mechanism, which states that if the system is coupled to a local gauge theory, then the massless gauge particle becomes massive and by a gauge transformation the Goldstone boson can be made to disappear. This Higgs-mechanism is often summarized by stating that the massless Goldstone boson is "eaten" by the gauge boson, which therefore acquires mass.

In the scenario of dynamical symmetry breaking the fundamental scalar field is replaced by a bound state of other constituents. This always occurs in combination with a phase transition, when below a certain energy the bound states of say particles A and B form a condensate given by

$$< AB > \neq 0 \tag{2.2}$$

and break the symmetry. If the theory is coupled to gauge fields with local gauge invariance the created Goldstone bosons again are eaten and the gauge particles become massive.

In the following sections we analyse three phenomena: superconductivity, quark-pion interaction and the electroweak symmetry breaking. For each of these, at first only spontaneous symmetry breaking by a fundamental scalar field was realized. One at a time these scalar fields turned out to be (possible) composites.

- The BCS theory was the first in which it became clear that the scalar field is a bound state: the field is an effective description of the Cooper pair consisting of two electrons. This pair breaks the $U(1)$ symmetry and generates a photon mass.
- In the strong force dynamics it is the pion which is a bound state of quarks. At a certain energy level this forms a condensate, breaking chiral symmetry and generating quark mass.
- The electroweak symmetry breaking is achieved through the scalar condensate of the so-called Higgs boson. This particle has however still not been detected and its exact composition, fundamental or composite, is unknown. A possible dynamical scenario goes under the name Technicolor.

These subjects are dealt with in many textbooks and reviews. For more details on symmetry breaking we refer to [18, 19]; information on pion physics and strong force dynamics was found in [30] and for reviews on Technicolor we used [19, 20].

In section 2.4 we then go from scalar to vector condensates and discuss the differences. As all these theories and scenario's partially overlap we give an overview in figure 2.1 on page 27.

2.1 Electromagnetic gauge symmetry breaking; Superconductivity

In this section we illustrate how the $U(1)$ electromagnetic symmetry is broken by a scalar field ϕ. Before the BCS theory of superconductivity the Landau-Ginzburg theory (or the abelian Higgs model) with scalar field ϕ carrying a charge $q = -2e$ was the only known model in which the phenomenon could be explained. What the field ϕ exactly was, was not clear and the breaking of gauge invariance was more seen as a problem than as a solution.

2.1.1 The abelian Higgs model

We consider in Minkowsky space a complex scalar field $\phi(x)$ coupled to the electromagnetic field A_μ by a charge q, with a quartic self interaction:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathcal{D}_\mu\phi)^* (\mathcal{D}^\mu\phi) - V(\phi) \ , \tag{2.3}$$

$$\mathcal{D}_\mu = \partial_\mu + iqA_\mu \tag{2.4}$$

$$V(\phi) = -\mu^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \ . \tag{2.5}$$

The system has a $U(1)$ symmetry and is invariant under the local gauge transformation

$$\phi(x) \to e^{iq\omega(x)}\phi(x) \quad , \quad A_\mu(x) \to A_\mu(x) - \frac{1}{q}\partial_\mu\omega(x) \ . \tag{2.6}$$

The Lagrangian of eq. (2.3) has two unusual aspects: the sign of the μ^2 term is not correct for it to be the mass term of the ϕ-field, and in this potential the groundstate is not when $<\phi>= 0$. The potential $V(\phi)$ has a minimum when

$$<\phi>= \left(\frac{\mu^2}{\lambda}\right)^{\frac{1}{2}} \equiv \phi_0 \ . \tag{2.7}$$

If we are to perform perturbation theory, we must analyse the perturbations around the groundstate. We therefore expand about this groundstate by writing the complex field $\phi(x)$ as

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \ . \tag{2.8}$$

In the potential we now see that the field ϕ_1 acquires a mass-term, while the ϕ_2 field does not:

$$V(\phi) = -\frac{1}{2\lambda}\mu^4 + \frac{1}{2}(2\mu^2)\phi_1^2 + \mathcal{O}\left(\phi_i^3\right) \ , \tag{2.9}$$

The two degrees of freedom of the complex field ϕ have reduced to one massive scalar field ϕ_1 and one massless scalar field ϕ_2. This last one is exactly the Goldstone boson as predicted by Goldstones theorem. The potential of eq. (2.9) is however not symmetric anymore under local gauge transformations. This is because the fields ϕ_1 and ϕ_2 are expansions about the groundstate which does not respect the symmetry.

The second term in Lagrangian (2.3), that is the kinetic part $(\mathcal{D}_\mu\phi)^2$, is now a mixture of many terms which can be rearrange by a gauge transformation. With the correct $\omega(x)$ in eq. (2.6) we can make ϕ in eq. (2.8) a real-valued field. In other words, we transform such that the field ϕ_2 completely disappears and the Lagrangian becomes, with $m_A^2 = 2q^2\phi_0^2$,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi_1)^*(\partial^\mu\phi_1) + \frac{1}{2}m_A^2A_\mu A^\mu - V(\phi_1) \ . \tag{2.10}$$

The disappearance of the Goldstone boson might seem as a loss of a degree of freedom, which is of course impossible. In fact what has happened is that the massless photon, which has two polarization states, gains an extra degree of freedom with its mass: it has three polarization states.

2.1.2 Bardeen-Cooper-Schieffer theory

In the technical description of the $U(1)$ symmetry breaking given above the phenomenon of superconductivity is not obvious. Unfortunately to do so requires highly detailed dynamics which we shall not handle here. We merely give a summary on how in the theory set forth by Bardeen, Cooper and Schieffer the symmetry breaking can be explained; for more details see [30] and references therein.

When a superconductor is cooled down to below a certain temperature T_c we see that electric currents can flow without resistance and that all magnetic fields are expelled. This latter effect is called the Meisner-effect. What happens is that by lattice vibrations an attractive force arises between two electrons, binding them into a Cooper pair. This bound state of electrons creates an energy-gap, lowering the ground state energy. The lattice vibrations appear as pseudo-particles and are called phonons.

In the abelian Higgs model the scalar field ϕ is an effective description of the Cooper pair. The condensate formed by these pairs breaks the gauge invariance, which means that inside the superconductor the gauge field gains a mass. An effective description of a massive mediator is given by the Yukawa potential

$$V(r) \sim \frac{e^{-mr}}{r} \tag{2.11}$$

and we see that the mass of a photon confines the range of the electromagnetic force to order $\mathcal{O}(1/m)$. This explains the Meisner effect: external fields can hardly penetrate the material and we see the superconductor expelling all external magnetic fields.

2.2 The breaking of chiral symmetry; Pions

In section 2.1 we saw how the $U(1)$ gauge symmetry was broken by the interaction with a scalar particle and how this generated mass for the gauge field. In this section we discuss the breaking of chiral symmetry and how this generates mass for the fermion fields. In contrast to the local gauge theory where the Goldstone boson disappears after the gauge mass generation, the Goldstone boson *does not* disappear after chiral symmetry breaking.

We first introduce the $U(1) \times U(1)$ symmetric σ-model in which the chiral symmetry is broken down to a residual $U(1)$ symmetry. We then show how the same phenomenon can dynamically be generated in the Nambu-Jona-Lasinio (NJL) model [1]. Both these models, the σ- and the NJL model, can easily be generalized to the $SU(2) \times SU(2)$ chiral symmetry being broken down to a residual $SU(2)$; this is the case in Quantum Chromodynamics with which we end this section.

2.2.1 The σ-model

A useful representation when analysing chiral symmetry is the left- and right-handed projection of the fermion field:

$$\psi_L = \frac{1}{2}(1-\gamma_5)\psi \quad ; \quad \psi_R = \frac{1}{2}(1+\gamma_5)\psi \ . \tag{2.12}$$

In the Lagrangian of a massive fermion field we see that in this representation the kinetic term decomposes into two independent terms, while the mass term mixes the left- and right-handed projections:

$$\mathcal{L} = \overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi \tag{2.13}$$
$$= \overline{\psi}_L\partial\!\!\!/\psi_L + \overline{\psi}_R\partial\!\!\!/\psi_R - m(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L) \ . \tag{2.14}$$

We now turn to the σ-model. The Lagrangian consists of a massless fermion field ψ coupled to a complex scalar field ϕ and is given by

$$\mathcal{L} = \overline{\psi}_L\partial\!\!\!/\psi_L + \overline{\psi}_R\partial\!\!\!/\psi_R + |\partial\phi|^2 - g(\overline{\psi}_L\psi_R\phi + \overline{\psi}_R\psi_L\phi^*) - V(\phi) \ , \tag{2.15}$$
$$V(\phi) = -\,M^2\phi^*\phi + \frac{\lambda}{2}\left(\phi^*\phi\right)^2 \ . \tag{2.16}$$

This system has two different $U(1)$ symmetries. The first is an ordinary global invariance and the transformation of the fields are given by

$$\psi \to e^{i\theta}\psi \quad \text{or} \quad \psi_L \to e^{i\theta}\psi_L \ ; \ \psi_R \to e^{i\theta}\psi_R \tag{2.17}$$
$$\phi \to \phi \ . \tag{2.18}$$

In other words only the fermion field transforms. The second invariance is a chiral symmetry which in contrast to the previous invariance does transform the scalar field and does treat the left- and right-handed fields different:

$$\psi \to e^{i\omega\gamma_5}\psi \quad \text{or} \quad \psi_L \to e^{-i\omega}\psi_L \ ; \ \psi_R \to e^{i\omega}\psi_R \tag{2.19}$$
$$\phi \to e^{-2i\omega}\phi \ . \tag{2.20}$$

Similar to the abelian Higgs model, see section 2.1.1, the vacuum built around the field configuration $<\phi>= 0$ is not stable. The real minimum of the potential is when the vacuum expectation value equals

$$<\phi>= \frac{M}{\sqrt{\lambda}} \equiv \frac{v}{\sqrt{2}} \ . \tag{2.21}$$

We expand about this condensate by rewriting the complex field $\phi(x)$. Defining $h(x)$ and $\sigma(x)$ as two real fields we redefine

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))e^{i\sigma(x)/v} \tag{2.22}$$

If we now substitute this in the Lagrangian (2.12) we see that $h(x)$ becomes a massive field and that $\sigma(x)$ is the massless Goldstone boson. The field $h(x)$ is however of no importance to us and we will write $\phi = (v/\sqrt{2})e^{i\sigma/v}$. The

Lagrangian then reads

$$\mathcal{L} - V = \overline{\psi}_L \partial\!\!\!/ \psi_L + \overline{\psi}_R \partial\!\!\!/ \psi_R + \frac{1}{2}(\partial\sigma)^2 - \frac{gv}{\sqrt{2}}\left(\overline{\psi}_L \psi_R e^{i\sigma/v} + \overline{\psi}_R \psi_L e^{-i\sigma/v}\right) \tag{2.23}$$

$$= \overline{\psi} \partial\!\!\!/ \psi + \frac{1}{2}(\partial\sigma)^2 - \frac{gv}{\sqrt{2}}\overline{\psi}\psi - i\left(\frac{g}{\sqrt{2}}\right)\sigma\overline{\psi}\gamma_5\psi + \dots \tag{2.24}$$

where in the last line we expanded in powers of σ/v. The fermion field has gained a mass $m = gv/\sqrt{2}$ and the field σ remains massless. That this breaks the chiral symmetry becomes clear if we try to transform $\overline{\psi}\psi$ by the transformation (2.19).

2.2.2 The Nambu-Jona-Lasinio model

In the σ-model the interaction of the fermion field with a scalar mode generates a mass for the fermion, but what the scalar mode represents is not specified. Only in 1961 could this question be answered when Nambu and Jona-Lasinio published two articles [1]: based on the BCS theory in which a bound state creates an energy gap, they came with the idea that a particles mass could be generated in a similar way.

The NJL theory starts with a massless fermion field, self-coupled via a four interaction:

$$\mathcal{L} = -\overline{\psi}\gamma_\mu\partial^\mu\psi + g_0\left[\left(\overline{\psi}\psi\right)^2 + \left(i\overline{\psi}\gamma_5\psi\right)^2\right] . \tag{2.25}$$

The Lagrangian is symmetric under two $U(1)$ symmetries: an ordinary global symmetry and a chiral symmetry, respectively given by

$$\psi \to e^{i\theta}\psi \quad ; \quad \overline{\psi} \to \overline{\psi}e^{-i\theta} \quad \text{and} \tag{2.26}$$

$$\psi \to e^{i\omega\gamma_5}\psi \quad ; \quad \overline{\psi} \to \overline{\psi}e^{i\omega\gamma_5} . \tag{2.27}$$

Note that the two terms $\left(\overline{\psi}\psi\right)^2$ and $\left(i\overline{\psi}\gamma_5\psi\right)^2$ are both not invariant under the chiral transformation. Their sum however is, as becomes clear from

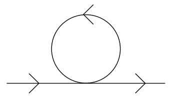
$$\overline{\psi}\psi \to \overline{\psi}\psi\cos 2\omega + i\overline{\psi}\gamma_5\psi\sin 2\omega \quad \text{and} \tag{2.28}$$

$$i\overline{\psi}\gamma_5\psi \to i\overline{\psi}\gamma_5\psi\cos 2\omega - \overline{\psi}\psi\sin 2\omega , \tag{2.29}$$

where we used the representation

$$e^{i\omega\gamma_5} = \mathbb{1}\cos\omega + i\gamma_5\sin\omega \text{ , with } \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.30}$$

Although the fermion starts massless, it turns out that the self-energy of the fermion propagator, which to the first order in the loop expansion receives a contribution of the single diagram



, (2.31)

can generate a mass m. With a high energy cut-off at $p^2 = \Lambda^2$ for the loop-integral we get the equation which determines the pole in the propagator

$$\frac{m^2}{\Lambda^2}\ln\left(\frac{\Lambda^2}{m^2} + 1\right) = 1 - \frac{2\pi^2}{g_0\Lambda^2} , \tag{2.32}$$

see [1]. Although an exact equation for the mass can not be formulated, we see that eq. (2.32) only has a solution for $0 < 2\pi^2/g_0\Lambda^2 < 1$ and that for a coupling constant g_0 higher than $2\pi^2/\Lambda^2$ a fermion mass is generated, increasing as the coupling gets stronger.

Creating a boson out of fermions

Chiral symmetry can only occur in massless fermionic systems and is broken when a mass term $m\overline{\psi}\psi$ enters in the Lagrangian, as we can see from eq. (2.28). Goldstones theorem predicts that we must obtain a massless boson the moment the mass is generated, but as we started with a purely fermionic theory, the only way to obtain a boson is by the formation of a bound state of fermions.

Nambu and Jona-Lasinio analysed this in the following way: with ψ_0 and ψ_m quantized fields satisfying respectively the massless and massive Dirac equation

$$\gamma_\mu \partial^\mu \psi_0(x) = 0 \quad \text{and} \quad (\gamma_\mu \partial^\mu + m)\, \psi_m(x) = 0 \;, \tag{2.33}$$

the massless field ψ_0 must at some point in time transform into the massive field ψ_m. If we set this at time $t = 0$ then we have the boundary condition

$$\psi_0(x) = \psi_m(x) \quad \text{when} \quad t = 0 \;. \tag{2.34}$$

Expanding both ψ_0 and ψ_m in their creation- and annihilation-operators we have two sets of operators, which we denote by respectively (a_0, b_0) and (a_m, b_m). We define the vacuum Ω_0 or Ω_m with respect to the field ψ_0 or ψ_m by

$$a_0\Omega_0 = b_0\Omega_0 = 0 \;, \tag{2.35}$$

$$a_m\Omega_m = b_m\Omega_m = 0 \;. \tag{2.36}$$

The two operator sets are related by the boundary condition of eq. (2.34) and this enables us to express the vacuum Ω_m in sets of the creation operators $a_0^\dagger$ and $b_0^\dagger$, see [1]. Thus the vacuum is a superposition of pairs of fermion and anti-fermions. An analysis shows that the energy of Ω_m is infinitely lower than Ω_0 and we can conclude that we have a new ground state with a condensate of bound state bosons.

The chiral current

Noether's theorem states that there is a conserved current associated with each generator of a symmetry. For the symmetries under the transformations of eq. (2.26) and (2.27) we have

$$\partial_\mu(\overline{\psi}\gamma^\mu\psi) \equiv \partial_\mu J^\mu = 0 \;, \tag{2.37}$$

$$\partial_\mu(\overline{\psi}\gamma_5^\mu\psi) \equiv \partial_\mu J_5^\mu = 0 \;. \tag{2.38}$$

In a system of free massive fermions there is no chiral symmetry and the divergence of the axial current reads

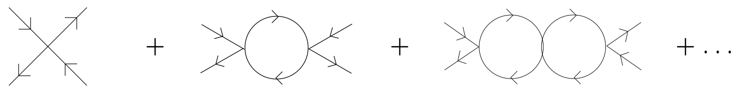
$$\partial_\mu(\overline{\psi}_m\gamma_5^\mu\psi_m) \equiv \partial_\mu {J_5}_m^\mu = 2m\overline{\psi}_m\psi_m \;. \tag{2.39}$$

With the generated mass in the NJL model we have that eq. (2.38) still holds, but only under specific conditions: Nambu and Jona-Lasinio show that a system can have mass generation with conserved axial current *only if* there is an interaction between the massive fermions and the condensate of massless fermion-antifermion pairs.

Identification of the particles

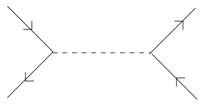
In 1961 the whole notion of quarks was still unknown and at that time the fermions in the NJL model represented nucleons: protons and neutrons. The authors continue to show their model to contain many bound states: not only scalar particles, but also pseudo-scalar, vector and axial vector particles. The "pseudo-" refers to the fact the particles have all normal particle properties, except for parity-symmetry and in the NJL model the scalar particles are massive, while the pseudo-scalar particles are not.

The next step was to identify the bound states. By analysing the full two-fermion scattering given by the summation of



(2.40)

they show the scattering amplitude to be equal to the scattering caused by the exchange of a single particle:



. (2.41)

Together with an analysis of the quantum numbers the pseudo-scalar was thus identified as the pion. There was however one problem: how could the massless bound state be the pion, which was known in nature to have a non-zero mass?

Approximate chiral symmetry

This problem was fixed in their second article [1], published in the same year: if the chirally symmetric Lagrangian of eq. (2.25) is extended with a small bare fermion mass, we can perform all previous taken steps ending up with a massive pion. With the important notion that for a small bare fermion mass, the pion mass is much higher. The inserted fermion mass can be seen as a perturbation to the chiral symmetry and this concept is the essence of chiral perturbation theory, also referred to as approximate chiral symmetry.

Nambu and Jona-Lasinio thus started with chiral symmetry and found a dynamical scheme for fermion mass generation. In their second article they however again insert a mass term by hand and find their result to comply better with nature. Although this is of course against the notion of the mass being dynamically generated, it turns out that this idea of chiral perturbation theory is of great use in QCD.

2.2.3 Quantum Chromodynamics

The theory of the strong force is based on the non-abelian gauge group $SU(3)$ of colors. It is an asymptotically free theory, which means that its coupling constant goes asymptotically to zero when the energy goes to infinity. Turned around, the coupling constant grows for lower energies and blows up at energies of order $\mathcal{O}(200\text{MeV})$. This energy is defined as the energy scale of the strong force Λ_c.

The $SU(3)$ color symmetry is an internal symmetry of each quark. In total there are six flavors of quarks divided into three generations of increasing mass. In general we can analyse any number of N flavors by a $SU(N)$ symmetry group. The internal $SU(3)$ color group of the quarks does not decouple, i.e. its symmetry does not break; it turns out that it is the flavor group of the multiple quarks which breaks up.

The theory of NJL applied to quarks

In the previous sections we analysed the symmetry breaking of $U(1)_L \times U(1)_R$ down to the residual $U(1)_{L+R}$ group, in short $U(1)_L \times U(1)_R > U(1)_V$, in both the σ-model and the NJL model. This can easily be extended to a higher group: instead of a single fermion field ψ, we analyse the doublet of the two flavors up u and down d

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix} . \tag{2.42}$$

These two quarks are the lightest of the quarks and form the first generation. With some changes we can now translate the NJL idea of strong coupling to that of QCD. Starting with massless u and d, we have a chiral $SU(2)_L \times SU(2)_R$ symmetry. At a certain energy the strong force binds a quark and an anti-quark into a bound state and a condensate with vacuum expectation value

$$< \overline{\Psi}\Psi > \neq 0 \tag{2.43}$$

is formed, breaking the symmetry group down to $SU(2)_V$. The quarks then become massive and three Goldstone bosons are formed. As we started with a purely fermionic theory these bosons are bound states and are the pions.

Following the steps of NJL even further, we can add to the system a small bare mass for the quarks and we see the pions gaining a much higher mass. It turns out that the quarks have a small mass of order $\mathcal{O}(5\text{MeV})$ and through the approximate chiral symmetry breaking scheme the pions gain a mass of order $\mathcal{O}(140\text{MeV})$. With the energy scale of the strong force at $\Lambda_c = \mathcal{O}(200\text{MeV})$ the perturbation of the fermion mass $\mathcal{O}(5\text{MeV})$ is only a few percent, justifying the use of the mass as a perturbation on chiral symmetry.

Origin of the quarks mass

The scheme of Nambu and Jona-Lasinio is an excellent scheme for the generation of fermion mass. Unfortunately we still must add the small bare mass term for the whole theory to comply with experiments. That is, the experiments tell us what the exact bare masses are. One of the big unsolved questions in physics is still what the origin of these masses is. Next to the quarks we also have the three generations of leptons. These six known flavors with increasing mass of both quarks and leptons seem to have a pattern and an open question in physics is still if this is evidence for the existence of any underlying theory. We return to this question in section 2.3.3 where we introduce the theory of Technicolor.

2.3 Electroweak symmetry breaking; The Higgs field

In the abelian model we saw the one generator of the electromagnetic gauge group $U(1)$ being broken and the photon acquire mass. The weak interaction is based on the $SU(2)$ gauge group which has three generators. This group can be broken in different ways: it can be broken down to the sub-group $U(1)$, leaving one gauge boson massless and two bosons acquire mass, or it can be fully broken in which case all three gauge bosons become massive.

This latter case can be established if we analyse a model with an $SU(2)$ gauge field coupled to a doublet ϕ of complex scalar fields

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} . \tag{2.44}$$

The field ϕ transforms as a spinor of $SU(2)$. The Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi) - V(\phi) , && (2.45) \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu , && (2.46) \\ \mathcal{D}_\mu &= \left(\partial_\mu - igW^a_\mu \tau^a\right) , && (2.47) \\ V(\phi) &= -\mu^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 , && (2.48) \end{aligned}$$

g being the coupling constant and the generators $\tau^a = \frac{1}{2}\sigma^a$. With $a = 1, 2, 3$ we have three gauge bosons W^a and the three pauli matrices $\{\sigma^1, \sigma^2, \sigma^3\}$.

The potential gives way to a groundstate with a non-zero expectation value $<\phi>$, which by a transformation in $SU(2)$ can be written as

$$<\phi> = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} , \tag{2.49}$$

with v real. Just as in the abelian case we expand about this groundstate, generating many new terms in the Lagrangian. The kinetic part $(\mathcal{D}_\mu \phi)^2$ leads among others to the quadratic term for the gauge field

$$\begin{aligned} \Delta\mathcal{L} &= \frac{1}{2} g^2 \begin{pmatrix} 0 & v \end{pmatrix} \tau^a \tau^b \begin{pmatrix} 0 \\ v \end{pmatrix} W^a_\mu W^{b\mu} = \frac{1}{2} \frac{g^2 v^2}{4} W^a_\mu W^{a\mu} && (2.50) \\ &\equiv \frac{1}{2} m^2_W \left(W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu} + W^3_\mu W^{3\mu} \right) && (2.51) \end{aligned}$$

and we see that all three gauge bosons have acquired the same mass m_W. The mere interaction of the gauge bosons with the scalar particle breaks all three degrees of symmetry and the full group $SU(2)$ is broken.

There are many ways of breaking symmetries. For example, had we considered a model in which the field ϕ is a triplet of real scalar fields transforming under the vector representation of $SU(2)$, we would have found that one gauge boson remains massless, implying that the group had been broken down to $U(1)$, see [18, 28].

2.3.1 The Glashow-Weinberg-Salam theory

The weak interaction is characterized by its weak force compared to the strong interaction, hence its name. Another important property is that it only acts over very small distances. This suggests that the mediators are massive, as this could yield a Yukawa potential $V(r) \sim e^{-mr}/r$ for particles interacting via the weak force and separated by a distance r. The mediators have mass m and carry the force between the particles. The negative exponential makes that the range of the force is of order m^{-1}.

It turns out that the three weak mediators and the photon can be unified into one theory of electroweak interaction, mediated by the $SU(2) \times U(1)$ gauge group. In this model spontaneous symmetry breaking occurs and the three weak mediators gain mass, leaving the photon massless. In other words the $SU(2) \times U(1)$ is broken down to $U(1)_{EM}$ and the three Goldstone bosons are eaten by the weak gauge bosons.

We saw in eq. (2.51) how the full breakdown of the $SU(2)$ group generated three masses. We extend this model with a $U(1)$ gauge symmetry, meaning that the doublet field ϕ transforms under two gauge transformations:

$$\phi \quad \to \quad e^{i\alpha^a \tau^a} e^{i\omega/2} \phi \, . \tag{2.52}$$

A groundstate with vacuum expectation value

$$< \phi >= \frac{1}{\sqrt{2}} \left(\begin{array}{c} 0 \\ v \end{array} \right) \tag{2.53}$$

breaks the three generators of the $SU(2)$ group. It is easy to see that it does not affect the $U(1)$ group: a simple gauge transformation as in eq. (2.52) with

$$\alpha^1 = \alpha^2 = 0 \quad \text{and} \quad \alpha^3 = \omega \tag{2.54}$$

leaves the vacuum expectation value $< \phi >$ invariant. [1] Thus the $U(1)$ gauge transformation is unaffected and the photon remains massless.

For the exact mass spectrum we need the correct covariant derivate $\mathcal{D}_\mu$ of ϕ with both the $SU(2)$ gauge bosons W^a_μ and the $U(1)$ boson B_μ:

$$\mathcal{D}_\mu = \partial_\mu - igW^a_\mu \tau^a - i\frac{1}{2}g' B_\mu Y \, , \qquad Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, . \tag{2.56}$$

Inserting the vacuum expectation value $< \phi >$ into the Lagrangian then yields the quadratic term of the gauge bosons

$$\Delta\mathcal{L} \;=\; \frac{1}{2}\begin{pmatrix} 0 & v \end{pmatrix} \left(gW^a_\mu \tau^a + \frac{1}{2}g' B_\mu \right) \left(gW^a_\mu \tau^a + \frac{1}{2}g' B^\mu \right) \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.57}$$

$$\;=\; \frac{v^2}{8} \begin{pmatrix} W^1_\mu & W^2_\mu & W^3_\mu & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \\ B_\mu \end{pmatrix} \tag{2.58}$$

[1] Remember that in a standard representation the pauli matrices are:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad ; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, .$$

In the last line we introduced a mass matrix which can be diagonalized by a transformation of our fields. We define

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) , \tag{2.59}$$

$$Z_\mu^0 = W_\mu^3 \cos\theta_w - B_\mu \sin\theta_w , \tag{2.60}$$

$$A_\mu = W_\mu^3 \sin\theta_w + B_\mu \cos\theta_w , \tag{2.61}$$

with the *weak mixing angle* θ_w given by

$$\cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}} , \quad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}} . \tag{2.62}$$

The mass spectrum becomes clear now and we end up with the three massive bosons W_μ^+, W_μ^- and Z_μ^0, while A_μ is massless:

$$m_{W^+} = m_{W^-} \equiv m_W = g\frac{v}{2} , \tag{2.63}$$

$$m_Z = \sqrt{g+g'^2}\frac{v}{2} , \tag{2.64}$$

$$m_A = 0 . \tag{2.65}$$

An important aspect which complies with experimental results is that the masses of the $W^\pm$ and Z bosons are related by

$$\frac{m_W}{m_Z} = \cos\theta_w \tag{2.66}$$

and they have been measured to be $m_W = \mathcal{O}(80\text{GeV})$ and $m_Z = \mathcal{O}(90\text{GeV})$.

2.3.2 The Higgs field

In the GWS theory a $SU(2) \times U(1)$ gauge group is coupled to a scalar field ϕ and the symmetry is spontaneously broken down to $U(1)_{EM}$. The complex doublet ϕ consists of four real fields ϕ_1, ϕ_2, ϕ_3 and ϕ_4 of which three can be gauged away, leaving only one scalar field to interact with.

The scalar particle associated with this field is the *Higgs particle*. Although it has not been found yet, the experimental measures up to high accuracy of the mass relation of eq. (2.66) indicate that the GWS theory is correct.

In the Standard Model the GWS theory is coupled to QCD, the theory of strong interaction, and in this model the Higgs boson is an elementary particle; this means it is not a composite of any other constituents. There are however aspects of a fundamental scalar particle which pose some problems, the most important being its unnaturalness: the expectation value breaking the electroweak symmetry and the mass of the fundamental Higgs boson are themselves unstable against radiative corrections.

The Higgs boson has therefore been speculated to be a composite particle. In 1978 Weinberg [21] and Susskind [22] proposed to again use the same techniques of the BCS and NJL model for finding a dynamical origin for the Higgs boson and came up with the idea of Technicolor: the Higgs boson is a composite in a whole new family of particles, the so called techni-quarks.

However, it was first tried to compose the Higgs out of quarks and although the applied scheme starts out promising, the end result is wrong as we shall now show.

A dynamically generated composite Higgs particle: the wrong way

We have a total of three generations of quarks and of leptons. For simplicity we restrict ourselves to the first generation with the two quark flavors up u and down d, and the two lepton flavors electron e and e-neutrino ν_e. We define

$$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad ; \quad l_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L . \tag{2.67}$$

With $\alpha = 1, \ldots, 8$ we have eight gluons G^α_μ and the eight Gell-Mann[2] matrices λ^α . We analyse the first generation of quarks and leptons coupled to the $SU(3) \times SU(2) \times SU(1)$ gauge group: the covariant derivative has the form

$$\mathcal{D}_\mu = \partial_\mu - i\frac{1}{2} g_S G^\alpha_\mu \lambda^\alpha - i\frac{1}{2} g W^a_\mu \tau^a - i\frac{1}{6} g' B_\mu Y . \tag{2.68}$$

In the Standard Model there is a left-right handedness asymmetry: the left-handed particles come in doublets, while the right-handed particles come in singlets and the neutrino's do not have a right-handed representation. This has no consequences for the current topic and we will not go into more details on this, although it is visible in the following Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} G^\alpha_{\mu\nu} G^{\alpha\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + i\bar{l}_L \gamma_\mu \mathcal{D}^\mu l_L + i\bar{e}_R \gamma_\mu \mathcal{D}^\mu e_R + \\ & + i\bar{q}_L \gamma_\mu \mathcal{D}^\mu q_L + i\bar{u}_R \gamma_\mu \mathcal{D}^\mu u_R + i\bar{d}_R \gamma_\mu \mathcal{D}^\mu d_R \end{aligned} \tag{2.69}$$

where $G^\alpha_{\mu\nu}, W^a_{\mu\nu}$ and $B_{\mu\nu}$ are respectively the field strengths for the gluon and the electroweak sector. The system consists of massless fermions coupled via the strong, weak and electromagnetic force.

In the Standard Model the energy scale of the electroweak force $\Lambda_{weak} = \mathcal{O}(175\text{GeV})$ is much higher than the energy scale of the strong force $\Lambda_c = \mathcal{O}(200\text{MeV})$ and normally the electroweak symmetry is therefore broken before the strong force creates pions and generates quark mass. If we however assume for a moment that the weak force is turned off we can study possible effects of the strong coupling.

As we have a doublet of massless quarks, we have a chiral flavor symmetry

$$SU(2)_L \times SU(2)_R . \tag{2.70}$$

We saw in the NJL model that if an attractive force is strong enough, the vacuum can destabilize and spontaneously generate bound states. In QCD we have a strong attractive force between quarks and anti-quarks, and not between leptons as these carry no color charge. We can now reasonably assume that the same phenomenon as in NJL model can occur: the vacuum spontaneously breaks down and generates bound states of quarks anti-quarks

$$< \bar{u}u > = < \bar{d}d > \neq 0 , \tag{2.71}$$

which breaks the chiral flavor symmetry down to the subgroup $SU(2)_{L+R}$. In contrast to the condensate $< \overline{\Psi}\Psi >$ in eq. (2.43) the different flavors of quarks

[2]These 3×3 matrices are the generators of the $SU(3)$ group; to be compared to the three matrices $\tau^a = \sigma^a/2$ generating the $SU(2)$ group.

do not mix: this leaves the chiral symmetry acting on the individual flavors intact, thus leaving both quarks massless.

The chiral symmetry $SU(2)_{L-R}$ on the doublet of flavors however has been broken, yielding three massless Goldstone bosons, which have to be bound states of fermions. If we now turn on the weak force again we have a system of fermions and three Goldstone bosons coupled with the three weak mediators. As in the GWS theory, see section 2.3.1, we can gauge the Goldstone bosons away and we are left with three massive gauge bosons.

Incorrect mass result

We can thus recreate the same symmetry breaking scheme as in the GWS theory, with the Higgs particle itself being generated by dynamics in the color gauge group. Unfortunately the resulting masses do not comply with experiments.

The generated m_W in the model we consider would be related to the QCD scale parameter $\Lambda_c = \mathcal{O}(200\text{MeV})$ and results in $m_W = \mathcal{O}(30\text{MeV})$, see [18]. This is a factor 10^3 too low compared to the real $m_W = \mathcal{O}(80\text{GeV})$. We have simplified the system by analyzing only one generation of flavors, but even if we consider the symmetry breaking of the three generations combined, the mass only increases to $m_W = \mathcal{O}(50\text{MeV})$, see [20].

As the breakdown of the electroweak force is via the same scheme as in the GWS theory, we do get the correct mass relation of eq. (2.66) with $m_W/m_Z = \cos\theta_w$. It is the scale of the mass which is not correct.

2.3.3 Technicolor

Although the strong color force mediated by the $SU(3)_c$ gauge group does not result in the correct mass, we have established in the previous section almost the same result as the GWS theory. Instead of a fundamental Higgs particle, we have broken the electroweak force through fermion-anti-fermion pairs, dynamically generated by the strong force. It almost worked: the correct mass relation between the gauge boson is established, but the scale of the mass however is too far off. The solution has been sought in what is called the Technicolor model, an exact copy of the asymptotically-free strong force $SU(3)_c$ interaction, with a higher energy scale parameter Λ_{TC} as to obtain the correct mass scale.

Instead of the group $SU(3)_c \times SU(2) \times U(1)$ we thus consider the group $SU(3)_{TC} \times SU(2) \times U(1)$. Equivalent to quarks or leptons we have the techni-quarks or techni-leptons and denoting these as we did with the QCD doublets in eq. (2.67) we have the first generation of techni-fermions

$$Q_L = \begin{pmatrix} U \\ D \end{pmatrix}_L \qquad \text{and} \qquad L_L = \begin{pmatrix} N_E \\ E \end{pmatrix}_L \ . \tag{2.72}$$

It is now the strong TC-force which breaks the chiral TC-flavor symmetry by forming bound states of the techni-quarks:

$$< \overline{U}U > = < \overline{D}D > \neq 0 \ . \tag{2.73}$$

Three bound states act as Goldstone bosons which when coupled to the electroweak gauge group break the weak generators, disappearing themselves and rendering the weak mediators massive. We saw that the mass m_W is related to the energy scale of the force triggering the symmetry breaking. If the $W^{\pm}$ pions are to have mass $m_W = \mathcal{O}(80\text{GeV})$ it turns out that $\Lambda_{TC} = \mathcal{O}(1\text{TeV})$.

Coupling with QCD in Extended Technicolor

The Technicolor model breaks the electroweak force exactly as we want it and without a fundamental Higgs boson, but through a techni-quark bound state coupled to the weak mediators.

Technicolor was not only established to get rid of a fundamental scalar Higgs particle. As mentioned, another problem in physics which has not been resolved is the flavor symmetry with fermion masses: the classification of the six flavors as we see them with three generations of increasing mass.

The idea is that by using the fermion mass generating scheme of NJL we can solve this problem. Embedding both the technicolor gauge group and color gauge group into a higher group of *Extended Technicolor* G_{ETC}, we introduce interactions between the techni-fermions F and familiar fermions f. This interaction then breaks the Extended TC gauge symmetry.

We saw in the NJL theory, see section 2.2.2, how a massless fermionic theory with quartic interaction can, if the coupling is strong enough, break the chiral symmetry and generate fermion mass. In a similar way Extended TC breaks down to TC at some energy level forming bound states and generating the fermion mass. The quartic interaction has the form, see eq. (2.25),

$$\mathcal{L}_{int} = g_{ETC}\left[\left(\overline{F}F\right)\left(\overline{f}f\right) + \left(i\overline{F}\gamma_5 F\right)\left(i\overline{f}\gamma_5 f\right)\right] \ . \tag{2.74}$$

Tumbling Extended Technicolor

The six flavors of fermions we know, can not be generated by one symmetry breaking and for the flavors to be grouped in generations, the ETC group needs to be broken multiple times down to TC. This phenomenon is called tumbling, first introduced in 1980 by Raby, Dimopoulos and Susskind [23] and its breaking in several successive stages creates apart from the fermion flavor symmetry many more particles and bound states.

The three known generations of mass could for example be generated by the sequential breakdown

$$SU(6)_{ETC} \ \rightarrow \ SU(5)_{E'TC} \ \rightarrow SU(4)_{E''TC} \ \rightarrow \ SU(3)_{TC} \ . \tag{2.75}$$

At each breaking the subgroup's gauge coupling becomes strong, raising the possibility of new condensates, which in turn generates the next breaking to another subgroup.

Of course the idea sounds simple, but in practice it is not. The breaking into subgroups is possible in many ways and the challenge is to find the so-called MAC's, or *most attractive channels.* For example, if a group G breaks down to subgroups G_1 and G_2, Raby *et al.* showed that the potential of the channel can be approximated by

$$V \sim C_G - C_{G_1} - C_{G_2} \ , \tag{2.76}$$

with C_G, C_{G_1} and C_{G_2} the eigenvalues of the Casimir operators for the combined group and subgroups. The MAC is the channel in which the combination of these operators yields the lowest possible potential.

This analysis results in many possible condensates and not only of scalar bound states. In 1981 Bais and Frere [24] analysed the possibility of vector channels and came to the conclusion that in many tumbling processes the vector condensates would create larger energy gaps than the scalar condensates. It could thus be that the gauge fields in the subgroups are dynamically generated, i.e. in a completely different way than the model set forth by Bjorken: not the breaking of Lorentz invariance, but the breaking of flavor symmetry generates effective gauge symmetries.

Objections to Technicolor

In this chapter we have discussed a few important symmetries in the Standard Model and along the way mentioned some of the unanswered questions still remaining: where does the bare quark mass come from, and is the Higgs boson fundamental or not? For a while, the answers to both questions were thought to be found with Technicolor, but unfortunately an exact ETC model turned out to be extremely complex and could never been fully accomplished.

The major obstacles to an adequate model were the possibility of Flavor Changing Neutral Currents, or FCNC's, and the anomaly cancellation by 't Hooft. The FCNC's make it possible for a quark q of a certain generation to transform into a quark q' of another generation, by the decay process

$$q \to Q \to q' \ , \tag{2.77}$$

with Q a techi-fermion. Certain experimentally verifiable processes receive contribution from these currents and consequently imply severe constraints on the theory.

The same goes for the anomaly cancellation by 't Hooft. An anomaly is a higher order contribution to an interaction which violates a certain symmetry and cancellations of these must be enforced as to keep the theory consistent. After more than two decades of research both the FCNC and the anomaly constraint could not be met in Technicolor, and by now Technicolor seems to be abandoned.

2.3.4 The Higgs sector

The search for the Higgs particle continues and research is still done on the possibility that the Higgs particle is a bound state. For example, in [11] and references therein the possibility is analysed that the Higgs boson is a top-quark condensate.

However, in the completely different approach by supersymmetry (SUSY) which implies an invariance under the interchange of fermions and bosons, the unnaturalness of the Higgs boson is not present and the scalar boson is an elementary particle. In this thesis we do not go into more details on this theory; we can only say that the idea of the Higgs being a fundamental particle seems to be favored by many at the moment.

Perhaps more important is that in two years the construction of the Large Hadron Collider at Cern in Geneva will be completed and the Higgs sector, i.e. energies of $\mathcal{O}(1\text{TeV})$ and higher, will come within experimental reach. The experimental verification of the existence of the Higgs particle is of course one of its first goals, but the energies will also be high enough to measure possible low-energy features of Technicolor and SUSY, see [25]. It is up to these experiments to shed light on the Higgs sector.

2.4 Dynamical symmetry breaking: the vector condensate

In contrast to a scalar condensate given by for example $<\overline{\psi}\psi>$ a vector condensate given by $<\overline{\psi}\gamma_\mu\psi>$ does not break chiral symmetry. We can see this in the chiral transformations of eq. (2.27), yielding

$$\overline{\psi}\gamma_\mu\psi \rightarrow \overline{\psi}\gamma_\mu\psi\ ; \tag{2.78}$$
$$\overline{\psi}\psi \rightarrow \overline{\psi}\psi\cos 2\alpha + i\overline{\psi}\gamma_5\psi\sin 2\alpha\ . \tag{2.79}$$

It does however break Lorentz invariance, as a non-zero vector condensate implies that space-time can have a preferred direction. In the theory set forth by Bjorken [2] this is not seen as a problem. The vector condensate $<\overline{\psi}\gamma_\mu\psi>$ fixes a gauge leaving all physical observables Lorentz invariant, and this interpretation has extensively been discussed and verified by Nambu [4]. The symmetry breaking by vector condensates is thus phenomenological totally different from the breaking by scalar condensates.

As we mentioned in section 1.1, after Bjorken discussed a possible composite photon the idea was generalized to the non-abelian case by Eguchi [6]. Yet in his second article [7] he argues that photons could not be composites in his model, because an UV fixed point is needed.

There is an essential difference between abelian and non-abelian gauge theories, namely the fact that when lowering the energy we see the coupling constant in abelian cases decrease, while in non-abelian cases the coupling constant increases. In any Renormalization Group [3] approach this distinction is essential: non-abelian gauge theories, or QCD-like theories have a UV fixed point, while abelian, or QED-like theories have an IR fixed point.

In the Standard Model QED is the only theory with an IR fixed point, and it is embedded into the GWS theory of electroweak force. As we discussed in section 2.3 the underlying dynamics of the electroweak symmetry breaking are still not fully understood. It is perhaps possible that in the unified electroweak theory mechanisms are possible which dynamically generate all gauge fields. For example, in Suzuki's model [8, 9] the electroweak symmetry is broken without any scalar Higgs particle and the $W^\pm, Z$ gauge fields are dynamically generated. In a study model by Dmitrasinovic [14] the symmetry is broken by a composite Higgs boson, dynamically generating all gauge fields, including the massless photon.

2.4.1 Historical overview

On page 27 we present a graphical overview on most of the theories discussed in this chapter. Before the formation of Cooper pairs was known, the breaking of the electromagnetic, the strong and the weak force were only understood

[3]See appendix C.2 for more details.

by means of spontaneous symmetry breaking models. This completely changed when the BCS theory of superconductivity was invented.

In the model by Nambu and Jona-Lasinio, based on the BCS theory, the pion was thought to be a composite of nucleons; this was written before quarks were known, but the model was perfectly translated to the idea that the pion is made up of quarks. The model also served as a prototype for the idea of Technicolor and many more theories, not listed in this figure, in which the Higgs boson is a composite of either techni-quarks, quarks, or other constituents.

The inspiration for Bjorken to come up with a composite photon also came from the NJL model. His idea was that by an unknown mechanism a vector condensate would form, breaking the Lorentz symmetry and creating the photon out of a fermion and an anti-fermion. Eguchi translated his idea to the non-abelian case and in Suzuki's model the electroweak symmetry is broken, forming massive composite $W^{\pm}$ and Z bosons.

In view of the Renormalization Group approach the equivalence between (non-)abelian gauge theories and purely fermionic theories has been analysed by, among others, A. and P. Hasenfratz for the $SU(N)$ symmetries and by Sonoda for the specific case of QED.

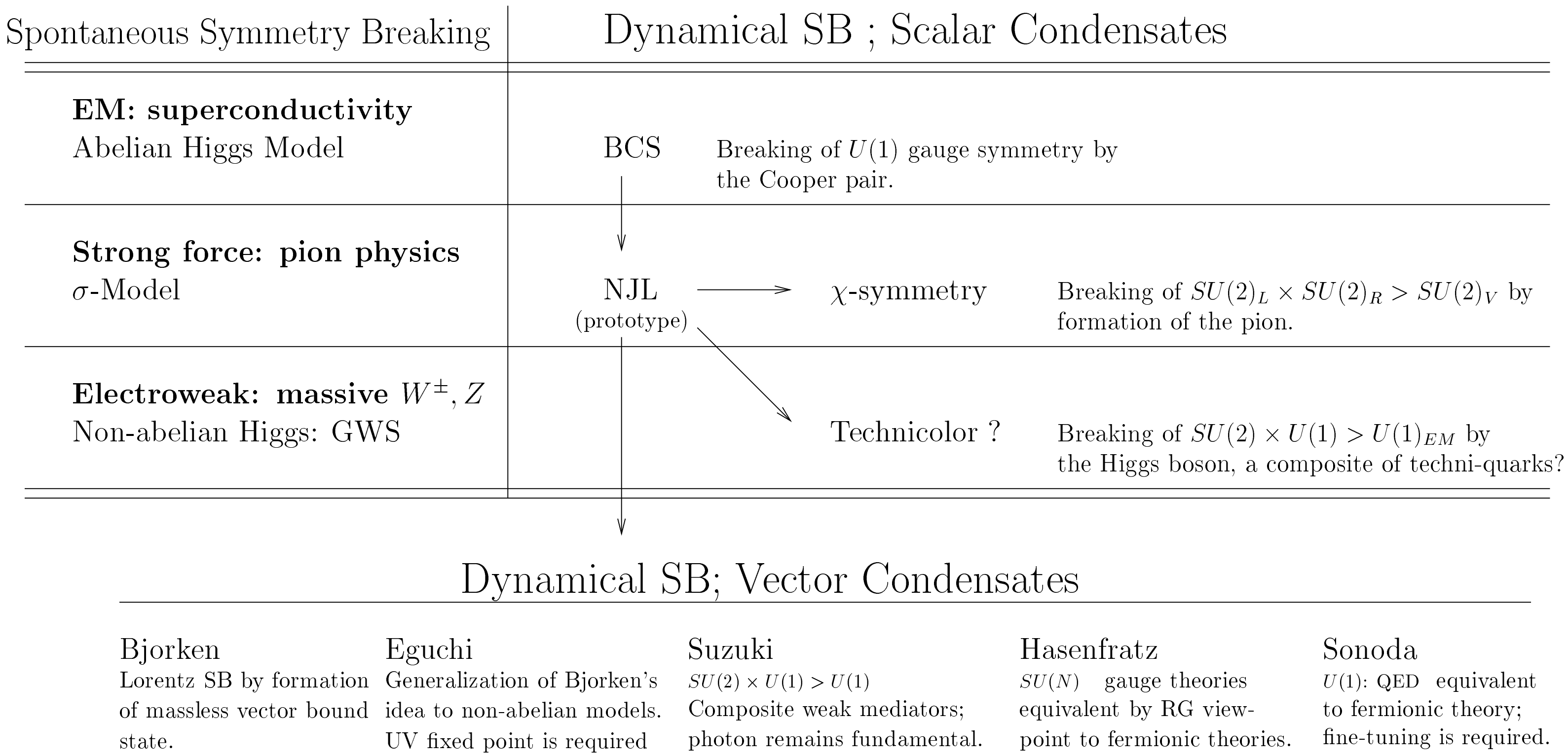


Figure 2.1: Historical Overview.

Chapter 3

Quantum Electrodynamics

In this chapter we discuss the QED Lagrangian and the precise form of it that we will be using. The use of the 1/N expansion is introduced and explained. In the second section we determine the Feynman rules, that is the rules for associating analytic expressions with pieces of Feynman diagrams.

3.1 The QED Lagrangian

Standard QED has only a few terms in its Lagrangian: a kinetic and mass term for the electron field, a kinetic term for the photon field such that only the transverse components remain, and an interaction term. We adopt the somewhat unconventional definition of

$$\overline{\psi} \equiv -i\psi^{\dagger}\gamma_0 \ , \tag{3.1}$$

which compared to the standard notation implies the transformation $\overline{\psi} \to -i\overline{\psi}$, see [28]. In Minkowski space the Lagrangian now reads:

$$\mathcal{L}_{\mathrm{M}} = \overline{\psi}\,(\partial\!\!\!/ + iM)\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + ie_0 A_\mu \overline{\psi}\gamma^\mu\psi \ . \tag{3.2}$$

We set out to show that at low energies a purely fermionic theory is equivalent to QED by analyzing the renormalization flows of both theories and concluding that these are the same. Plain perturbation theory [1] restricted to the electrodynamics of a single electron field does not suffice. We therefore extend our system to N fermion fields, i.e. N different flavors of fermions with the same mass and electric charge. In addition to plain perturbation theory we need the $1/N$-expansion, see [32].

For convenience we will not be working in Minkowski space, but in Euclidean space. This is achieved by the transformation $t \to it_{\mathrm{E}}$, which has implications for the definitions of several inner products, see appendix A.1. We also absorb the constant e_0 into the gauge field A_μ. With $I = 1, \ldots, N$ and summing over I this results in the Lagrangian

$$\mathcal{L}_{\mathrm{E}} = \overline{\psi}^I\,(-i\partial\!\!\!/ + iM)\,\psi^I + \frac{1}{4e_0^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{\sqrt{N}}A_\mu\overline{\psi}^I\gamma^\mu\psi^I \ , \tag{3.3}$$

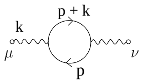
[1] we define *plain perturbation theory* as the commonly used perturbative approach by means of expansion of the electric charge parameter e. In section 3.1.1 we return to this point.

invariant under the local gauge transformations given by eq. (1.1) and (1.2).

The coupling constant

Comparing eq. (3.2) and (3.3) we see that the coefficient of the interaction term has gone from $ie_0 \to N^{-1/2}$. The i has disappeared through the change of the inner product $A_\mu \gamma^\mu$ and the e_0 has been absorbed into the fields A_μ. The order of N however needs extra explanation.

In section 4.2 it will become clear that together with the redefinition $e_0 A_\mu \to A_\mu$ and a coefficient with the correct order N we can perform the renormalization of most parameters by analyzing the one loop correction of the photon propagator



. (3.4)

The photon propagator itself is of order $\mathcal{O}(N^0)$, see eq. (3.29) in table 3.1, and the renormalization procedure will be performed up to the highest order in $\mathcal{O}(1/N)$. The diagram of eq. (3.4) is thus to be of the same order.

The single fermion loop is of order $\mathcal{O}(N^1)$ due to the summation of all N flavors as explained in section 3.1.1, and for the total expression of the diagram to be of order $\mathcal{O}(N^0)$ the vertex of one gauge and two fermion fields must be $\mathcal{O}(N^{-1/2})$; hence the coefficient $1/\sqrt{N}$ for the interaction term.

3.1.1 The $1/N$ expansion

When applying plain perturbation theory to the electrodynamics of a single electron the coupling constant between the fermion fields and the gauge field is the electric charge e. The analytic expression for the self-energy of the electron or the photon consists of twice the interaction vertex and is thus of order e^2. In fact plain perturbation theory is based on the expansion in

$$\alpha \equiv \frac{e^2}{4\pi} = \frac{1}{137} \tag{3.5}$$

and one can show that the series is asymptotically convergent. This expansion in terms of the coupling constant α coincides with the expansion in the number of loops in the diagrams and is hence often called loop-expansion.

For the $1/N$ expansion we extend the system to N flavors of fermions, all with the same mass and electric charge, and redefine the interaction term, see the Lagrangian (3.3). With $N \to \infty$ an expansion in powers of $1/N$ would be justified, hence the name. However, N does not have to go to infinity. Compared to an expansion in e we can conclude from eq. (3.5) that $N \geqq 3$ would generate comparable coefficients and be acceptable. See also [32].

This is of course still no proof that the $1/N$ expansion works or that it is justified. Especially in the case of QED one can wonder what the N fermion fields represent, for there are no N different flavors of fermions in nature with equal mass.

G. 't Hooft [31] was the first to introduce this expansion in chromodynamics where instead of three colors he extended his system to N colors. It turns out that the mathematics becomes much simpler with a large N while at the same time, dependence on the exact value of N is minimal; whether N is $2, 3$ or 4 has little impact on the physical observables.

The $1/N$ expansion has thus become a very useful method for analysing theories, even if there do not really exist N different flavors. For the case of QED, the subject of this thesis, it is argued in [12] that by the Renormalization Group viewpoint on the equivalence with a Four Fermi Theory, it is even justified in the end to set $N = 1$.

The N-order of diagrams

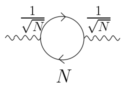
The coefficient of fermion-photon interaction has changed to $1/\sqrt{N}$. The total power of a diagram does not only depend on the number of interaction vertices, it also depends on the presence of fermion loops.

In the Lagrangian we have a summation over repeated I indices: not only in the interaction term, also in the terms generating the fermion propagator. We therefore have several possibilities for a fermion line, which vary depending whether the line is external or internal, and whether it is or is not part of a closed loop.

- A single fermion propagator; this is the propagator of one flavor and thus of order N^0.
- An external fermion line is connected to a vertex. The external line is again one single fermion and is of order N^0.

From the summation over $I = 1, \ldots, N$ in the interaction term we can see that all flavors of fermions can interact with a photon. Different flavored fermions do not interact with each other and we have flavor conservation at each vertex. This has consequences for internal fermion lines:

- An internal fermion line inside a closed fermion loop: combining the summations over I with the flavor conservations at each vertex we have a single summation of all N flavors going round once. Independent of the number of fermion lines the loop is made of, each closed loop is of order N. As an example, let us analyse the following diagram



$$\sim \mathcal{O}(N^0). \tag{3.6}$$

  Both vertices are of order $N^{-1/2}$. The analytic expression for this diagram has two summations over I coming from the two vertices, but the combined flavor conservation at each vertex cancels one: the closed loop is of order N.

- An internal fermion line not part of a closed fermion loop: both ends of the line are directly or indirectly connected to an external fermion line. Because of the flavor conservation there is therefore no summation over N and the line is of order N^0.

In our renormalization procedure we will be performing all calculations up to the leading order in $1/N$.

3.1.2 Regulating higher order shifts

Renormalizing a theory is all about defining the real physical parameters which can be compared with experimental results. The value of a physical parameter is a function of the energy of the system and information can be extracted out of the gradient of this function, the so-called renormalization flow, see appendix C. When defining the physical parameter we have to take into account higher order terms. At the same time it may be that higher order terms generate parameters which were not present at first, for example a mass term.

In the $1/N$ expansion in QED we need to add three parameters to the Lagrangian of eq. (3.3): a bare photon mass, a quartic gauge interaction parameter and a gauge fixing parameter. The first two are due to the fact that we will be using the cut-off regularization scheme which does not respect gauge invariance. The third is necessary to deal with an excess of degrees of freedom, a consequence of the gauge invariance. We now introduce these three terms one by one.

A photon mass

When regularizing we can use gauge invariance as a given fact and impose the condition that our regularization is to respect the invariance, as is the case with dimensional regularization. Or we can regularize without it and use our renormalization conditions afterwards to impose the physical quantities to respect gauge invariance. This is the case in the cut-off regularization scheme which breaks the invariance by generating a photon mass.

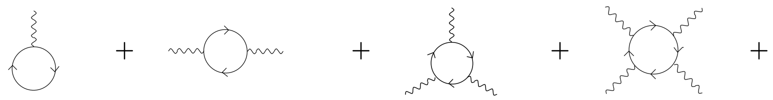
Our aim in this thesis is to fine-tune a purely fermionic theory such that its properties are indistinguishable from the ones in QED. To facilitate the comparison we set up our QED such that we have to renormalize its photon mass to zero. To do so, we introduce explicitly a photon mass parameter

$$\mathcal{L} \to \mathcal{L} + \frac{m_0^2}{2e_0^2} A^2 \tag{3.7}$$

and use the cut-off instead of the dimensional regularization scheme.

The box diagram

The box diagram is a higher order diagram with four external photon lines and one fermion loop. The name was introduced as the loop is often drawn as a square, see [33]. It is the fourth order term in the one loop expansion of the effective action given by



$\dots$ (3.8)

The expansion continues with more and more photon fields coupled by one fermion loop and at the same time the loop of course involves more fermion propagators. The cut-off dependence of a loop integral of n fermion propagators $S(p)$ can in short be summarized by

$$\int_{|p|<\Lambda} \mathrm{d}^4 p\, [S(p)]^n = \mathcal{O}(\Lambda^{4-n}) \,, \tag{3.9}$$

with Λ the high energy cut-off. By this rule the second diagram in eq. (3.8) is of order $\mathcal{O}(\Lambda^2)$ and diverges quadratically. It is the first correction term in the self energy of the photon propagator.

The first and third diagram in the expansion of eq. (3.8) are respectively of order $\mathcal{O}(\Lambda^3)$ and $\mathcal{O}(\Lambda^1)$. It can however easily be shown by Furry's theorem that an odd number of fermion propagators in a closed fermion loop always yields zero, see [28].

The fourth diagram is the box diagram, it is $\mathcal{O}(\Lambda^0)$ and non-zero. This interaction has four fermion-photon vertices, each of order $\mathcal{O}(N^{-1/2})$ and one fermion loop of order $\mathcal{O}(N^1)$ due to a summation over all N flavors in the loop, see section 3.1.1. Hence the diagram is of order $\mathcal{O}(N^{-1})$. As we will be performing all calculations up to the leading order in $1/N$, we need a counterterm of the same order in N to renormalize this diagram. We therefore expand our Lagrangian with a quartic gauge counterterm

$$\mathcal{L} \to \mathcal{L} + \Delta_4 \frac{1}{N8(4\pi)^2}(A_\mu A^\mu)^2 \ , \tag{3.10}$$

where the factor $1/8(4\pi)^2$ is purely a normalization factor and the Δ_4 is the coefficient to be defined by renormalization. From eq. (3.9) it becomes clear that if the loop consists of five or more fermion propagators, the expression converges. We therefore do not require any counterterms for these diagrams.

3.1.3 The gauge fixing method

Gauge invariance is not a symmetry, in the sense that two physical states have the same properties. The invariance of the system under the transformation $A_\mu \to A_\mu - \partial_\mu \omega$ merely states that two different gauge potentials describe the same physical state. There is a difference. The invariance is a result of using Lorentz vector fields; for a massive vector particle has three polarization states, a massless vector field has but two. There is an excess of degrees of freedom in Lorentz vector fields for describing massless photons and *that* is what the invariance exposes [27]. This redundancy in describing photons leads to problems in calculations and has to be eliminated by fixing the gauge.

In the calculation of the propagator $D^{\mu\nu}$ for the gauge field the redundancy makes it impossible to exactly define $D^{\mu\nu}$. This problem can be avoided by simply adding a gauge fixing term to the Lagrangian, which is guaranteed by the Faddeev-Popov not to have any measurable effects on physical observables, see [28].

Let us try to compute the photon propagator. We start with the free electromagnetic field and its action

$$\begin{aligned} \mathcal{S} &= \int \mathrm{d}^4x \left(\frac{1}{4e_0^2} F_{\mu\nu} F^{\mu\nu} \right) \\ &= -\int \mathrm{d}^4x \frac{1}{2e_0^2} A^\mu(x) \left(\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu\right) A^\nu(x) \ , \end{aligned} \tag{3.11}$$

where in the last equation we performed a partial integration. After standard procedures this leads to

$$-\left(\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu\right) D^{\nu\alpha}(x-y) = -\,\delta_\mu^\alpha \delta^{(4)}(x-y) \ , \tag{3.12}$$

with $D^{\nu\alpha}$ the photon propagator, $\delta_{\mu\nu}$ the Euclidean metric and $\delta^{(4)}(x-y)$ a four-delta function, only non-zero when $x-y=0$.

By a Fourier transformation we can express this in momenta [2]:

$$(k^2\delta_{\mu\nu} - k_\mu k_\nu) D^{\nu\alpha}(k) = -\,\delta_\mu^\alpha\ . \tag{3.13}$$

But the matrix $(k^2\delta_{\mu\nu} - k_\mu k_\nu)$ has eigenvalues zero, thus no inverse and the propagator cannot be computed. The gauge fixing method now consists of changing the action by:

$$\mathcal{S} \to \mathcal{S} + \frac{1}{2{e_0}^2\xi_0}\int \mathrm{d}^4x\,(\partial^\mu A_\mu \partial^\nu A_\nu)\ . \tag{3.14}$$

This extra term breaks the gauge invariance as it is not invariant under the transformation of eq. (1.2). But the equation (3.12) for the propagator becomes

$$\begin{aligned} &\left(\partial^2\delta_{\mu\nu} - (1-\frac{1}{{e_0}^2\xi_0})\partial_\mu\partial_\nu\right) D^{\nu\alpha}(x-y) = \delta_\mu^\alpha \delta^{(4)}(x-y) \\ \to\ &\left(-k^2\delta_{\mu\nu} + (1-\frac{1}{{e_0}^2\xi_0})k_\mu k_\nu\right) D^{\nu\alpha}(k) = \delta_\mu^\alpha \end{aligned} \tag{3.15}$$

and the inverse can now be computed: the propagator equals

$$D^{\mu\nu} = -\frac{1}{k^2}\left(\delta^{\mu\nu} - (1-{e_0}^2\xi_0)\frac{k^\mu k^\nu}{k^2}\right)\ . \tag{3.16}$$

The gauge invariance has not really been broken, i.e. at a physical level. Only in the action and in the path integral have certain gauge fields been eliminated. For the path integral takes into account all fields, that is all the fields ${A_\mu}' = A_\mu - \partial_\mu\omega$ with all possible values for ω. But all these fields describe the same physical situation as A_μ. With the gauge fixing parameter only one field is taken into account and not all gauge copies of it. When calculating physical quantities the parameter ξ_0 drops out. Thus the gauge invariance is indeed still present, for ξ_0 can afterwards be changed without changing results.

3.2 The Feynman rules

With all the extra terms of eq. (3.7), (3.10) and (3.14) the Lagrangian of eq. (3.3) becomes

$$\begin{aligned} \mathcal{L} \;=\; &\overline{\psi}^I(-i\partial\!\!\!/ + iM)\psi^I + \frac{1}{4e_0^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{\sqrt{N}}A_\mu\overline{\psi}^I\gamma^\mu\psi^I + \\ &+\frac{1}{2e_0^2\xi_0}(\partial_\mu A^\mu)^2 + \frac{m_0^2}{2e_0^2}A_\mu A^\mu + \Delta_4\frac{1}{N8(4\pi)^2}(A_\mu A^\mu)^2\ . \end{aligned} \tag{3.17}$$

The Feynman rules are the rules for associating analytic expressions with pieces of Feynman diagrams. We now derive these rules including those following from the counterterms. The corresponding coefficients are to be determined by the renormalization conditions. The use of $1/N$ expansion instead of plain

[2] We use the same letter D for the propagator before and after the Fourier transformation; the dependence on $(x-y)$ or on (k) states which function we are using, see appendix A.2.

perturbation theory makes this renormalization procedure somewhat different from the conventional scheme. With counterterms the Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2e^2\xi}\left(\partial_\mu A^\mu\right)^2 + \frac{m_\gamma^2}{2e^2}A^2 + \overline{\psi}^I\left(-i\partial\!\!\!/ + iM'\right)\psi^I + \\
&\quad + \frac{1}{\sqrt{N}}A_\mu\overline{\psi}^I\gamma^\mu\psi^I + \\
&+ \Delta_1\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \Delta_2\frac{1}{2}\left(\partial_\mu A^\mu\right)^2 + \Delta_3\frac{1}{2}A^2 + \Delta_4\frac{1}{N8(4\pi)^2}(A_\mu A^\mu)^2 + \\
&\quad \overline{\psi}^I\left(-\Delta_5 i\partial\!\!\!/ + \Delta_6 iM'\right)\psi^I + \Delta_7\frac{1}{\sqrt{N}}A_\mu\overline{\psi}^I\gamma^\mu\psi^I \; . \qquad (3.18)
\end{aligned}$$

For each term in the Lagrangian of eq. (3.17), we have in the Lagrangian (3.18) one dressed term and one counterterm, with the exception for the quartic gauge interaction, which we introduced as a counterterm to compensate for the box diagram.

This results in the definition of the following bare parameters:

$$\frac{1}{e_0^2} = \frac{1}{e^2} + \Delta_1 \; ; \qquad (3.19)$$

$$\frac{1}{e_0^2\xi_0} = \frac{1}{2e^2\xi} + \Delta_2 \; ; \qquad (3.20)$$

$$\frac{m_0^2}{e_0^2} = \frac{m_\gamma^2}{e^2} + \Delta_3 \; ; \qquad (3.21)$$

$$M = M'(1+\Delta_6) \; . \qquad (3.22)$$

3.2.1 The photon diagrams

The Feynman rules can be extracted from eq. (3.18). The first three terms yield the photon propagator. Performing partial integrations and going to momentum space we have that

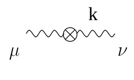
$$\begin{aligned}
&\frac{1}{2}A_\mu\frac{1}{e^2}\left(-\partial^2\delta^{\mu\nu} + \partial^\mu\partial^\nu - \frac{1}{\xi}\partial^\mu\partial^\nu + m_\gamma^2\delta^{\mu\nu}\right)A_\nu \quad \rightarrow \\
&\quad \frac{1}{2}A_\mu\frac{1}{e^2}\left(\delta^{\mu\nu}k^2 - (1-\frac{1}{\xi})k^\mu k^\nu + m_\gamma^2\delta^{\mu\nu}\right)A_\nu \; . \qquad (3.23)
\end{aligned}$$

In eq. (3.16) we already partially calculated the propagator. Our expression has now been extended with one term, the mass component. Comparing with eq. (3.16) we can easily see that the inverse propagator is:

$$(D^{-1})^{\mu\nu}(k) = -\left\{\frac{1}{e^2}(\delta^{\mu\nu}k^2 - k^\mu k^\nu) + \frac{1}{e^2\xi}k^\mu k^\nu + \frac{m_\gamma^2}{e^2}\delta^{\mu\nu}\right\} \; . \qquad (3.24)$$

The overall factor (-1) is a consequence of our Wick rotation. In Minkowski space it would have been a factor $(-i)$.

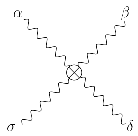
In a same manner the counterterms for the quadratic gauge field terms, that is the terms with coefficients Δ_1, Δ_2 and Δ_3 in Lagrangian (3.18), result in:



$$= -\left\{\Delta_1(\delta^{\mu\nu}k^2 - k^\mu k^\nu) + \Delta_2 k^\mu k^\nu + \Delta_3\delta^{\mu\nu}\right\} \; . \qquad (3.25)$$

The quartic gauge interaction

In the quartic gauge vertex we have to take into account the possible combinations of the 4 lines α, β, δ and σ, of which there are $4! = 24$ combinations. Due to momentum conservation these 24 combinations are a summation of three topologically different permutations:



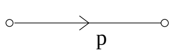
$$
\begin{aligned}
&= 4!\,(-1)\frac{\Delta_4}{N(4\pi)^2 8}\left\{\frac{1}{3}(\delta_{\alpha\beta}\delta_{\delta\sigma}+\delta_{\alpha\delta}\delta_{\beta\sigma}+\delta_{\alpha\sigma}\delta_{\beta\delta})\right\} \\
&= \frac{-\Delta_4}{N(4\pi)^2}(\delta_{\alpha\beta}\delta_{\delta\sigma}+\delta_{\alpha\delta}\delta_{\beta\sigma}+\delta_{\alpha\sigma}\delta_{\beta\delta})\ . \qquad (3.26)
\end{aligned}
$$

3.2.2 The fermion propagator

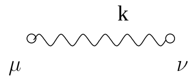
Obtaining the fermion propagator is now straightforward: a factor (-1) from the Euclidean weight, and the inverse of the Fourier transformed part of the quadratic fermion term:

$$
S(p) \equiv \frac{+1}{\not{p} - iM'} = \frac{\not{p} + iM'}{p^2 + (M')^2}\ . \qquad (3.27)
$$

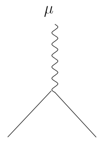
Table 3.1 contains an overview of all Feynman rules. It also contains the expressions whose explicit calculation we did not show: the quadratic fermion counter vertex, the fermion-photon vertex and the fermion-photon counter vertex.



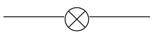
$$= \frac{+1}{\not{p} - iM'} = \frac{\not{p} + iM'}{p^2 + (M')^2} \tag{3.28}$$



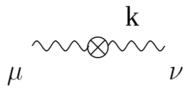
$$= -\left\{ \frac{1}{e^2}(\delta^{\mu\nu}k^2 - k^\mu k^\nu) + \frac{1}{e^2\xi}k^\mu k^\nu + \frac{m_\gamma^2}{e^2}\delta^{\mu\nu} \right\}^{-1} \tag{3.29}$$



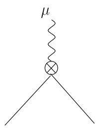
$$= -\frac{\gamma^\mu}{\sqrt{N}} \tag{3.30}$$



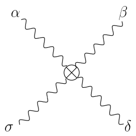
$$= \left\{ \Delta_5 \not{p} - i\Delta_6 M' \right\} \tag{3.31}$$



$$= -\left\{ \Delta_1(\delta^{\mu\nu}k^2 - k^\mu k^\nu) + \Delta_2 k^\mu k^\nu + \Delta_3 \delta^{\mu\nu} \right\} \tag{3.32}$$



$$= -\Delta_7 \frac{\gamma^\mu}{\sqrt{N}} \tag{3.33}$$



$$= \frac{-\Delta_4}{N(4\pi)^2}(\delta_{\alpha\beta}\delta_{\delta\sigma} + \delta_{\alpha\delta}\delta_{\beta\sigma} + \delta_{\alpha\sigma}\delta_{\beta\delta}) \tag{3.34}$$

Table 3.1: *A total list of all Feynman rules. Beware of the difference between lines with a dot at the end and lines without. The first are lines with propagators attached, the second are lines without and represent the various interaction vertices.*

Chapter 4

Renormalization of QED

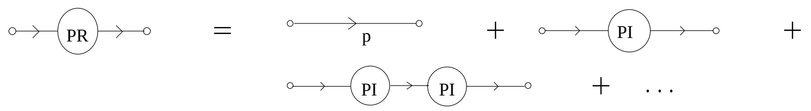
In this chapter we first perform the renormalization of the QED parameters to lowest order in the $1/N$ expansion. We then make sure our results satisfy the Ward identities and finally we analyse the obtained renormalization flows with resulting fixed points.

4.1 Renormalization conditions

Renormalization is the process of defining the physical parameters of a theory. These definitions have to comply with experimental results and are dependent on the energies of the system, i.e. the parameters are functions of the momenta. The renormalization conditions state these definitions: at a certain energy the physical parameter has to have the appropriate experimental value.

The fermion mass

Starting with the renormalization condition of the fermion mass we analyse the summation of:



(4.1)

The first diagram of the summation is the dressed fermion propagator given in eq. (3.28) of table 3.1. A "PI-bubble" is the summation of all "one Particle Irreducible" diagrams and is the self energy Σ of the fermion propagator. The endless summation of more and more PI-bubbles is represented by the "PR-bubble"; the abbreviation PR stands for "one Particle Reducible". We define the left-hand side of eq. (4.1), i.e. the PR-bubble with on both sides fermion propagators attached as the *full fermion propagator* S' and eq. (4.1) can thus be written as

$$S^{'}(p) = S(p) + S(p)\Sigma(p)S(p) + S(p)\Sigma(p)S(p)\Sigma(p)S(p) + \dots \tag{4.2}$$

Multiplying this equation with $(S^{-1}(p) - \Sigma(p))$ the right-hand side goes to unity. In other words

$$S^{'}(p) = \frac{1}{S^{-1}(p) - \Sigma(p)} = \frac{1}{\not{p} - iM' - \Sigma(p)} \,. \tag{4.3}$$

As a condition we want M' to be equal to the physical mass of the fermion. This means that the full propagator $S^{'}(p)$ has to have a pole at

$$\not{p} = iM^{'} \ . \tag{4.4}$$

The i in the right-hand side may seem strange but comes from being in Euclidean space, see last example in eq. (A.6).

Expanding $\Sigma(\not{p}) = A + B(\not{p} - iM^{'}) + C(\not{p} - iM^{'})^2 + \ldots$ we want both A and B zero at $\not{p} = iM^{'}$:

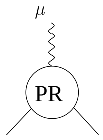
$$\Sigma(\not{p} = iM^{'}) \quad = \quad 0 \tag{4.5}$$

$$\left[\frac{\partial}{\partial(\not{p})}\Sigma(\not{p})\right]_{\not{p}\,=\,iM'} \quad = \quad 0 \ . \tag{4.6}$$

The second order term with coefficient C and the higher order terms are of no importance, because the inverse propagator $S^{-1}(p)$ is of first order in p. Therefore in Feynman integrals the order $\mathcal{O}(p^2)$ terms of $\Sigma(p)$ are suppressed by Λ^{-1}, with Λ the high momentum cut-off.

The electric charge

The coupling constant e, which we absorbed into the gauge field A_μ, is fixed via the fermion-photon interaction vertex, coming from the term $\frac{1}{\sqrt{n}}A_\mu \overline{\psi}^I \gamma^\mu \psi^I$. This is clearer when looking at standard QED, see eq. (3.2). With $p_{\rm in}$ and $p_{\rm out}$ defined as the momenta of respectively the incoming and outgoing fermion we have as renormalization condition that when working on-shell the full fermion-photon vertex

$\Gamma^\mu_{\overline{\psi}\psi} \equiv$  $= \dfrac{-\gamma^\mu}{\sqrt{N}}$ when $\not{p}_{\rm in} = \not{p}_{\rm out} = iM'$. (4.7)

The photon propagator

Just like the fermion self-energy was fixed on-shell at $p^2 = -{M'}^2$, see eq. (4.4), for the photon propagator one would expect fixing the self-energy $\Pi^{\mu\nu}$ at $k^2 = 0$ or for a massive photon at $k^2 = -m_\gamma^2$. But this turns out to be not useful and generates singularities, therefore we fix the self energy at the imaginary point $k^2 = \mu^2$, with μ the renormalization scale. This scale, which is the energy level of the theory, is to be low: $\mu \ll \Lambda$. The condition states

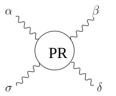
$$\Pi^{\mu\nu}(k^2 = \mu^2) = 0 \ , \tag{4.8}$$

and determines all three counter-coefficients Δ_1, Δ_2 and Δ_3.

The quartic gauge vertex

As a final condition we use the Ward identity concerning the quartic gauge vertex, which states $k_\alpha \Pi^{\alpha\beta\delta\sigma} = 0$, see section 4.3.1. As was explained when introducing the counterterm in eq. (3.10), the four photon interaction is a higher order process, the so-called box diagram.

It turns out that this is satisfied if we set as condition the vertex to be zero when all four external photon momenta $k_{(1)} = k_{(2)} = k_{(3)} = k_{(4)} = 0$, that is the full quartic gauge vertex

$\Pi^{\alpha\beta\delta\sigma} \equiv$ 

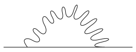
$= 0\ .$ (4.9)

4.2 Fixing the counterterms in the $1/N$ expansion

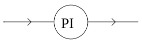
From here on we perform all our expansions to their leading order in $1/N$. With N high enough the coefficient of the higher order terms are suppressed and we can justify the discarding of these terms, yet as explained in section 3.1.1, the theory remains valid even if N is set to 1.

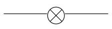
4.2.1 Fermion self energy

In the case of the fermion self energy the $1/N$ expansion approach significantly simplifies the calculations compared to the plain perturbation theory method. The fermion propagator is of leading order N^0. Each possible correction to this needs the triple interaction vertex and is thus of higher order in $1/N$. For example, the diagram

 . (4.10)

This diagram has twice a fermion-photon vertex. Its total order is thus $\mathcal{O}(N^{-1})$ and the diagram can be neglected. To leading order in $1/N$ the expression for the self energy is no more than the counterterm:



$=$  $+ \quad \mathcal{O}(N^{-1})\ .$ (4.11)

The first and second renormalization conditions, eq. (4.5) and eq. (4.6), state that M' has to be equal to the physical mass and the self energy has to vanish when $\not{p} = iM^{'}$. This leads to the conclusion that the counterterm has to equal zero. We set

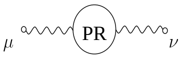
$$\boxed{\Delta_5 = \Delta_6 = 0}\ . \tag{4.12}$$

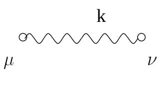
As a result the bare parameter $M = M'(1 + \Delta_6)$, see eq. (3.22), does not renormalize and the physical mass parameter M' is consequently equal to the bare parameter:

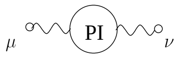
$$M' = M\ . \tag{4.13}$$

4.2.2 Photon self energy

The full photon propagator $(D')^{\mu\nu}$ is defined by the summation of the iteration of PI-bubbles:



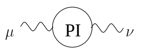
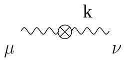
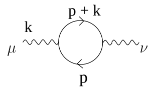
$=$ 

$+$ 

$+ \quad \ldots$ (4.14)

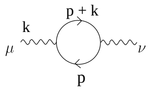
with the dressed photon propagator $D^{\mu\nu}$ given in eq. (3.29) in table 3.1.

To leading order in $1/N$ the photon self energy consists of two terms both of order $\mathcal{O}(N^0)$: the counterterm and the one fermion loop correction. This last diagram has two vertices of order $N^{-1/2}$, yet with the one loop being a summation over all flavors and yielding a factor N its total order is of $\mathcal{O}(N^{-1})$. The self energy

$\Pi^{\mu\nu} \equiv$  $=$  $+$  $+\ \mathcal{O}(N^{-1})\ .$ (4.15)

This one loop diagram is not different from the one which appears in calculations of standard QED, see [28], even though our calculation of the expression is different. We use the cut-off regularization scheme in contrast to the commonly used dimensional regularization. The reason is that we want to compare the renormalizable QED theory to the non-renormalizable Four Fermi Theory and by definition it is impossible not to introduce a cut-off when regularizing a non-renormalizable theory, see section C.4.

Closed fermion loops always give a factor (-1), see [28]. Applying the Feynman rules to the diagram results in the expression for the one loop integral:

 $= \left(\frac{-1}{\sqrt{N}}\right)^2 N(-1)\mathrm{Tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\mu S(p)\gamma^\nu S(p+k)$ (4.16)

$$\begin{aligned}
&= -\mathrm{Tr}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\mu \frac{\gamma^\lambda p_\lambda + iM}{p^2+M^2}\gamma^\nu \frac{\gamma^\delta p_\delta + \gamma^\delta k_\delta + iM}{(p+k)^2+M^2} && (4.17)\\
&= -\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left[\left(p^2+M^2\right)\left((p+k)^2+M^2\right)\right]^{-1} \\
&\quad \left[-4\delta^{\mu\nu}M + 4(p_\lambda p_\delta + p_\lambda k_\delta)(\delta^{\mu\lambda}\delta^{\nu\delta} - \delta^{\mu\nu}\delta^{\lambda\delta} + \delta^{\mu\delta}\delta^{\lambda\nu})\right] && (4.18)\\
&= -4\int \frac{\mathrm{d}^4p}{(2\pi)^4}\int_0^1 \mathrm{d}x\left[x(p^2+M^2) + (1-x)\left((p+k)^2+M^2\right)\right]^{-2} \\
&\quad \left[-\delta^{\mu\nu}p^2 - \delta^{\mu\nu}p\cdot k - \delta^{\mu\nu}M^2 + p^\mu p^\nu + p^\mu k^\nu + p^\nu p^\mu + p^\nu k^\mu\right]. && (4.19)
\end{aligned}$$

In the final expression we used integrals over Feynman parameters. Switching from variable p to $l = p + (1-x)k$ we can next leave out in the numerator all terms linear in l: the Euclidean integral over l in all directions and the quadratic term of l in the denominator make all odd powers of l disappear through symmetry. Through symmetry we can also replace

$$l^\mu l^\nu \to \frac{1}{4}\delta^{\mu\nu}l^2\ . \tag{4.20}$$

Performing these several steps our expression in eq. (4.19) becomes the divergent integral

$$-4\int \frac{\mathrm{d}^4l}{(2\pi)^4}\int_0^1 \mathrm{d}x \frac{-\frac{1}{2}\delta^{\mu\nu}l^2 - \delta^{\mu\nu}\left(M^2 - x(1-x)k^2\right) - 2k^\mu k^\nu(1-x)x}{\left[l^2+M^2+x(1-x)k^2\right]^2}\ . \tag{4.21}$$

At this point the cut-off must be introduced. The 4 dimensional integral over l can be computed by taking

$$\int \mathrm{d}^4 l \to 2\pi^2 \int_0^\Lambda \mathrm{d}l\, l^3 \,. \tag{4.22}$$

Defining the one loop integral by $I^{\mu\nu}$ and computing the integral this results in the regularized expression

$$\begin{aligned} I^{\mu\nu}(k) =& \frac{2}{(4\pi)^2}\delta^{\mu\nu}\left(\Lambda^2 - M^2\right) + \frac{1}{(4\pi)^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right) - \frac{1}{3(4\pi)^2}k^\mu k^\nu \\ &+ \frac{-8}{(4\pi)^2}\int_0^1 \mathrm{d}x \left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)(1-x)x \ln \frac{\Lambda^2}{M^2 + x(1-x)k^2} \,. \end{aligned} \tag{4.23}$$

The renormalization condition in eq. (4.8) states that for $k^2 = \mu^2$ the self energy must vanish. Thus the sum of the counterterm and the one loop integral in eq. (4.15) must be equal to zero:

$$\begin{aligned} 0 \;=\; & -\Delta_1(\delta^{\mu\nu}k^2 - k^\mu k^\nu) - \Delta_2 k^\mu k^\nu - \Delta_3 \delta^{\mu\nu} + \\ & \frac{1}{(4\pi)^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right) - \frac{1}{3(4\pi)^2}k^\mu k^\nu + \frac{2}{(4\pi)^2}\delta^{\mu\nu}\left(\Lambda^2 - M^2\right) + \\ & \frac{-8}{(4\pi)^2}\int_0^1 \mathrm{d}x \left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)(1-x)x \ln \frac{\Lambda^2}{M^2 + x(1-x)\mu^2} \,. \end{aligned} \tag{4.24}$$

The condition was set at the point $k^2 = \mu^2$. We have only replaced one k^2 term. This way it is easy to see that the cancellations by the counterterms of respectively $(\delta^{\mu\nu}k^2 - k^\mu k^\nu)$, $k^\mu k^\nu$ and $\delta^{\mu\nu}$ occur if we set:

$$\Delta_1 = \frac{1}{(4\pi)^2} - \frac{8}{(4\pi)^2}\int_0^1 \mathrm{d}x (1-x)x \ln \frac{\Lambda^2}{M^2 + x(1-x)\mu^2} \,; \tag{4.25}$$

$$\boxed{\Delta_2 = -\frac{1}{3}\frac{1}{(4\pi)^2}}\,; \tag{4.26}$$

$$\boxed{\Delta_3 = \frac{2}{(4\pi)^2}(\Lambda^2 - M^2)}\,. \tag{4.27}$$

If we now assume $\mu^2 \gg M^2$ we can neglect in eq. (4.25) the contribution from the fermion mass in the integral over the Feynman parameter x and we can rewrite eq. (4.25) as

$$\Delta_1 \approx \frac{1}{(4\pi)^2}\left(1 - 8\int_0^1 \mathrm{d}x \left[(1-x)x \ln\frac{\Lambda^2}{\mu^2} - (1-x)x\ln\left((1-x)x\right)\right]\right) \tag{4.28}$$

$$= \frac{1}{(4\pi)^2}\left(1 - 8\left[\frac{1}{6}\ln\frac{\Lambda^2}{\mu^2} + \frac{5}{18}\right]\right) \tag{4.29}$$

$$\to \boxed{\Delta_1 = -\frac{1}{3(4\pi)^2}\left(\frac{11}{3} + 4\ln\frac{\Lambda^2}{\mu^2}\right)}\,. \tag{4.30}$$

In eq. (3.19), (3.20) and (3.21) we defined the relation between dressed parameters, bare parameters and the counterterms. Thus we finally arrive at the following renormalization equations:

$$\frac{1}{e^2} = \frac{1}{e_0^2} + \frac{1}{3(4\pi)^2}\left(\frac{11}{3} + 4\ln\frac{\Lambda^2}{\mu^2}\right) ; \tag{4.31}$$

$$\frac{m_\gamma^2}{e^2} = \frac{m_0^2}{e_0^2} - \frac{2}{(4\pi)^2}\left(\Lambda^2 - M^2\right) ; \tag{4.32}$$

$$\frac{1}{e^2\xi} = \frac{1}{e_0^2\xi_0} + \frac{1}{3}\frac{1}{(4\pi)^2} . \tag{4.33}$$

Different regularization schemes

In appendix B.1 we analyse the photon self energy with the dimensional regularization scheme. There is an important difference in the result of both schemes: after dimensional regularization the leading divergence is logarithmic, while after the cut-off regularization the leading divergence is quadratic. In appendix B.2 we discuss this contract in greater detail.

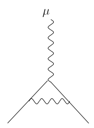
These two schemes of course still yield the same physical end-results, for after the two regularization schemes come two different renormalization approaches, see appendix C.

4.2.3 Vertices

Having renormalized the full fermion and the full photon propagators, we now focus on the two interaction vertices.

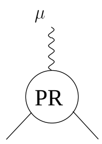
The fermion-photon vertex

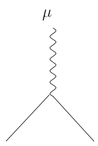
The fermion-photon vertex is of order $\frac{1}{\sqrt{N}}$ and so is its counterterm. It is easy to see that each possible correction to these is of higher order in $1/N$. For example, the diagram of

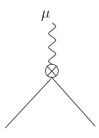


(4.34)

consists of three vertices and is of order $\mathcal{O}(N^{-3/2})$. The full fermion-photon interaction vertex $\Gamma^\mu_{\bar{\psi}\psi}$ is therefore no more than



$=$


$+$


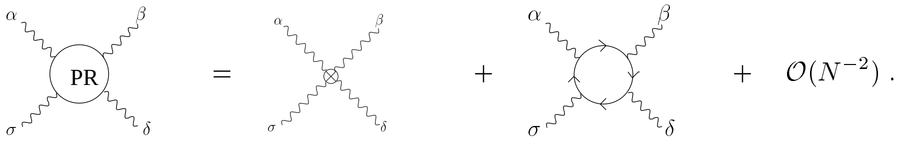
$+ \quad \mathcal{O}(N^{-3/2})$. (4.35)

The first diagram of this summation is the dressed vertex, see eq. (3.30). The renormalization condition in eq. (4.7) states that the full vertex is to be $-\gamma^\mu/\sqrt{N}$ when the momenta of incoming and outgoing fermion are equal and on-shell. As the triple vertex and its counterterm are independent of momenta this condition is easily satisfied by setting

$$\boxed{\Delta_7 = 0} . \tag{4.36}$$

The quartic gauge vertex

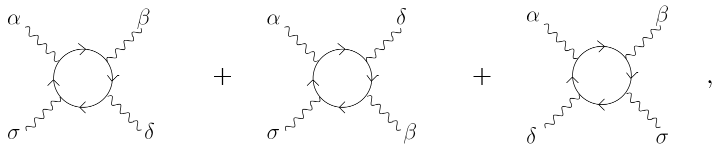
The analytic expression for the quartic counter vertex is of order $1/N$, see eq. (3.34). It was introduced to renormalize the higher order one loop diagram. Other diagrams with four external photon lines can be neglected and the full quartic gauge vertex $\Pi^{\alpha\beta\delta\sigma}$ is



(4.37)

The renormalization condition in eq. (4.9) states that the full vertex has to disappear if the external photons have zero momenta.

The one loop diagram is actually a sum of six permutations, see [33]: starting with a vertex in the lower left corner, we have $3! = 6$ possible ways of connecting the other three vertices. Yet, only three permutations are topologically different and these are



(4.38)

the other three are the same with the loop current turned around. The four photon lines can be attached in 4! possible ways to the four vertices; this is exactly cancelled by a 1/4! coming from the expansion of e^{-S}.

Defining the sum of all six permutations of the one loop diagram by $I^{\alpha\beta\delta\sigma}$, we get:

$$\begin{aligned} I^{\alpha\beta\delta\sigma} &= 2\Bigg\{ \frac{-1}{N}\mathrm{Tr} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left(\gamma^\alpha \frac{\not{p} + iM}{p^2 + M^2} \gamma^\beta \frac{\not{p} + \not{k}_1 + iM}{(p+k_1)^2 + M^2} \times \right. \\ & \left. \gamma^\delta \frac{\not{p} + \not{k}_1 - \not{k}_2 + iM}{(p+k_1-k_2)^2 + M^2} \gamma^\sigma \frac{\not{p} - \not{k}_4 + iM}{(p-k_4)^2 + M^2} \right) \\ & + \quad 2 \text{ permutations} \Bigg\} \, . \end{aligned} \tag{4.39}$$

Setting all external momenta to zero we can immediately use the symmetry of the Euclidean integral and drop all terms in the numerator with odd powers of p. In eq. (4.20) we used the symmetry for the quadratic terms. The symmetry for the quartic terms yields

$$p^\alpha p^\beta p^\delta p^\sigma \to \frac{1}{24} \left(\delta^{\alpha\beta}\delta^{\delta\sigma} + \delta^{\alpha\delta}\delta^{\beta\sigma} + \delta^{\alpha\sigma}\delta^{\beta\delta} \right) . \tag{4.40}$$

Let us now summarize the steps for the calculation of the Trace. The first diagram of eq. (4.38) equals

$$
\begin{aligned}
&\frac{-1}{N}\mathrm{Tr}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[p^2+M^2\right]^{-4}\times\\
&\Big\{M^4\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\sigma+\frac{1}{24}p^4\left(\delta_{\mu\lambda}\delta_{\eta\omega}+\delta_{\mu\eta}\delta_{\lambda\omega}+\delta_{\mu\omega}\delta_{\lambda\eta}\right)\left(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\lambda\gamma^\delta\gamma^\eta\gamma^\sigma\gamma^\omega\right)+\\
&\frac{-M^2}{4}p^2\Big[\delta_{\mu\lambda}\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\lambda\gamma^\delta\gamma^\sigma+\delta_{\mu\eta}\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\delta\gamma^\eta\gamma^\sigma+\delta_{\mu\omega}\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\delta\gamma^\sigma\gamma^\omega+\\
&\qquad\delta_{\lambda\eta}\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\delta\gamma^\eta\gamma^\sigma+\delta_{\lambda\omega}\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\delta\gamma^\sigma\gamma^\omega+\delta_{\eta\omega}\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\eta\gamma^\sigma\gamma^\omega\Big]\Big\}\\
&=\frac{-1}{N}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[p^2+M^2\right]^{-4}\Big\{4M^4\left(\delta^{\alpha\beta}\delta^{\delta\sigma}-\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}\right)+\\
&\qquad\left(\frac{4}{3}p^4+4M^2p^2\right)\left(\delta^{\alpha\beta}\delta^{\delta\sigma}-2\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}\right)\Big\}\ . \qquad (4.41)
\end{aligned}
$$

The three different permutations of eq. (4.38) yield three different coefficients of the M^4 term in the numerator of eq. (4.41). The sum of the three Traces of γ matrices equals

$$
\begin{aligned}
\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\sigma+\gamma^\alpha\gamma^\delta\gamma^\beta\gamma^\sigma+\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\delta\right]&=\delta^{\alpha\beta}\delta^{\delta\sigma}-\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}+\\
&\quad\delta^{\alpha\delta}\delta^{\beta\sigma}-\delta^{\alpha\beta}\delta^{\delta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}+\\
&\quad\delta^{\alpha\beta}\delta^{\sigma\delta}-\delta^{\alpha\sigma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\sigma}\\
&=\delta^{\alpha\beta}\delta^{\delta\sigma}+\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}\ .
\end{aligned}
\qquad (4.42)
$$

The same goes for the coefficient of $(\frac{4}{3}p^4+4M^2p^2)$, yet the end result of the summation of the three permutations is different, namely

$$
\begin{aligned}
&\delta^{\alpha\beta}\delta^{\delta\sigma}-2\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}+\\
&\delta^{\alpha\delta}\delta^{\beta\sigma}-2\delta^{\alpha\beta}\delta^{\delta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}+\\
&\delta^{\alpha\beta}\delta^{\sigma\delta}-2\delta^{\alpha\sigma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\sigma}=0\ .
\end{aligned}
\qquad (4.43)
$$

Thus only the three permutations $(\delta^{\alpha\beta}\delta^{\delta\sigma}+\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta})$ of eq. (4.42) times an Euclidean integral over a single term of order M^4p^{-8} remain. This integral converges and is

$$
-\frac{1}{N}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{4M^4}{[p^2+M^2]^4}=-\frac{2}{3}\frac{1}{(4\pi)^2N}\ . \qquad (4.44)
$$

Going back to eq. (4.39) the total value of $I^{\alpha\beta\delta\sigma}$ with external momenta set to zero is two times the three permutations $(\delta^{\alpha\beta}\delta^{\delta\sigma}+\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta})$ of the integral of eq. (4.44) and equals

$$
I^{\alpha\beta\delta\sigma}=-2\frac{2}{3}\frac{1}{(4\pi)^2N}\left(\delta^{\alpha\beta}\delta^{\delta\sigma}+\delta^{\alpha\delta}\delta^{\beta\sigma}+\delta^{\alpha\sigma}\delta^{\beta\delta}\right)\ . \qquad (4.45)
$$

The renormalization condition of eq. (4.9) states that the counter vertex, its exact expression given in eq. (3.26), has to cancel $I^{\alpha\beta\delta\sigma}$ and we therefore set

$$
\boxed{\Delta_4=-\frac{4}{3}}\ . \qquad (4.46)
$$

4.3 Ward identities

Noether's theorem states that a conserved current is associated with each generator of a continuous symmetry. We can apply this theorem to the gauge invariance which results in the conserved electric current. Often in literature the terms gauge invariance and Ward identity are used as if representing the same phenomenon. This is not exactly the case: the Ward identity is a diagrammatic expression of the electric current conservation; a consequence of the gauge invariance.

The Ward identity is a simplification of a more general identity: the Ward-Takahashi identity. This complete identity is given by, see [28],

$$\begin{aligned} k_\mu \mathcal{M}^\mu \left(k\,; p_1 \ldots p_n\,; q_1 \ldots q_n\right) = \frac{1}{\sqrt{N}} \sum_i \big[&\mathcal{M}_0 \left(p_1 \ldots p_n\,; q_1 \ldots (q_i - k) \ldots\right) \\ &- \mathcal{M}_0 \left(p_1 \ldots (p_i + k) \ldots p_n\,; q_1 \ldots q_n \ldots\right) \big] \,. \end{aligned} \tag{4.47}$$

$\mathcal{M}^\mu$ is the amplitude for an arbitrary process involving an external photon line, *without* propagator attached, with momentum k_μ. When removing this photon line we get the amplitude $\mathcal{M}_0$. The p_i are all momenta of incoming fermions, the q_i are momenta of outgoing fermions.

Instead of looking at full processes we can implement the identity on the basic parts, meaning that we only have to look at vertices with at least one photon line. We have the full quartic gauge vertex, the full fermion-photon interaction vertex and the full photon propagator. If these satisfy the identity, all other processes do too.

4.3.1 The quartic gauge vertex

For the full quartic vertex the identity is simple: there are no external fermion lines, so the right-hand side of eq. (4.47) is zero. Thus the identity states that

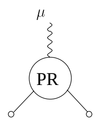
$$k_\alpha \Pi^{\alpha\beta\delta\sigma} = 0\,. \tag{4.48}$$

We use this identity to implement the renormalization condition of eq. (4.9). Although this condition is set at the specific point with all four external photon momenta equal to zero, it can be shown that by the fixing of the coefficient Δ_4 in eq. (4.46) the Ward identity for the quartic interaction is satisfied.

4.3.2 The fermion-photon vertex

In case of the fermion-photon interaction the right-hand side of eq. (4.47) is not zero. With $\Gamma^\mu_{\bar\psi\psi}$ the full vertex, see eq. (4.7), the amplitude for the interaction of two fermion propagators and one photon line is given by

$$\mathrm{G}^\mu_{\bar\psi\psi} = S(p+k)\Gamma^\mu_{\bar\psi\psi} S(p) =$$



$$. \tag{4.49}$$

Applying this to the Ward-Takahashi identity we identify $\mathcal{M}^\mu = \mathrm{G}^\mu_{\bar\psi\psi}$ and $\mathcal{M}_0 = S$. Eq. (4.47) becomes

$$k_\mu \mathrm{G}^\mu_{\bar\psi\psi}(k\,;p\,;p+k) = \frac{1}{\sqrt{N}}\big[S(p) - S(p+k)\big] \ . \tag{4.50}$$

Multiplying from the left with $S^{-1}(p+k)$ and from the right with $S^{-1}(p)$ this simplifies to

$$k_\mu \Gamma^\mu_{\bar\psi\psi}(k\,;p\,;p+k) = \frac{1}{\sqrt{N}}\big[S^{-1}(p+k) - S^{-1}(p)\big] \ . \tag{4.51}$$

Using eq. (3.27) for the expression for the propagator $S(p)$ the identity is

$$\begin{aligned} k_\mu \Gamma^\mu_{\bar\psi\psi}(k\,;p\,;p+k) &= \frac{1}{\sqrt{N}}\big[(\not p + \not k - iM) - (\not p - iM)\big] \\ &= \frac{1}{\sqrt{N}}\not k \ , \end{aligned} \tag{4.52}$$

which indeed corresponds to the value of our vertex in eq. (4.7). The fact that eq. (4.7) is a renormalization condition at the point $p_{\text{in}} = p_{\text{out}}$ is not relevant. The vertex is independent of momenta and thus is always correct.

4.3.3 The photon propagator

As $\mathcal{M}^\mu$ is the amplitude with an external photon line carrying momentum k^μ, without propagator attached, we do not apply the Ward identity to the full propagator $(D')^{\mu\nu}$, but to $(D'^{-1})^{\mu\nu}$. We have that the right-hand side of eq. (4.47) is zero due to a lack of fermion lines and therefore

$$k_\mu (D'^{-1})^{\mu\nu} = 0 \ . \tag{4.53}$$

By the renormalization of the photon self energy to zero in section 4.2.2 the full propagator equals the dressed propagator $D^{\mu\nu}$. Hence eq. (4.53) is the same as

$$k_\mu (D^{-1})^{\mu\nu} = 0 \ . \tag{4.54}$$

Yet our result in eq. (3.24) does not satisfy this identity: the inproduct of k_μ with the inverse propagator of eq. (3.24) yields

$$k_\mu (D^{-1})^{\mu\nu} = k^\nu\big[\frac{1}{e^2\xi}k^2 + \frac{m_\gamma^2}{e^2}\big] \ . \tag{4.55}$$

This can be understood as follows: the Ward identity is a consequence of the gauge invariance. Our system, the Lagrangian in eq. (3.18), has a photon mass parameter m_γ. A mass parameter immediately breaks the gauge invariance so if we were to keep massive photons, i.e. a non-zero m_γ, it would be futile to even try to satisfy the identity.

Yet we did not introduce the mass parameter to gain physical massive photons, but to construct a renormalization flow of this parameter. Although this flow gives us information on our system, it is the fixed point with $m_\gamma = 0$ which represents the physical system we know: a system of massless photons. The moment we regain our physical system with $m_\gamma = 0$, we see the right-hand side of eq. (4.55) go to zero, if working on-shell $k^2 = 0$, and the Ward Takahashi identity is satisfied.

4.4 Renormalization flows

Having renormalized all the parameters we can now analyse the resulting renormalization flows. The Lagrangian (3.18) holds a total of seven counterterm coefficients which have been defined by the renormalization procedure. Summarizing the result we have for $\mu^2 \gg M^2$ that

$$\Delta_1 = -\frac{1}{3(4\pi)^2}\left(\frac{11}{3} + 4\ln\frac{\Lambda^2}{\mu^2}\right) ; \tag{4.56}$$

$$\Delta_2 = -\frac{1}{3}\frac{1}{(4\pi)^2} ; \tag{4.57}$$

$$\Delta_3 = \frac{2}{(4\pi)^2}(\Lambda^2 - M^2) ; \tag{4.58}$$

$$\Delta_4 = -\frac{4}{3} ; \tag{4.59}$$

$$\Delta_5 = \Delta_6 = \Delta_7 = 0 . \tag{4.60}$$

These coefficients determine the renormalization equations of the Lagrangian's parameters, i.e. the fermion mass M', the gauge fixing term ξ, the electric charge e and the photon mass m_γ. Summarizing the calculated renormalization equations yields

$$M' = M ; \tag{4.61}$$

$$\frac{1}{e^2} = \frac{1}{e_0^2} + \frac{1}{3(4\pi)^2}\left(\frac{11}{3} + 4\ln\frac{\Lambda^2}{\mu^2}\right) ; \tag{4.62}$$

$$\frac{m_\gamma^2}{e^2} = \frac{m_0^2}{e_0^2} - \frac{2}{(4\pi)^2}(\Lambda^2 - M^2) ; \tag{4.63}$$

$$\frac{1}{e^2\xi} = \frac{1}{e_0^2\xi_0} + \frac{1}{3}\frac{1}{(4\pi)^2} . \tag{4.64}$$

The fermion mass does not renormalize, that is to the leading order in our expansion and is therefore a marginal parameter, see appendix C. In plain perturbation theory however it does renormalize and is a relevant parameter, see [28], with fixed point at $M' = 0$. It is safe to say that calculations of higher orders in the $1/N$ expansion will hence result in a flow, such that we have a marginally relevant parameter. For now it is irrelevant whether we use M' or M, as both are equal.

The three other parameters yield the renormalization equations given in eq. (4.62), (4.63) and (4.64). For any parameter λ_i renormalized at a scale μ with $\mu^2 \gg M^2$ the flow equation β_{λ_i} is defined by

$$\beta_{\lambda_i}(\lambda_i) = \mu\frac{\partial}{\partial\mu}\lambda_i(\mu) . \tag{4.65}$$

A positive β function implies an increasing parameter when the renormalization scale μ is raised. The renormalization group approach by Wilson and its renormalization flows consist of analyzing the system by lowering μ, therefore a positive β function implies we have a flow of a decreasing parameter. A fixed point is reached when $\beta = 0$ and there the flow terminates.

4.4.1 The electric charge

We can express the β_e function in terms of the scale μ, or we can express it in terms of the charge e by

$$\mu\frac{\partial}{\partial\mu}(e^2) = \frac{\frac{8}{3(4\pi)^2}}{\left\{\frac{1}{e_0^2} + \frac{1}{3(4\pi)^2}\left(\frac{11}{3} + 4\ln\frac{\Lambda^2}{\mu^2}\right)\right\}^2} = \frac{8}{3(4\pi)^2}e^4 \tag{4.66}$$

From eq. (4.66) we can see that we only have one fixed point, the trivial case of an interaction free theory with $e = 0$. We can extract more information from the renormalization equation (4.62) by analysing its cut-off dependence.

- When keeping the bare charge e_0 fixed, we have in the limit of $\Lambda \to \infty$ the trivial point $e = 0$.

- When keeping the renormalized charge e non-zero and fixed, we see we cannot set $\Lambda \to \infty$. We cannot raise the cut-off higher than the so-called Landau pole
$$\Lambda_{\mathrm{L}} \equiv \mu^2 e^{3/4\left(1/e^2+1\right)} \;. \tag{4.67}$$
In the limit $\Lambda \to \Lambda_{\mathrm{L}}$ we have $e_0 \to \infty$ and a cut-off higher than Λ_{L} would imply an imaginary bare charge.

The Landau pole sets a boundary on the validity of QED: for energies higher than Λ_{L} the theory does not hold. Although this poses serious problems, we know that the theory of electromagnetism is embedded in the electroweak theory, see section 2.3.1. This theory unifies the electromagnetic and weak nuclear force and states that both are low energy manifestations of one electroweak force.

The unification energy is far below the Landau pole and the singularity is therefore never reached. The electroweak theory itself has a much higher Landau pole $\Lambda_{\mathrm{L}_{weak}}$, i.e. an energy above which it does not hold; this is again never reached as the electroweak theory is embedded in the Standard Model, unifying the electroweak and the strong force.

If we keep $\Lambda < \Lambda_{\mathrm{L}}$ and lower our renormalization scale μ, i.e. we analyse the theory at lower energies, we see from eq. (4.62) and (4.66) that the electric charge decreases: at higher distances the absolute value of the charge is smaller. This is often interpreted as a bare charge being surrounded by virtual positron-electron pairs: the vacuum polarizes and screens the bare charge. Hence a test-particle at a certain distance sees a charge of lower value. In the region $\mu^2 \ll \Lambda^2 < \Lambda_{\mathrm{L}}$ the renormalized charge parameter is of order $e = \mathcal{O}(10^{-1})$, see eq. (3.5), validating the perturbative approach of QED at low energies.

4.4.2 The photon mass

As eq. (4.63) does not explicitly show any μ dependence we can conclude that the combination $m_\gamma^2/e^2\xi$ does not renormalize. Yet with the photon mass of great importance with regard to the gauge invariance we express m_γ in terms of $e^2(\mu)$ such that we can analyse the β_{m_γ} function

$$\begin{aligned}
\mu\frac{\partial}{\partial\mu}(m_\gamma^2) &= \frac{\frac{8}{3(4\pi)^2}}{\left\{\frac{1}{e_0^2}+\frac{1}{3(4\pi)^2}\left(\frac{11}{3}+4\ln\frac{\Lambda^2}{\mu^2}\right)\right\}^2}\left(\frac{m_0^2}{e_0^2}-\frac{2}{(4\pi)^2}\left(\Lambda^2-M^2\right)\right) \\
&= \frac{8}{3(4\pi)^2}e^4\left(\frac{m_0^2}{e_0^2}-\frac{2}{(4\pi)^2}\left(\Lambda^2-M^2\right)\right) \qquad (4.68) \\
&= \frac{8}{3(4\pi)^2}e^2 m_\gamma^2 \qquad (4.69)
\end{aligned}$$

We see that the renormalization flow of the photon mass reaches a fixed point only if either e^2 or m_γ^2 is zero. We have that e^2 is always > 0 in a non-trivial theory and therefore require for a fixed point that $m_\gamma = 0$. This can easily be accomplished by setting

$$\frac{m_0^2}{e_0^2} = \frac{2}{(4\pi)^2}\left(\Lambda^2-M^2\right) . \qquad (4.70)$$

The fermion mass parameter is marginally relevant, with a fixed point at $M = 0$ and compared to the high energy cut-off the physical fermion mass can be neglected. Without loss of information this gives us the possibility of taking the limit of $M = 0$. The fixed point of $m_\gamma = 0$ can now be reached by setting

$$\frac{m_0^2}{e_0^2} = \frac{2\Lambda^2}{(4\pi)^2} . \qquad (4.71)$$

If the bare parameter m_0 is not exactly set at this point it is easy to see that the flow of m_γ for a decreasing μ is towards the fixed point. Consequently m_γ is by definition an irrelevant parameter.

This definition can be misleading, in that irrelevant in the renormalization group theory does not mean it is of no importance to our theory. We discussed in section 4.4.1 that to have a non-trivial theory it is of importance to keep $e^2 > 0$. Therefore if we do not start the renormalization flow at exactly the value in eq. (4.70), the flow of the photon mass parameter can never exactly reach $m_\gamma = 0$. As was concluded in section 4.3.3 a non-zero photon mass breaks the gauge invariance. If the theory is to describe physical photons, we must start the flow in its fixed point: the bare mass parameter must be fine-tuned by eq. (4.70), or by eq. (4.71) if the fermion mass is set to zero.

4.4.3 The gauge fixing parameter

From eq. (4.64) it is easy to see that the combination $1/e^2\xi$ does not renormalize. ξ can nevertheless still be expressed in terms of $e^2(\mu)$ yielding the β_ξ function

$$\mu\frac{\partial}{\partial\mu}(\xi) = -\frac{8}{3(4\pi)^2}\left(\frac{1}{e_0^2\xi_0}+\frac{1}{3}\frac{1}{(4\pi)^2}\right)^{-1} = -\frac{8}{3(4\pi)^2}e^2\xi . \qquad (4.72)$$

From eq. (4.64) and (4.72) we can conclude that $\xi_0 = 0$ yields $\xi = 0$ and that this is a fixed point. If however $\xi_0 > 0$ or $\xi_0 < 0$ we have respectively a negative or a positive β function and the flow, generated by a decreasing μ, is always away from the fixed point. In other words we have an unstable fixed point and by definition ξ is therefore a relevant parameter.

This does not anyhow affect our gauge invariance. The parameter ξ does not appear in the end result of physical calculations and its value is thus of no importance, even if $\xi \to \pm\infty$.

In conclusion, we have a perturbative gauge invariant theory of N massless fermions and a massless photon at low energies, under the conditions that the electric charge parameter is kept non-zero and the bare photon mass parameter is correctly fine-tuned; no condition is required for the gauge fixing parameter. We see that the complete fixed point is given by

$$(e,\, \xi,\, m_\gamma) \;=\; (0,0,0) \; . \tag{4.73}$$

In the limit of $\mu \to 0$ we have $(e, m_\gamma) \to (0,0)$; if the starting point was not the fixed point these two parameters shall come closer and closer to their fixed point. The gauge fixing parameter ξ however only stays zero if ξ_0, that is the initial value of ξ, was set to zero. Otherwise we have in the limit of $\mu \to 0$ that $\xi \to \pm\infty$.

Chapter 5

Four Fermi Theory

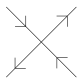
We will first analyse the purely fermionic theory: its properties concerning the $1/N$ expansion and all the possible higher order interactions. We then introduce an auxiliary field and briefly state how it may become an effective photon field. We conclude with determining the Feynman rules concerning all fields and interactions.

5.1 The purely fermionic theory

The purely fermionic theory we use, is based on a current-current interaction of N flavors of fermions denoted by

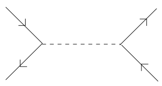
$$\left(\overline{\psi}^I \gamma_\mu \psi^I\right)\left(\overline{\psi}^J \gamma^\mu \psi^J\right) \equiv J_\mu J^\mu \ , \tag{5.1}$$

with a summation over the repeated $I, J = 1, \ldots, N$. It is similar to the theory set forth by Bjorken [2], see eq. (1.3), with the difference that Bjorken analysed the model with only one flavor. A current-current interaction is different from the $(\overline{\psi}\psi)^2$ interaction in a NJL model, see section 2.2.2, even though both can be represented in Feynman diagrams by



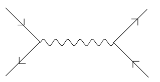
(5.2)

and denote an interaction of four fermion fields. The simple fact that $\overline{\psi}\psi$ is a scalar and $\overline{\psi}\gamma_\mu\psi$ a vector changes the possible physical identification. Nambu and Jona-Lasinio [1] concluded that by fine-tuning the parameters this vertex plus its higher order corrections were equal to a four fermion vertex describing the interchange of a scalar particle, i.e. to the exchange of a pion by



. (5.3)

Our aim however is to show that with a correct fine-tuning of a current-current interaction the exchange is that of a vector particle, namely a photon. Instead of the diagram (5.3) we want the full four fermi vertex to be equivalent to



. (5.4)

We start with the following Lagrangian in Euclidean space

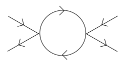
$$\mathcal{L} = \overline{\psi}^I \left(-i\partial\!\!\!/ + iM\right)\psi^I - \frac{\alpha}{2N\Lambda^2}\left(\overline{\psi}^I \gamma_\mu \psi^I\right)\left(\overline{\psi}^J \gamma^\mu \psi^J\right) . \tag{5.5}$$

In the interaction term we have summations over the indices $I, J = 1, \ldots, N$. Thus fermions and anti-fermions of the same flavor can interact, but also a fermion and anti-fermion of one certain flavor can interact with a fermion and anti-fermion of another flavor. The differently flavored fermions have the same mass M.

The interaction coefficient

All Lagrangians in this thesis are action densities, see appendix A.1, and therefore of order $[length]^{-4} = [mass]^4$. From the first terms in eq. (5.5), leading to the kinetic part of the fermion propagator, we can see that the fermion field ψ thus is of dimension $[mass]^{3/2}$. In the interaction we have $\mathcal{O}(\psi^4)$ which has a dimension of $[mass]^6$. For the whole term with coefficient to be of dimension $[mass]^4$ we must consequently divide the dimensionless parameter α by Λ^2. This choice of the high energy cut-off seems to imply that the interaction is greatly suppressed. This may not be the case: the coefficient α is still unknown and still can be set such that α/Λ^2 can be of any value.

The reason that the coefficient is of order $\mathcal{O}(N^{-1})$ is the following: the higher order corrections to the four fermion vertex must be fine-tuned to result in a full vertex equivalent to diagram (5.4). As we perform all calculations to leading order in $1/N$, these correction terms must be of the same order in N. The first order correction is


(5.6)

and consists of two times the four fermion vertex. The loop can be of maximum $\mathcal{O}(N)$, see section 5.1.1. If we assume the vertex to be of order a in N then the one loop correction is $\mathcal{O}(N^{2a+1})$. By the simple equation

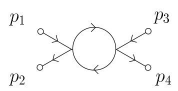
$$\mathcal{O}(N^{2a+1}) = \mathcal{O}(N^a) \tag{5.7}$$

we see that $a = -1$: the four fermion interaction coefficient must be $\mathcal{O}(N^{-1})$.

We study the theory's low energy behavior, i.e. at energies μ with $\mu \ll \Lambda$. As explained in section 3.1.1 we perform all calculations to leading order in $1/N$. As a second approximation we neglect all terms suppressed by negative powers of Λ^2, that is all expressions of order $\mathcal{O}(\mu^2/\Lambda^2)$.

5.1.1 The N-order of diagrams

The current-current interaction with a summation over $I$and J has some implications for the counting of the order of N in a Feynman diagram. In this theory a closed fermion loop is no longer immediately of order N^1, as was the case in the Lagrangian of QED, see section 3.1.1. For example, let us analyse the following diagram



, (5.8)

which is a correction to the four fermi interaction. With the definition of the current

$$J^\mu(x) \;\equiv\; \overline{\psi}^I(x)\gamma^\mu\psi^I(x)\;, \tag{5.9}$$

and with T defined as the time-ordering operator, the corresponding analytic expression for the diagram is

$$\int \mathrm{d}^4x\mathrm{d}^4y\frac{1}{2}\left(\frac{\alpha}{2N\Lambda^2}\right)^2 < 0|T\psi_1\overline{\psi}_2\psi_3\overline{\psi}_4\; J_\mu(x)J^\mu(x)\; J_\nu(y)J^\nu(y)|0>_{conn}\;. \tag{5.10}$$

If we assume that the time runs from left to right in the diagram, we see that we have an interaction of a fermion and an anti-fermion. We leave out the coefficient and write the four different summations in the different currents J's as summations over I_1, I_2, I_3 and I_4. This yields

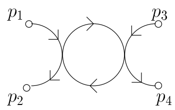
$$< 0|\overline{\psi}_3\psi_4\;\overline{\psi}_x^{I_1}\gamma_\mu\psi_x^{I_1}\;\overline{\psi}_x^{I_2}\gamma^\mu\psi_x^{I_2}\;\overline{\psi}_y^{I_3}\gamma_\nu\psi_y^{I_3}\;\overline{\psi}_y^{I_4}\gamma^\nu\psi_y^{I_4}\;\overline{\psi}_1\psi_2|0>. \tag{5.11}$$

For this expression there are three distinguishable ways of connecting the external $\overline{\psi}_1$, ψ_2, $\overline{\psi}_3$ and ψ_4 to the $\overline{\psi}_x$'s and ψ_y's:

Type A We connect $\overline{\psi}_1$ to $\psi_y^{I_4}$, i.e. to one certain flavor, and we connect ψ_2 to $\overline{\psi}_y^{I_4}$. Because of the summation over I_4, ψ_2 can only have the same flavor as $\overline{\psi}_1$. At the same time we connect ψ_4 to $\overline{\psi}_x^{I_1}$ and $\overline{\psi}_3$ to $\psi_x^{I_1}$; again both have one and the same flavor. This however does not have to be the same as the flavor of $\overline{\psi}_1$ and ψ_2.

The loop itself consists of connecting $\psi_x^{I_2}$ with $\overline{\psi}_y^{I_3}$ and connecting $\overline{\psi}_x^{I_2}$ with $\psi_y^{I_3}$. The combination of the two summations over I_2 and I_3 make that we have once all flavors in the loop; i.e. we have one summation over all flavors $1,\ldots,N$. Each interaction vertex is of order N^{-1}, making the total order to be $N^{-2}N^1 = N^{-1}$.

This specific Type A with flavor conservation at vertices is often drawn as

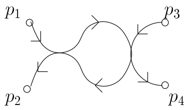


$$= \mathcal{O}(N^{-1})\;. \tag{5.12}$$

Type B We connect $\overline{\psi}_1$ to $\psi_y^{I_4}$ and we connect ψ_2 to $\overline{\psi}_y^{I_3}$. At the point y we now have one $\overline{\psi}_y^{I_4}$ and one $\psi_y^{I_3}$ left, which form the loop by connecting to respectively $\psi_x^{I_1}$ and $\overline{\psi}_x^{I_1}$. At the other side of the diagram we connect $\overline{\psi}_3$ and ψ_4 to respectively $\psi_x^{I_2}$ and $\overline{\psi}_x^{I_2}$.

The flavor belonging to fermion field $\overline{\psi}_1$ is passed on through the summation over I_4 into the loop. This makes it impossible to perform a summation over I_1: only one flavor can go round in the loop. This in turn implies that the same flavor is to be taken in I_3. At the same time the external fermion fields $\overline{\psi}_3$ and ψ_4 have one and the same flavor; there is no summation over I_2.

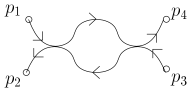
All in all there is no summation at all over the flavors. We have a fermion and anti-fermion of the same flavor coming in, and a fermion and anti-fermion of the same flavor coming out; incoming and outgoing flavors do not have to be the same. Due to the two vertices we have a total order of N^{-2} . This Type B is drawn as



$$= \mathcal{O}(N^{-2}) \ . \tag{5.13}$$

Type C As in Type B we connect $\overline{\psi}_1$ to $\psi_y^{I_4}$ and ψ_2 to $\overline{\psi}_y^{I_3}$. The loop is formed by connecting $\overline{\psi}_y^{I_4}$ and $\psi_y^{I_3}$ to respectively $\psi_x^{I_1}$ and $\overline{\psi}_x^{I_2}$. For the outgoing fermion we connect ψ_4 and $\overline{\psi}_3$ to $\overline{\psi}_x^{I_1}$ and $\psi_x^{I_2}$.

The flavor of the fermion fields $\overline{\psi}_1$ and ψ_4 are now the same. Also the flavors of ψ_2 and $\overline{\psi}_3$ are equal. This particluar Type C is drawn as



$$= \mathcal{O}(N^{-2}) \ . \tag{5.14}$$

The difference between Type A and B or C is that in A the $\overline{\psi}$ and ψ belonging to one certain current J are either both inside or both outside the loop. In Type B or C this is not the case. The summation over flavors acts up as flavor conservation: in Type B the flavor of the loop lines is the same as the flavor of the external fermions with momentum $\overline{\psi}_1$ and ψ_2. There is no summation over flavors inside the loop. Between Type B and C the difference is that in B incoming and outgoing fermions can be of different flavors. In Type C the two incoming flavors equal the two outgoing flavors; again there is no summation over flavors inside the loop.

Only in the case of Type A does the loop gain an extra order N^1. As we perform all calculations to the leading order in the $1/N$ expansion, we only need the contractions of spinors which correspond to Type A. We therefore do not use the different representations of eq. (5.12), (5.13) or (5.14) in this thesis.

5.1.2 Structure of the four fermion interaction

In this section we discuss the general structure of the four fermion interaction. As explained when discussing the diagram (5.4) the interchanged effective field can be shown to be a scalar field if the interaction is given by $(\overline{\psi}\psi)^2$, a product of scalars; for it to be a vector field the interaction must be an inproduct of vectors. Which the term $\overline{\psi}\gamma_\mu\psi$ indeed is, but in fact it is only the zeroth order of the expansion

$$\overline{\psi}\left(a + b(\overleftarrow{\not\partial} + \overrightarrow{\not\partial}) + c(\overleftarrow{\not\partial} + \overrightarrow{\not\partial})^2 + \dots\right)\gamma_\mu\psi \tag{5.15}$$

of which all terms are vectors. The $\overleftarrow{\not\partial}$ operates on $\overline{\psi}$ and $\overrightarrow{\not\partial}$ operates on ψ. The higher order terms were not included in the Lagrangian (5.5).

We note that an infinite number of interaction terms is of course impossible to deal with. We do however need extra parameters in this theory if we are to show it equivalent to QED: for now the theory only has M and α as free parameters, while the theory of QED has M, e, ξ, m_γ and Δ_4. There are more than one way of bringing in the extra parameters needed and we introduce them now by means of counterterms; in section 5.2 we discuss one other method. Defining as counterterms

$$\frac{\beta_a}{2N\Lambda^2}J_\mu J^\mu + \frac{\beta_b}{2N\Lambda^2}\frac{1}{\Lambda^2}\left(\partial_\nu J_\mu\right)\left(\partial^\nu J^\mu\right) + \frac{\beta_c}{2N\Lambda^2}\frac{1}{\Lambda^4}\left(\partial^2 J_\mu\right)\left(\partial^2 J^\mu\right) \tag{5.16}$$

it is easy to see that as the dimension of $\partial_\mu = [mass]$, the coefficient of the second and third counterterm must be of dimension $[mass]^{-4}$ and $[mass]^{-6}$; the parameters β_a, β_b and β_c are dimensionless. Counterterms with even more derivatives are of course possible, but it turns out that due to the increasing negative power of Λ in the coefficient the higher derivative terms are suppressed and we can neglect the $\sim (\partial^2 J_\mu)^2$ and higher terms. To prove this we do take the third counterterm $\sim (\partial^2 J_\mu)^2$ in eq. (5.16) into account and in the end show it to be negligable.

The second counterterm $\sim (\partial_\nu J_\mu)^2$ however does not suffice and it is more convenient to decompose this term into its Lorentz components. The tensor

$$T_{\mu\nu} \equiv \partial_\mu\left[\overline{\psi}^I \gamma_\nu \psi^I\right] = \partial_\mu J_\nu \tag{5.17}$$

has 16 degrees of freedom, which we can split in three different tensors, namely

$$\frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right) \qquad \text{the anti}-\text{symmetric tensor}\ , \tag{5.18}$$

$$T_\eta{}^\eta \qquad \text{the trace,} \tag{5.19}$$

$$\frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right) - \frac{1}{4}T_\eta{}^\eta\, \mathbb{1}_{\mu\nu} \qquad \text{the traceless symmetric tensor.} \tag{5.20}$$

These tensors are orthogonal to each other, i.e. contractions of different terms vanish. Instead of the three counterterms (5.16) we therefore define as the five counterterms for the four fermion interaction

$$\frac{\beta_0}{2N\Lambda^2}J_\mu J^\mu + \frac{\beta_1}{4N\Lambda^4}\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right)^2 + \frac{\beta_2}{2N\Lambda^4}\left(\partial_\eta J^\eta\right)^2 + \tag{5.21}$$

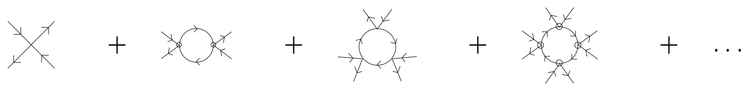
$$\frac{\beta_3}{4N\Lambda^4}\left(\partial_\mu J_\nu + \partial_\nu J_\mu - \frac{1}{2}\partial_\eta J^\eta\, \mathbb{1}_{\mu\nu}\right)^2 + \frac{\beta_4}{2N\Lambda^6}\left(\partial^2 J_\mu\right)\left(\partial^2 J^\mu\right)$$

$$\equiv \beta_i' J_i^2\ . \tag{5.22}$$

We have redefined the parameters and in the last line we introduce a shorthand notation for the summation of all five counterterms.

5.1.3 Higher order interactions

We still only have one interaction, connecting a fermion and a anti-fermion of the same flavor i with another fermion and anti-fermion of the same flavor j; the flavors i and j do not have to be same. Higher order interactions are present however in the one loop expansion of the effective action



(5.23)

By the rule in eq. (3.9) we see that if the fermion loop consists of four or less propagators the expression for the diagram diverges and has to be regularized.

The second diagram is a correction term to the four fermion vertex itself and is part of the renormalization of this vertex.

The third diagram in eq. (5.23) is a six fermion interaction, with its loop consisting of an odd number of fermion propagators. This is just as in the case of a three gauge interaction with one loop, see eq. (3.8): for all loops consisting of an odd number of propagators, the expression for the diagram vanishes.

The fourth diagram does not yield zero and is of order $\mathcal{O}(\Lambda^0)$. For renormalization of this eight fermion interaction we thus have to add a counterterm $\sim (J_\mu J^\mu)^2$ to the Lagrangian (5.5). With four times a four fermion vertex and one closed loop the diagram is of order $(-1/2)^4 - 1 = -3$ in N and as the counterterm consists of eight fermion fields its coefficient must be of a dimension $[mass]^{-8}$. We add to the Lagrangian the counterterm with dimensionless parameter γ and transform

$$\mathcal{L} \to \mathcal{L} + \frac{\gamma}{4N^3\Lambda^8} \left(J_\mu J^\mu\right)^2 \ . \tag{5.24}$$

In the higher order diagrams of eq. (5.23) the fermion loop is convergent the moment it consists of five or more fermion propagators, see eq. (3.9) and so for the interactions of more than eight fermion fields in the expansion of eq. (5.23) no counterterms are needed.

With the ordering of the fermion interaction by the couterterms $\beta_i' J_i^2$ in eq. (5.22) and the determination of possible divergent diagrams in the expansion of (5.23) we can now conduct a proper analysis of the Four Fermi Theory. The Lagrangian (5.5) has to be expanded to

$$\begin{aligned}\mathcal{L}_{FFT} =& \overline{\psi}^I \left(-i\partial\!\!\!/ + iM\right)\psi^I - \frac{\alpha}{2N\Lambda^2} J_\mu J^\mu + \frac{\gamma}{4N^3\Lambda^8}\left(J_\mu J^\mu\right)^2 \\ &+ \frac{\beta_0}{2N\Lambda^2} J_\mu J^\mu + \frac{\beta_1}{4N\Lambda^4}\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right)^2 + \frac{\beta_2}{2N\Lambda^4}\left(\partial_\eta J^\eta\right)^2 + \\ &+ \frac{\beta_3}{4N\Lambda^4}\left(\partial_\mu J_\nu + \partial_\nu J_\mu - \frac{1}{2}\partial_\eta J^\eta\, \mathbb{1}_{\mu\nu}\right)^2 + \frac{\beta_4}{2N\Lambda^6}\left(\partial^2 J_\mu\right)\left(\partial^2 J^\mu\right) \ .\end{aligned} \tag{5.25}$$

5.2 The effective photon field

By dynamical symmetry breaking in a NJL model [1] a Goldstone boson is formed in a purely fermionic theory and this boson must hence be a bound state of fermions. For a photon to arise dynamically in the FFT it too must be a composite state of an even number of fermion fields: a massless vector field is to be formed out of interacting spinor fields.

One composite vector boson field in a fermion theory is the current term $J_\mu = \overline{\psi}^I \gamma_\mu \psi^I$, but we see in the summation of (5.15) that this is merely the zeroth order of a series. As explained in section 5.1.2 the general structure of the four fermion interaction is more than only $J_\mu J^\mu$, and there are several methods for introducing extra parameters. One method is to identify the effective photon field as the whole series

$$A_\mu \sim \frac{1}{\Lambda^2}\overline{\psi}\left(a + b(\overleftarrow{\partial\!\!\!/} + \overrightarrow{\partial\!\!\!/}) + c(\overleftarrow{\partial\!\!\!/} + \overrightarrow{\partial\!\!\!/})^2 + \ldots\right)\gamma_\mu\psi \tag{5.26}$$

and determine the significant parameters. All possible forms of a current-current interaction are then implemented by the definition of A_μ. We have not chosen this method, but instead we organized the possibility of derivative terms by the definition of the counterterms $\beta'_i J_i^2$ in eq. (5.22). Consequently we define the effective field as merely

$$A_\mu \sim \frac{1}{\Lambda^2}\overline{\psi}^I \gamma_\mu \psi^I \ , \tag{5.27}$$

where the factor Λ^{-2} is due to dimensional reasons: in QED the gauge field $A_\mu = [mass]^1$ and the fermion field has dimension $[mass]^{3/2}$.

Identification of the effective field

The effective photon can not simply be added to the theory, it must come forth out of the system. One way to see this happen, is by rewriting the functional integral

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-\int \mathrm{d}^4x\mathcal{L}} = C^{-1}\int \mathcal{D}\overline{\psi}\mathcal{D}\psi[\mathcal{D}A]e^{-\int \mathrm{d}^4x\mathcal{L}}\ e^{-\int \mathrm{d}^4x\frac{1}{2}s\left(A_\mu-\frac{t}{\Lambda^2}\overline{\psi}^I\gamma_\mu\psi^I\right)^2} \ , \tag{5.28}$$

$$\text{where } C = \int [\mathrm{D}A]e^{-\int \mathrm{d}^4x\frac{1}{2}s\left(A_\mu-\frac{t}{\Lambda^2}\overline{\psi}^I\gamma_\mu\psi^I\right)^2} \ . \tag{5.29}$$

The constant C is a gaussian integral with two undefined parameters s and t. Without changing the functional integral we can thus rewrite the Lagrangian

$$\mathcal{L} \to \mathcal{L} + \frac{1}{2}s\left(A_\mu - \frac{t}{\Lambda^2}\overline{\psi}^I\gamma_\mu\psi^I\right)^2 \ . \tag{5.30}$$

Applying this to the Lagrangian $\mathcal{L}_{FFT}$ in eq. (5.25) we define

$$\mathcal{L}_A \equiv \mathcal{L}_{FFT} + \frac{1}{2}s\left(A_\mu A^\mu - \frac{2t}{\Lambda^2}A_\mu J^\mu + \frac{t^2}{\Lambda^4}J_\mu J^\mu\right) \ . \tag{5.31}$$

We see that if the first of these three new terms is to be of dimension $[mass]^4$ the parameter $s = [mass]^2$, and if the second term is to regain the structure of the interaction term in QED, see eq.(3.17), the parameter t should be of order $\mathcal{O}(N^{-1/2})$.

If now via this auxiliary field A_μ the physics of QED is to be regained, the interaction of A_μ with J^μ should be the only main interaction. By properly adjusting the parameters s and t it should be possible to cancel out $-\frac{\alpha}{2N\Lambda^2}J_\mu J^\mu$ in Lagrangian 5.25. It is easy to see that this is the case if we set

$$\mathcal{L}_A - \mathcal{L}_{FFT} = \frac{\Lambda^2}{2\alpha}\left(A_\mu + \frac{\alpha}{\sqrt{N}\Lambda^2}\overline{\psi}^I\gamma_\mu\psi^I\right)^2 \ . \tag{5.32}$$

Adding these three terms to the Lagrangian of eq. (5.25) the bare J^2 term vanishes and a mass term for the effective field arises together with an interaction term similar to the one in QED. We have

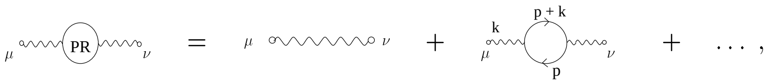
$$\begin{aligned}\mathcal{L}_A \ &= \ \overline{\psi}^I\left(-i\partial\!\!\!/ + iM\right)\psi^I + \frac{\Lambda^2}{2\alpha}A_\mu A^\mu + \frac{1}{\sqrt{N}}A_\mu J^\mu + \\ &\quad +\beta'_i J_i^2 + \frac{\gamma}{4N^3\Lambda^8}\left(J_\mu J^\mu\right)^2 \ .\end{aligned} \tag{5.33}$$

The effective photon propagator

A new field has now appeared into the fermionic theory. It is an auxiliary field, the representation of a bound state of fermions, and its exact expression is

$$A_\mu = -\frac{\alpha}{\sqrt{N}\Lambda^2} J_\mu \ . \tag{5.34}$$

In Lagrangian (5.33) we see it has no kinetic term, its bare propagator therefore carries no momentum dependence and is merely defined by mass term. Nevertheless we will see that its effective propagator, i.e. the propagator including quantum corrections

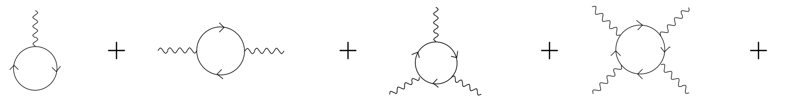


(5.35)

gains a momentum dependence and after the correct fine-tuning of all parameters exactly behaves like a photon: the photon field results from an interaction between two fermion fields.

Effective photon interactions

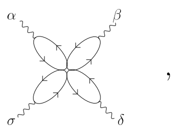
With the new interaction term in the Lagrangian (5.33) we have to beware of new possible higher order interactions in the expansion of e^{-S}. Analyzing the effective action



(5.36)

we come to the same conclusion as taken from eq. (3.8): due to the properties of the fermion loop, the tadpole disappears and there is no three photon interaction. Neither is there an interaction with five or more photons, only the four photon interaction is non-zero and possibly divergent. Equivalent to the case in QED this vertex is of order $\mathcal{O}(\Lambda^0)$ and must be cancelled.

In turns out however we do not need an extra counterterm $\sim (A_\mu A^\mu)^2$ as we introduced in QED, see eq. (3.10), this due to the counterterm of eq. (5.24) representing an eight fermion interaction. In combination with the fermion-photon interaction this generates the correction term



, (5.37)

which can be used to cancel the four photon vertex. This is *not* a coincidence as the A_μ field is a composite field similar to J_μ. The eight fermion term can be seen as a four current interaction $J_\mu^4 \sim A_\mu^4$ and thus it might as well be a counterterm to the four photon interaction.

5.3 The Feynman rules

For the derivation of the Feynman rules we follow the same steps taken in the analysis of QED, see section 3.2. The full Lagrangian with all counterterms is

$$
\begin{aligned}
\mathcal{L}_A = {} & \overline{\psi}^I \left(-i\partial\!\!\!/ + iM'\right)\psi^I + \frac{m_\gamma^2}{2e^2}A_\mu A^\mu + \frac{1}{\sqrt{N}}A_\mu J^\mu + \frac{\gamma}{4N^3\Lambda^8}\left(J_\mu J^\mu\right)^2 + \\
& + \overline{\psi}^I \left(-\omega_1 i\partial\!\!\!/ + \omega_2 iM'\right)\psi^I + \frac{\omega_3}{2}A_\mu A^\mu + \frac{\omega_4}{\sqrt{N}}A_\mu J^\mu + \\
& + \frac{\beta_0}{2N\Lambda^2}J_\mu J^\mu + \frac{\beta_1}{4N\Lambda^4}\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right)^2 + \frac{\beta_2}{2N\Lambda^4}\left(\partial_\eta J^\eta\right)^2 + \\
& + \frac{\beta_3}{4N\Lambda^4}\left(\partial_\mu J_\nu + \partial_\nu J_\mu - \frac{1}{2}\partial_\eta J^\eta\, \mathbb{1}_{\mu\nu}\right)^2 + \frac{\beta_4}{2N\Lambda^6}\left(\partial^2 J_\mu\right)\left(\partial^2 J^\mu\right). \qquad (5.38)
\end{aligned}
$$

For the first four terms in eq. (5.33) we have in eq. (5.38) a dressed term and a counterterm, respectively in the first and second line of eq. (5.38). We define the following bare parameters

$$\frac{\Lambda^2}{\alpha} = \frac{m_\gamma^2}{e^2} + \omega_3 \; ; \qquad (5.39)$$

$$M = M'\left(1 + \omega_2\right) \; . \qquad (5.40)$$

The derivation of the Feynman rules is a straightforward job now of which we state the results in table 5.1. Except for the four fermion countervertex, for this vertex's expression is complicated by the presence of the derivatives in the counterterms $\beta_i' J_i^2$ and depends on the orientation of the vertex in a Feynman diagram. Instead of defining a Feynman rule we calculate each diagram with this vertex by starting with the correlation function in position space.

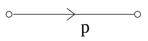
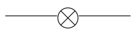
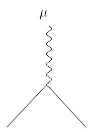
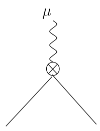
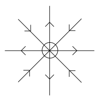
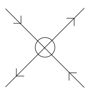
	$= \dfrac{+1}{p\!\!\!/ - iM'}$	;		$= \left\{\omega_1 p\!\!\!/ - i\omega_2 M'\right\}$	(5.41)
μ ν	$= -\dfrac{e^2}{m_\gamma^2}\delta^{\mu\nu}$	;	μ  ν	$= -\omega_3\delta^{\mu\nu}$	(5.42)
	$= -\dfrac{\gamma^\mu}{\sqrt{N}}$	;		$= -\omega_4\dfrac{\gamma^\mu}{\sqrt{N}}$	(5.43)
	$= -\dfrac{\gamma}{4N^3\Lambda^8}$	;		$= \ldots$	(5.44)

Table 5.1: *Eight Feynman rules.*

Chapter 6

The Fine Tuning of the Four Fermi Theory

In this chapter we first define the FFT renormalization and fine-tuning conditions. We then apply these conditions and determine all renormalization equations. In the final section we fine-tune all parameters and we establish the equivalence of the purely fermionic theory with QED.

6.1 Renormalization and tuning conditions

If by a correct fine-tuning the FFT can be reduced to QED in the low energy limit, the renormalization flows and all processes in both theories must be the same. In chapter 4 we saw that QED contains four basic processes: the full fermion propagator (4.1), the full photon propagator (4.14), the full fermion-photon interaction (4.35) and the full quartic gauge interaction (4.37). In this section we define the conditions needed for the FFT to be equivalent to QED at low energies so that only the same interactions as in QED survive.

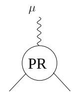
The full fermion propagator

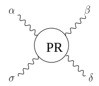
The fermion fields in the purely fermionic theory must represent the same fermion particles as in QED. The physical mass in both theories is thus the same M' and it is easy to see that the renormalization of the fermion mass leads us to exactly the same conditions as in QED, see section 4.1. With the fermion self energy Σ, see eq. (4.5) and (4.6), we fix the fermion mass in the FFT with

$$\begin{aligned} \Sigma(\not{p} = iM^{'}) &= 0\,; && (6.1)\\ \left[\frac{\partial}{\partial(\not{p})}\Sigma(\not{p})\right]_{\not{p}\,=\,iM'} &= 0\,. && (6.2) \end{aligned}$$

Gauge interactions

As for the gauge interactions we have to demand them to be equal to respectively eq. (4.7) and (4.9)

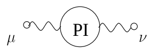
 $= \dfrac{-\gamma^\mu}{\sqrt{N}}$ when $\not{p}_{\rm in} = \not{p}_{\rm out} = iM'$; (6.3)

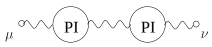
 $= 0$ when $k_{(1)} = k_{(2)} = k_{(3)} = k_{(4)} = 0$. (6.4)

The first is a condition which indirectly renormalizes the coupling constant; the second is to ensure that the Ward identity concerning the quartic gauge interaction (4.48) is satisfied. Although gauge invariance is not explicitly implemented in the FFT, by this condition we indirectly do implement the possible consequences of the invariance.

The full effective photon propagator

We recall that the full effective photon propagator is defined by the summation of more and more PI-bubbles in a row, see section 4.1:

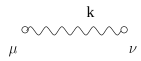
 $=$ $+$  $+$

$+$  $+ \ \dots$ (6.5)

The first term of this summation is the dressed propagator, that is the analytic expression for the gaussian part excluding the counterterm. In table 5.1 we defined

 $= \quad -\dfrac{e^2}{m_\gamma^2}\delta^{\mu\nu}$. (6.6)

This carries no momentum dependence and is by far not the effective photon propagator. The renormalization of QED is defined such that the full propagator equals the dressed propagator and at the renormalization scale $k^2 = \mu^2$ we have

 $\overset{\text{in QED}}{=}$  (6.7)

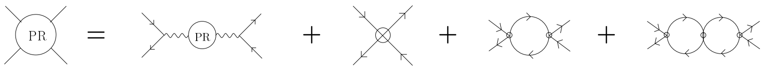
The condition for the same full propagator in the FFT is therefore that at the renormalization scale $k^2 = \mu^2$ it has to be equal

 $\overset{\text{in FFT}}{=} -\left\{\dfrac{1}{e^2}(\delta^{\mu\nu}k^2 - k^\mu k^\nu) + \dfrac{1}{e^2\xi}k^\mu k^\nu + \dfrac{(m_\gamma)^2}{e^2}\delta^{\mu\nu}\right\}^{-1}$, (6.8)

where we used the Feynman rule (3.29) for the dressed propagator in QED.

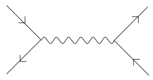
The full four fermion vertex

The full four fermion interaction consists of two infinite series. First, the summation of all possible expressions composed of two fermion-photon vertices connected by the photon lines: this is the diagram of four fermions interacting through the exchange of the full effective photon (6.5). Secondly, a purely fermionic series of more and more fermion loops in a row; in this series the four fermion counter vertex (5.44) is the zero loop term. We have



$$+\ldots \quad (6.9)$$

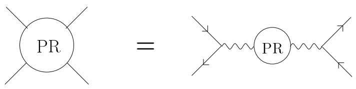
Exactly as Nambu and Jona-Lasinio proved in the case of the identification with the pion, this full vertex should be equal to the same expression in QED in which an exchange of a photon takes place, i.e. the expression



in QED. (6.10)

The reason that we use the dressed propagator in this expression in contrast to the full propagator in the first term of eq. (6.9), is because in QED the full equals the dressed propagator, see eq. (6.7). In the FFT this is not the case, because in the Lagrangian (5.38) the auxiliary field A_μ has no kinetic terms and it is only the full summation of eq. (6.5) that can be identified as a physical propagating effective photon.

Comparing the summation (6.9) with the expression (6.10) we come to the renormalization condition

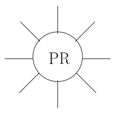


$$+\,\mathcal{O}\left(\mu^2/\Lambda^2\right) \; . \quad (6.11)$$

At low energies, which means that the external fermions have momenta of order $\mathcal{O}(\mu)$, and up to suppressed contributions the full four fermion vertex has to equal the exchange of the full effective photon propagator defined by the condition (6.8).

The full eight fermion vertex

As there is no eight fermion interaction in the theory of QED, it is easy to see that for QED to be a low energy limit of the FFT, the eight fermion vertex must be zero or suppressed. The renormalization condition thus states



$$= \; \mathcal{O}\left(\mu^2/\Lambda^2\right) \; . \quad (6.12)$$

Of course all expressions in this section are up to $\mathcal{O}(\mu^2/\Lambda^2)$ and we might as well have stated in this last equation that the full eight interaction should be zero. This notation however is merely to emphasize that it is the focus on low energy processes which makes that the FFT effectively behaves like QED.

Renormalizable or not?

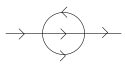
By the power counting rule, see section C.4, we should conclude from the Lagrangian (5.5) that the interaction in the FFT is non renormalizable. As a whole the theory indeed is, but we only analyse the low energy effects of the theory. Would we perform a full analyses, which includes all energies possible, the theory would become non renormalizable. It is the focus on a limiting situation which makes the analysis possible and which allowes a non renormalizable theory to gain a physical meaning.

6.2 The renormalization

We now apply the conditions and determine the exact value of all counterterm coefficients. All terms of order $\mathcal{O}(\mu^2/\Lambda^2)$ are neglected and all calculations are made up to the leading order in the $1/N$ expansion. For computing the exact order in $1/N$ of each diagram we must carefully take into account the contribution of the fermion lines and of the fermion loops. The rules for the fermion lines are stated in section 3.1.1, the rules for the loops in section 5.1.1.

6.2.1 The fermion self energy

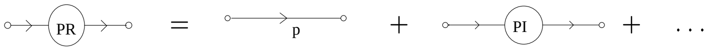
Due to the four fermion interaction, diagrams contributing to the fermion self energy can take on many forms. The diagram



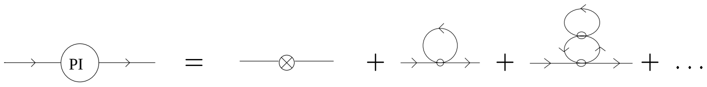
(6.13)

for example consists of two times the four fermion vertex and of two fermion loops. But these loops share one fermion line and this implies that both are of the same flavor at all time: there can be only one summation over all N flavors and the total order of the diagram is therefore N^{-1}. As the propagator is of order N^0 this correction term can be neglected in the expansion.

To leading order in the $1/N$ expansion the number of terms contributing to the fermion self energy is still infinite. We have



(6.14)

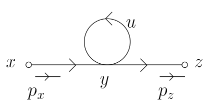


(6.15)

The summation in the self energy of eq. (6.15) continues over more and more stacked loops, for if these are of Type A as in section 5.1.1, all diagrams are of order N^0.

The stacked loops

The diagrams with stacked loops must all be taken into account. For the calculation of the one loop correction we start in the position space: the expression for the propagator with one loop is, with again T as the time-ordering operator,



$$
=< 0|T\psi_z\overline{\psi}_x \frac{-1}{2N\Lambda^2}\int \mathrm{d}^4y \left[\beta_0 J_\mu J^\mu + \frac{\beta_1}{2\Lambda^2}\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right)^2 + \right.
$$
$$
\left. + \frac{\beta_2}{\Lambda^2}\left(\partial_\eta J^\eta\right)^2 + \frac{\beta_3}{2\Lambda^2}\left(\partial_\mu J_\nu + \partial_\nu J_\mu - \frac{1}{2}\partial_\eta J^\eta\,\mathbb{1}_{\mu\nu}\right)^2 + \frac{\beta_4}{\Lambda^4}\left(\partial^2 J_\mu\right)^2\right]|0> . \tag{6.16}
$$

Instead of continuing this calculation with all five counterterm, we first calculate the contribution of the β_0- and β_2-term. With summations over repeated indices we can write

$$-\frac{1}{2N\Lambda^2}\int \mathrm{d}^4y < 0|T\psi_z\overline{\psi}_x\Big[\beta_0\left(\overline{\psi}^I_a(y)(\gamma_\mu)_{ab}\psi^I_b(y)\right)\left(\overline{\psi}^I_c(y)(\gamma^\mu)_{cd}\psi^I_d(y)\right)+$$
$$+\frac{\beta_2}{\Lambda^2}\partial_{y_\eta}\left(\overline{\psi}^I_a(y)(\gamma^\eta)_{ab}\psi^I_b(y)\right)\partial_{y_\eta}\left(\overline{\psi}^I_c(y)(\gamma^\eta)_{cd}\psi^I_d(y)\right)\Big]|0> \ . \tag{6.17}$$

For the loop intergral to be of Type A we must connect the ψ^I_b with the $\overline{\psi}^I_a$. The loop can also be constructed by connecting ψ^I_d with the $\overline{\psi}^I_c$. These two combinations are completely similar and we therefore get a factor 2 in the expression for eq. (6.17). With a factor (-1) every time a ψ^I and $\overline{\psi}^I$ switch place and with $S^I_{ba}(m-n)$ the fermion propagator for flavor I from coordinate m to n we have for eq. (6.17):

$$-\frac{2}{2N\Lambda^2}\int \mathrm{d}^4y < 0|T\psi_z\overline{\psi}_x\Big[\beta_0(-1)\left(\sum_{I=1}^{N}S^I_{ba}(y-y)\right)(\gamma_\mu)_{ab}\overline{\psi}^I_c(y)(\gamma^\mu)_{cd}\psi^I_d(y)+$$
$$+\frac{\beta_2}{\Lambda^2}\partial_{y_\eta}\left((-1)\sum_{I=1}^{N}S^I_{ba}(y-y)(\gamma^\eta)_{ab}\right)\ \partial_{y_\eta}\left(\overline{\psi}^I_c(y)(\gamma^\eta)_{cd}\psi^I_d(y)\right)\Big]|0> \ . \tag{6.18}$$

With all N flavors having the same propagator and assuming that in the diagram of (6.16) time runs from left to right, this last equations equals

$$=\frac{1}{\Lambda^2}\int \mathrm{d}^4y < 0|\Big[\beta_0 Tr\left[S(y-y)\gamma_\mu\right]\psi_z\ \overline{\psi}^I_y\gamma^\mu\psi^I_y\ \overline{\psi}_x+$$
$$+\frac{\beta_2}{\Lambda^2}\partial_{y_\eta}\left(Tr\left[S(y-y)\gamma^\eta\right]\right)\partial_{y_\eta}\left(\psi_z\ \overline{\psi}^I_y\gamma^\eta\psi^I_y\ \overline{\psi}_x\right)\Big]|0> \ . \tag{6.19}$$

We note that it is not completely correct to replace only some of the field operators by propagators. Yet in this notation we can clearly see that both terms have an integral over a Trace of one propagator which goes to zero by

$$\begin{aligned}\mathrm{Tr}\left[S(y-y)\gamma_\mu\right]=\mathrm{Tr}\int\frac{\mathrm{d}^4u}{(2\pi)^4}e^{iu\cdot(y-y)}\frac{u_\kappa\gamma^\kappa\gamma_\mu+iM'\gamma_\mu}{u^2+M'^2}=\\ =\int\frac{\mathrm{d}^4u}{(2\pi)^4}\frac{4u_\mu+0}{u^2+M'^2}=0\ .\end{aligned} \tag{6.20}$$

In the last line we used the fact that the Euclidean integral over u in all directions together with the quadratic denominator make that through symmetry the linear numerator goes to zero. Without letting the derivatives act, we have that the β_0- and β_2-term of eq. (6.16) are therefore equal to zero.

The β_1-, β_3- and β_4-term in eq. (6.16) which are all current-current terms, contribute in exactly the same way. Thus all are proportional to eq. (6.20) and equal to zero.

The counterterm

All in all, the one loop coontribution to the self energy of the fermion is zero. The same goes for the two- or more stacked loops: each diagram has one loop made up of a single propagator and therefore equals to zero. We see that the self energy only consists of the fermion counterterm and that the renormalization conditions eq. (6.1) and (6.2), which state that M' has to be equal to the physical mass can be easily satisfied by

$$\boxed{\omega_1 = \omega_2 = 0}\ . \tag{6.21}$$

The bare parameter $M = M'(1+\omega_2)$, see eq. (5.40), does not renormalize and the physical mass M' is equal to the bare parameter: $M' = M$.

The partial derivative

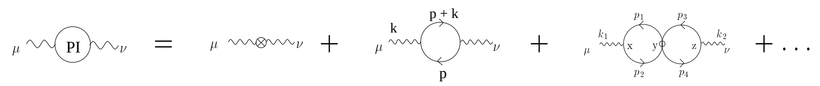
To see how the derivative operator does act, we can contract the remaining four field operators in the last term of eq. (6.19):

$$\begin{aligned}
&\int \mathrm{d}^4 y < 0|\psi_z \left(\partial_{y_\eta} J^\eta\right) \overline{\psi}_x |0> = \int \mathrm{d}^4 y < 0|\partial_{y_\eta} \left(\psi_z\, \overline{\psi}^I_y \gamma^\eta \psi^I_y\, \overline{\psi}_x\right) |0> = \\
&= \int \mathrm{d}^4 y\ \partial_{y_\eta} \Big(S(z-y)\gamma^\eta S(y-x) \Big) \\
&= \int \mathrm{d}^4 y\ \partial_{y_\eta} \left(\int \frac{\mathrm{d}^4 p'}{(2\pi)^4} e^{-ip'(z-y)} S(p')\, \gamma^\eta \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip(y-x)} S(p) \right) \\
&= \int \mathrm{d}^4 y\ i(p'-p)_\eta \int \frac{\mathrm{d}^4 p'}{(2\pi)^4} \frac{\mathrm{d}^4 p}{(2\pi)^4} S(p')\gamma^\eta S(p) e^{-iy(p-p')} e^{-i(p'z-px)} \\
&= i(p-p)_\eta \int \frac{\mathrm{d}^4 p}{(2\pi)^4} S(p)\gamma^\eta S(p) e^{-i(pz-px)}\ .
\end{aligned} \tag{6.22}$$

Although this last equation of course equals zero, it is clear now that the derivative acting on the current J^η simply yields the difference in momenta of the connected fermions. We have a factor $(+ip)_\eta$ for lines away from the vertex and a factor $(-ip)_\eta$ for lines towards the vertex. This rule will apply to all the diagrams in the remainder of this chapter.

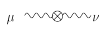
6.2.2 The effective photon self energy

To leading order in the $1/N$ expansion the photon self energy consists of the counterterm and an infinite summation of more and more loops:



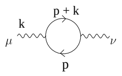
(6.23)

The analytic expression for the counterterm is given in eq. (5.42):



$$= -\omega_3 \delta^{\mu\nu}\ . \tag{6.24}$$

As the Feynman rules for the fermion propagator and the fermion-photon vertex are exactly the same as in QED the expression for the one loop diagram is the same as in eq. (4.23) and we have that

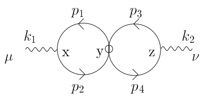


$$
\begin{aligned}
&= -\,\mathrm{Tr}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\gamma^\mu S(p)\gamma^\nu S(p+k) && (6.25)\\
&= \frac{2}{(4\pi)^2}\delta^{\mu\nu}\left(\Lambda^2 - M^2\right) + \frac{1}{(4\pi)^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right) - \frac{1}{3(4\pi)^2}k^\mu k^\nu && (6.26)\\
&\quad - \frac{4}{3(4\pi)^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)\left(\ln\frac{\Lambda^2}{\mu^2} + \frac{5}{3}\right)\,, && (6.27)
\end{aligned}
$$

where in the line (6.27) we already made the same steps as in eq. (4.28 - 4.30): assuming that $\mu^2 \gg M^2$ the integral over the Feynman parameter x results in a logarithmic term plus a constant.

The two loop diagram

For the calculation of the two loop diagram we start in coordinate space; we have no photon propagators attached to both sides and therefore write



$$
\begin{aligned}
&= \int \frac{\mathrm{d}^4 x}{(2\pi)^4}e^{ik_1\cdot x}\int \frac{\mathrm{d}^4 z}{(2\pi)^4}e^{-ik_2\cdot z}\left(\frac{-1}{\sqrt{N}}\right)^2 \overline{\psi}^I_a(x)\gamma^\mu_{ab}\psi^I_b(x)\times\\
&\quad \int \mathrm{d}^4 y\left(-\beta_i J_i^2\right)\overline{\psi}^I_c(z)\gamma^\mu_{cd}\psi^I_d(z)
\end{aligned}
\tag{6.28}
$$

with the shorthand notation for all five counterterms. We continue this calculation only for the β_1 term; the four other terms are to be dealth with in a similar way.

$$
\begin{aligned}
= \;& \frac{1}{N}\int \frac{\mathrm{d}^4 x}{(2\pi)^4}e^{ik_1\cdot x}\int \frac{\mathrm{d}^4 z}{(2\pi)^4}e^{-ik_2\cdot z}\overline{\psi}^I_a(x)\gamma^\mu_{ab}\psi^I_b(x)\Big\{\int \mathrm{d}^4 y\frac{-\beta_1}{4N\Lambda^4}\times\\
&\Big\{\partial_\alpha\left[\overline{\psi}^I_e(y)\gamma_{\beta\, ef}\psi^I_f(y)\right] - \partial_\beta\left[\overline{\psi}^I_e(y)\gamma_{\alpha\, ef}\psi^I_f(y)\right]\Big\}\times\\
&\Big\{\partial^\alpha\left[\overline{\psi}^I_g(y)\gamma^\beta{}_{gh}\psi^I_h(y)\right] - \partial^\beta\left[\overline{\psi}^I_g(y)\gamma^\alpha{}_{gh}\psi^I_h(y)\right]\Big\} + \ldots\Big\}\overline{\psi}^I_c(z)\gamma^\mu_{cd}\psi^I_d(z)
\end{aligned}
\tag{6.29}
$$

$$
\begin{aligned}
= \;& -2\int \frac{\mathrm{d}^4 x}{(2\pi)^4}e^{ik_1\cdot x}\int \frac{\mathrm{d}^4 z}{(2\pi)^4}e^{-ik_2\cdot z}\Big\{\int \mathrm{d}^4 y\frac{\beta_1}{4\Lambda^4}\times\\
&\big\{(-1)\partial_\alpha\left[S_{fa}(y-x)\gamma^\mu{}_{ab}S_{be}(x-y)\gamma_{\beta\, ef}\right] +\\
&-(-1)\partial_\beta\left[S_{fa}(y-x)\gamma^\mu{}_{ab}S_{be}(x-y)\gamma_{\alpha\, ef}\right]\big\}\times\\
&\big\{(-1)\partial^\alpha\left[S_{dg}(z-y)\gamma^\beta{}_{gh}S_{hc}(y-z)\gamma^\nu{}_{cd}\right] +\\
&-(-1)\partial^\beta\left[S_{dg}(z-y)\gamma^\alpha{}_{gh}S_{hc}(y-z)\gamma^\nu{}_{cd}\right]\big\} + \ldots\Big\}\,.
\end{aligned}
\tag{6.30}
$$

This is a combination of four Traces; going to momentum space we can write the last of these as

$$\begin{aligned}&\mathrm{Tr}\;\partial_y^\beta \int \frac{\mathrm{d}^4 p_4}{(2\pi)^4} S(p_4) e^{-ip_4\cdot(z-y)}\gamma^\alpha \int \frac{\mathrm{d}^4 p_3}{(2\pi)^4} S(p_3) e^{-ip_3\cdot(y-z)}\gamma^\nu = \\ &\mathrm{Tr}\; i\,(p_4-p_3)^\beta \int \frac{\mathrm{d}^4 p_4 \mathrm{d}^4 p_3}{(2\pi)^8} S(p_4)\gamma^\alpha S(p_3)\gamma^\nu e^{-ip_4(z-y)-ip_3(y-z)}\;.\end{aligned} \tag{6.31}$$

Applying this to all four Traces we then perform the integrals over x, y and z. These yield momentum conservating δ-functions such that

$$k_1 = -k_2 \equiv k \quad ; \quad p_2 = p_1 + k \quad ; \quad p_4 = p_3 + k \tag{6.32}$$

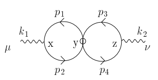
and the expression of eq. (6.30) is reduced to a combination of four momentum integrals:

$$\begin{aligned}&\frac{-2\beta_1}{4\Lambda^4}\left[\mathrm{Tr}k_\alpha \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} S(p_1)\gamma^\mu S(p_1+k)\gamma_\beta - \mathrm{Tr}k_\beta \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} S(p_1)\gamma^\mu S(p_1+k)\gamma_\alpha\right] \\ &\left[\mathrm{Tr}k^\alpha \int \frac{\mathrm{d}^4 p_4}{(2\pi)^4} S(p_4)\gamma^\beta S(p_4-k)\gamma_\nu - \mathrm{Tr}k^\beta \int \frac{\mathrm{d}^4 p_4}{(2\pi)^4} S(p_4)\gamma^\alpha S(p_4-k)\gamma_\nu\right] + \ldots\end{aligned} \tag{6.33}$$

These integrals are equal to the integral of eq. (6.25) up to interchanging indices. However, in eq. (6.33) the coefficient is of order $\mathcal{O}(\Lambda^{-4})$ and in the result of the integral (6.25) the highest order is of $\mathcal{O}(\Lambda^2)$. As we perform all calculations up to $\mathcal{O}(\mu^2/\Lambda^2)$ we can therefore approximate in the expression of eq. (6.33) each integral by its highest order. Eq. (6.33) becomes:

$$\begin{aligned}&= \left\{\frac{-\beta_1}{2\Lambda^4}\left[k_\alpha \frac{-2\Lambda^2}{(4\pi)^2}\delta^\mu_{\ \beta} - k_\beta \frac{-2\Lambda^2}{(4\pi)^2}\delta^\mu_{\ \alpha}\right]\left[k^\alpha \frac{-2\Lambda^2}{(4\pi)^2}\delta^{\beta\nu} - k^\beta \frac{-2\Lambda^2}{(4\pi)^2}\delta^{\alpha\nu}\right] + \ldots\right\} \\ &= \left\{\frac{-\beta_1}{2\Lambda^4}\left(\frac{2\Lambda^2}{(4\pi)^2}\right)^2 \left[2k^2\delta^{\mu\nu} - 2k^\mu k^\nu\right] + \ldots\right\}\;.\end{aligned} \tag{6.34}$$

The contributions of the four other β_i-counterterms can be calculated following the same steps taken in ths section for the β_1 term. The final result for the two loop diagram is



$$\begin{aligned}&= \left(\frac{2\Lambda^2}{(4\pi)^2}\right)^2 \left\{\frac{\beta_0}{\Lambda^2}\delta^{\mu\nu} - \frac{\beta_1}{\Lambda^4}\left(k^2\delta^{\mu\nu} - k^\mu k^\nu\right) - \frac{\beta_2}{\Lambda^4}k^\mu k^\nu + \right. \\ &\left. - \frac{\beta_3}{\Lambda^4}\left(k^2\delta^{\mu\nu} + \frac{1}{2}k^\mu k^\nu\right) + \frac{\beta_4}{\Lambda^6}k^4\delta^{\mu\nu}\right\}\;.\end{aligned} \tag{6.35}$$

The three or more loop diagrams

From the calculation of this two loop diagram it is easy to see that the leading term of the four fermion vertex is the β_0 term; it is $\mathcal{O}(\Lambda^{-2})$ in contrast to the other terms which are $\mathcal{O}(\mu^2/\Lambda^4)$. The highest order of a loop integral is

$\mathcal{O}(\Lambda^2)$ and thus the three loop diagram is of order $(\Lambda^2)^3(\Lambda^{-2})^2 = \Lambda^2$; a simple calculation shows that all possible number of n loops are of order

$$\mathcal{O}\left(\left(\Lambda^2\right)^n\left(\Lambda^{-2}\right)^{n-1}\right) = \mathcal{O}\left(\Lambda^2\right) \tag{6.36}$$

and we have an infinite summation of quadratically divergent diagrams.

One divergent term can be renormalized by one counterterm; for an infinite number of divergent integrals we need an infinite number of counterterms, which is of course impossible. This problem is easily solved by setting

$$\boxed{\beta_0 = 0}\,. \tag{6.37}$$

By this definition the four fermion vertex is of maximum order $\mathcal{O}(\mu^2/\Lambda^4)$ and an n-loop diagram is of order

$$\mathcal{O}\left(\left(\Lambda^2\right)^n\left(\frac{\mu^2}{\Lambda^4}\right)^{n-1}\right) = \mathcal{O}\left(\frac{\mu^{2n-1}}{\Lambda^{2n-4}}\right)\,. \tag{6.38}$$

Terms with three or more loops are thus suppressed and we can see that the self energy (6.23) consists of only the counterterm and the one- and two loop.

Applying the renormalization condition

We define the self energy by $\Pi^{\alpha\beta}$, the full effective photon propagator by $D'^{\mu\nu}$ and the dressed propagator by $D^{\mu\nu} = -\frac{e^2}{m_\gamma^2}\delta^{\mu\nu}$. With these definitions the full propagator given by the series (6.5) is

$$\begin{aligned} D'^{\mu\nu} &= D^{\mu\nu} + D^{\mu\alpha}\Pi_{\alpha\beta}D^{\beta\nu} + D^{\mu\alpha}\Pi_{\alpha\beta}D^{\beta\kappa}\Pi_{\kappa\lambda}D^{\lambda\nu} + \ldots & (6.39)\\ &= \frac{1}{D^{-1\mu\nu} - \Pi^{\mu\nu}}\,. & (6.40)\end{aligned}$$

The renormalization condition (6.8) states that at the renormalization scale $k^2 = \mu^2$ we have

$$D'^{-1\mu\nu} = -\left\{\frac{1}{e^2}(\delta^{\mu\nu}k^2 - k^\mu k^\nu) + \frac{1}{e^2\xi}k^\mu k^\nu + \frac{(m_\gamma)^2}{e^2}\delta^{\mu\nu}\right\}^{-1}\,. \tag{6.41}$$

In this equation the inverse full propagator is the sum of the three terms $\sim (\delta^{\mu\nu}k^2 - k^\mu k^\nu)$, $\sim k^\mu k^\nu$ and $\sim \delta^{\mu\nu}$. By arranging the right-hand side of eq. (6.40) in these same terms we can determine the coefficients. The self energy $\Pi^{\mu\nu}$ is the sum of the counterterm (6.24), the one loop (6.25) and the two loop correction (6.35). In the two loop correction we also have a term $\sim \beta_3\left(k^2\delta^{\mu\nu} + \frac{1}{2}k^\mu k^\nu\right)$ and as this is not present in the left-hand side of eq. (6.40), we set the coefficient

$$\boxed{\beta_3 = 0}\,. \tag{6.42}$$

Organizing all terms with $\sim (\delta^{\mu\nu}k^2 - k^\mu k^\nu)$, $\sim k^\mu k^\nu$ and $\sim \delta^{\mu\nu}$ we can now see that at the scale $k^2 = \mu^2$ with $\mu^2 \gg M^2$ we have

$$-\frac{1}{e^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right) = \left(-\frac{1}{(4\pi)^2} + \frac{4}{3(4\pi)^2}\left(\ln\frac{\Lambda^2}{\mu^2} + \frac{5}{3}\right) + \beta_1\frac{4}{(4\pi)^4}\right) \times \left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right) \tag{6.43}$$

$$\rightarrow \boxed{\beta_1 = \frac{(4\pi)^4}{4}\left(-\frac{1}{3(4\pi)^2}\left(4\ln\frac{\Lambda^2}{\mu^2} + \frac{11}{3}\right) - \frac{1}{e^2}\right)}; \tag{6.44}$$

$$-\frac{1}{e^2\xi}k^\mu k^\nu = \left(\frac{1}{3(4\pi)^2} + \beta_2\frac{4}{(4\pi)^4}\right)k^\mu k^\nu \tag{6.45}$$

$$\rightarrow \boxed{\beta_2 = \frac{(4\pi)^4}{4}\left(\frac{1}{3(4\pi)^2} - \frac{1}{e^2\xi}\right)}; \tag{6.46}$$

$$-\frac{m_\gamma^2}{e^2}\delta^{\mu\nu} = \left(-\frac{m_\gamma^2}{e^2} + \omega_3 - \frac{2(\Lambda^2 - M^2)}{(4\pi)^2} - \frac{\beta_4}{\Lambda^2}k^2\right)\delta^{\mu\nu} \tag{6.47}$$

$$\rightarrow \omega_3 = \frac{2(\Lambda^2 - M^2)}{(4\pi)^2} + \frac{\beta_4}{\Lambda^2}k^2 . \tag{6.48}$$

In the last equation we see that at the renormalization scale $k^2 = \mu^2$ the coefficient β_4 is suppressed by a factor μ^2/Λ^2, exactly as claimed in section 5.1.2. As it is suppressed we can easily set

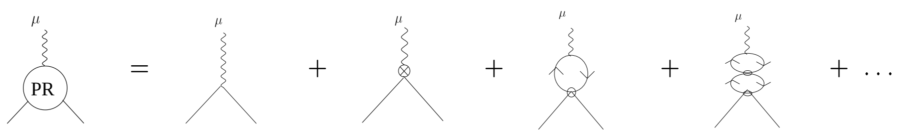
$$\boxed{\beta_4 = 0}, \tag{6.49}$$

hence justifying the statement in section 5.1.2 that counterterms with more than two derivatives are not necessary. At the same time this fixes the coefficient ω_3 at

$$\boxed{\omega_3 = \frac{2(\Lambda^2 - M^2)}{(4\pi)^2}}. \tag{6.50}$$

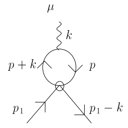
6.2.3 The fermion-photon vertex

To leading order in the $1/N$ expansion the fermion-photon interaction consists of the infinite summation of more and more loops:



(6.51)

The computation of the one loop diagram is however straightforward. We have set the parameters $\beta_0 = \beta_3 = \beta_4 = 0$ and the four fermion counter vertex consists of two terms with derivatives. We saw in the calculations of the one loop correction to the fermion propagator, see eq (6.16), and of the two loop correction to the photon propagator, see eq. (6.28), that if the loop is to be of order N the derivatives in the counterterms lead to the difference in momenta of either the two fermion lines inside a loop or of the two external lines. We can thus easily see that when $p_{in} = p_{out}$, the point of renormalization as stated in eq. (6.3), we have



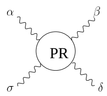
$$= \mathcal{O}\left((p-(p-k))^2\right) = \mathcal{O}\left(k^2\right) = 0 \quad \text{when } p_{in} = p_{out} \,. \tag{6.52}$$

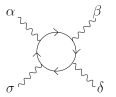
The diagrams with more and more loops in the summation of eq. (6.51) yield exactly the same result. Thus the renormalization condition, which states that the full PR vertex is to be equal to the first diagram of eq. (6.51) when working on-shell, is easily satisfied by setting the fermion-photon counter vertex to zero:

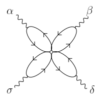
$$\boxed{\omega_4 = 0}\,. \tag{6.53}$$

6.2.4 The quartic photon vertex

To leading order in the $1/N$ expansion we have that the quartic photon interaction is the sum of two diagrams:



$=$


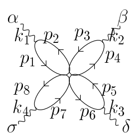
$+$


. (6.54)

The expression for the one loop diagram is given in eq. (4.45) and when $k_{(1)} = k_{(2)} = k_{(3)} = k_{(4)} = 0$ equals

$$I^{\alpha\beta\delta\sigma} = -\frac{4}{3}\frac{1}{(4\pi)^2 N}\left(\delta^{\alpha\beta}\delta^{\delta\sigma} + \delta^{\alpha\delta}\delta^{\beta\sigma} + \delta^{\alpha\sigma}\delta^{\beta\delta}\right) . \tag{6.55}$$

The calculation of the four loop diagram is not as difficult as it may seem. We start with the fermion fields necessary for the calculations: we define $\overline{\psi}^I\gamma^\alpha\psi^I$ for the current term belonging to the fermion-photon interaction of the α corner and $J_\mu J^\mu J_\nu J^\nu$ for the eight fermion interaction in the center. The three other corners contribute with respectively J^β, J^δ and J^σ. We have



$$= \left(\frac{-1}{\sqrt{N}}\right)^4\left(\frac{-\gamma}{4N^3\Lambda^8}\right)\overline{\psi}^I\gamma^\alpha\psi^I\,\overline{\psi}^I\gamma^\beta\psi^I\,\overline{\psi}^I\gamma^\delta\psi^I\,\overline{\psi}^I\gamma^\sigma\psi^I$$
$$\times\,\overline{\psi}^I\gamma_\mu\psi^I\,\overline{\psi}^I\gamma^\mu\psi^I\,\overline{\psi}^I\gamma_\nu\psi^I\,\overline{\psi}^I\gamma^\nu\psi^I\,. \tag{6.56}$$

Each loop is to be of order N, thus each corner current is to be connected to one of the four currents in the center of the diagram and we have $4! = 24$ ways of connecting these terms. Continuing the calculation with one of these

24 permutations we have

$$
\begin{aligned}
&= \frac{-\gamma}{4N\Lambda^8}\Big\{ \mathrm{Tr}\int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} S(p_1)\gamma^\alpha S(p_1+k_1)\gamma_\mu \ \mathrm{Tr}\int \frac{\mathrm{d}^4 p_3}{(2\pi)^4} S(p_3)\gamma^\beta S(p_3+k_2)\gamma^\mu \\
&\qquad \times \mathrm{Tr}\int \frac{\mathrm{d}^4 p_5}{(2\pi)^4} S(p_5)\gamma^\delta S(p_5+k_3)\gamma_\nu \ \mathrm{Tr}\int \frac{\mathrm{d}^4 p_7}{(2\pi)^4} S(p_7)\gamma^\sigma S(p_7+k_4)\gamma^\nu \\
&\qquad + 23 \text{ permutations} \Big\} \\
&= -\frac{\gamma}{4N\Lambda^8}\left\{ \frac{-2\Lambda^2}{(4\pi)^2}\delta^\alpha_\mu \ \frac{-2\Lambda^2}{(4\pi)^2}\delta^{\beta\mu} \ \frac{-2\Lambda^2}{(4\pi)^2}\delta^\delta_\nu \ \frac{-2\Lambda^2}{(4\pi)^2}\delta^{\sigma\nu} + 23 \text{ permutations} \right\}
\end{aligned}
\tag{6.57}
$$

In the last line we approximated each loop integral by its highest order term; all other terms only contribute to the diagram in the order of $\mathcal{O}(\mu^2/\Lambda^2)$. The different permutations result in 24 sequences for $\alpha\beta\delta\gamma$. It turns out that among these 24 permutations we have three different results; defining the four loop diagram by $\Gamma^{\alpha\beta\delta\sigma}$ we have

$$
\Gamma^{\alpha\beta\delta\sigma} = -\frac{\gamma}{4N\Lambda^8}\left\{ \left(\frac{2\Lambda^2}{(4\pi)^2}\right)^4 8\left[\delta^{\alpha\beta}\delta^{\delta\sigma} + \delta^{\alpha\delta}\delta^{\beta\sigma} + \delta^{\alpha\sigma}\delta^{\beta\delta}\right]\right\} \tag{6.58}
$$

The renormalization condition of eq. (6.4) states that when all four external momenta are zero the quartic photon PR vertex is to be zero. Combining eq. (6.55) and (6.58) we easily see that this is the case if we set

$$
\boxed{\gamma = -\frac{1}{24}(4\pi)^6} \, . \tag{6.59}
$$

6.2.5 The four and eight fermion vertex

In eq. (6.9) we saw that the full four fermion vertex is a summation of two infinite series. As a tuning condition we stated in eq. (6.11) that when all external fermions are of low energy, the PR vertex is to be equal to the first diagram in the summation (6.9); all other diagrams are to be of order $\mathcal{O}(\mu^2/\Lambda^2)$.

As we have set the parameters $\beta_0 = \beta_3 = \beta_4 = 0$ we have that in the FFT Lagrangian (5.38) only two counterterms yield the four fermion vertex:

$$
\mathcal{L}_{4int} = \frac{\beta_1}{4N\Lambda^4}\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right)^2 + \frac{\beta_2}{2N\Lambda^4}\left(\partial_\eta J^\eta\right)^2 \, . \tag{6.60}
$$

We have seen that the derivates lead to the difference in momenta and thus we can safely state that the four fermion vertex, that is the second diagram in the summation of eq. (6.9), is of order $\mathcal{O}(\mu^2/\Lambda^4)$. With each fermion loop of maximum order of $\mathcal{O}(\Lambda^2)$ we see that a diagram of n loops and $n+1$ vertices is of order

$$
\mathcal{O}\left(\left(\Lambda^2\right)^n \left(\frac{\mu^2}{\Lambda^4}\right)^{n+1}\right) = \mathcal{O}\left(\frac{\mu^{2n+2}}{\Lambda^{4n+4}}\right) \, . \tag{6.61}
$$

It is easy to see that the whole series of loop diagrams is suppressed and that therefore the tuning condition is satisfied.

As for the eight fermion vertex we set as condition in eq. (6.12) that the full vertex is to be of order $\mathcal{O}(\mu^2/\Lambda^2)$. With the parameter γ set in eq. (6.59) we have that the eight fermion counterterm itself is of order $\mathcal{O}(\Lambda^{-8})$, see eq. (5.44). We can easily see that any correction to this term is of even lower order and we thus conclude that the tuning condition is satisfied.

6.3 Fine tuning of the free parameters

In this section we analyse all parameters present in the FFT and fine-tune them such that the low energy limit of the theory reduces to Quantum Electrodynamics. Summarizing all counterterm coefficients we have in the FFT

$$\omega_1 = \omega_2 = \omega_4 = 0 \; ; \tag{6.62}$$

$$\omega_3 = \frac{2(\Lambda^2 - M^2)}{(4\pi)^2} \; ; \tag{6.63}$$

$$\beta_0 = \beta_3 = \beta_4 = 0 \; ; \tag{6.64}$$

$$\beta_1 = \frac{(4\pi)^4}{4}\left(-\frac{1}{3(4\pi)^2}\left(4\ln\frac{\Lambda^2}{\mu^2} + \frac{11}{3}\right) - \frac{1}{e^2}\right) \; ; \tag{6.65}$$

$$\beta_2 = \frac{(4\pi)^4}{4}\left(\frac{1}{3(4\pi)^2} - \frac{1}{e^2\xi}\right) \; ; \tag{6.66}$$

$$\gamma = -\frac{1}{24}(4\pi)^6 \; . \tag{6.67}$$

The parameters e^2 and ξ are defined by the coefficients β_1 and β_2: the eq. (6.65) and (6.66) are in fact renormalization equations. In section 6.2.1 we concluded that in the low energy approximation the fermion mass does not renormalize. With the definition of m_γ^2 in eq. (5.39) we have the four renormalization equations

$$M' = M \; ; \tag{6.68}$$

$$\frac{1}{e^2} = -\beta_1\frac{4}{(4\pi)^4} - \frac{1}{3(4\pi)^2}\left(4\ln\frac{\Lambda^2}{\mu^2} + \frac{11}{3}\right) \; ; \tag{6.69}$$

$$\frac{m_\gamma^2}{e^2} = \frac{\Lambda^2}{\alpha} - \frac{2}{(4\pi)^2}(\Lambda^2 - M^2) \; ; \tag{6.70}$$

$$\frac{1}{e^2\xi} = -\beta_2\frac{4}{(4\pi)^4} - \frac{1}{3}\frac{1}{(4\pi)^2} \; . \tag{6.71}$$

Comparing these to the renormalization equations (4.61), (4.62), (4.63) and (4.64) in QED, we see that the four equations become identical by fine-tuning the parameters at

$$\beta_1 = \frac{(4\pi)^4}{4}\left[-\frac{2}{3(4\pi)^2}\left(4\ln\frac{\Lambda^2}{\mu^2} + \frac{11}{3}\right) - \frac{1}{e_0^2}\right] \; ; \tag{6.72}$$

$$\frac{\Lambda^2}{\alpha} = \frac{m_0^2}{e_0^2} \; ; \tag{6.73}$$

$$\beta_2 = \frac{(4\pi)^4}{4}\left[-2\frac{1}{3(4\pi)^2} - \frac{1}{e_0^2\xi_0}\right] \; . \tag{6.74}$$

Although there is no flow as a consequence of the four gauge interaction, the Ward identity (4.48) sets as condition that the quartic interaction in QED must be zero. Extending this identity to the FFT we also fine-tune the eight fermion interaction at

$$\gamma = \frac{(4\pi)^6}{32}\Delta_4 \ . \tag{6.75}$$

With this last identification we now have an exact mapping of all parameters between both theories and all full propagators and interactions in both theories have become completely identical.

Renormalization flows

The renormalization flows are defined by the scale dependence of the renormalization equations, see eq. (4.65). As we have regained the exact same renormalization equations (4.61), (4.62), (4.63) and (4.64) we can directly conclude that the flows of the FFT are now equivalent to those in QED.

Chapter 7

Discussion and Conclusion

7.1 Discussion

In this thesis we have analysed many different bound states and extensively discussed the special case of the bound photon, formed in a purely fermionic theory. For the massless photon to be formed out of two fermions we implied the technique of fine-tuning as to reach the specific low energy point in the parameter space of the fermionic theory corresponding to an effective QED. We achieved this in several steps.

In chapter 3 and 4 we performed the renormalization of QED in the non-conventional case with N flavors and by use of the cut-off regularization scheme. With an $1/N$ expansion we discussed the 'marginal' fermion mass M and the renormalization flows of the 'irrelevant' coupling constant e, the 'irrelevant' photon mass m_γ and the 'relevant' gauge fixing parameter ξ, see section 4.4.

In chapter 5 we introduced the four fermi theory (FFT) with current-current interaction and we elaborated on how this non-renormalizable theory could be tuned such that its low energy behavior turns out to be identical to the renormalizable QED. In chapter 6 we renormalized for low energies the FFT and finally in section 6.3 we fine-tuned all free parameters. By this tuning the four renormalization flows in the FFT became at low energies an exact copy of the flows in QED mentioned above and the propagators plus interactions of both theories became the same, to leading non-trivial order in a $1/N$ expansion.

7.1.1 Gauge invariance and the Ward identities

We have not explicitly calculated or checked the Ward identities in the FFT, but we do not have to. By the exact mapping of the parameters it is immediately clear that all expressions for the Ward identities will be equal to the same expressions in QED in section 4.3.

The Ward identities are a consequence of gauge invariance, so without having mentioned gauge invariance in the FFT we have regained the same results at the special point in parameter space.

7.1.2 Higher order terms

In this thesis we have used two approximations: we neglected in the Feynman diagrams all contributions of order $\mathcal{O}(\mu^2/\Lambda^2)$ and we performed all calculations up to the leading order in the $1/N$ expansion.

We must emphasize at this point that the N flavored QED analysed in this thesis is a study model; there are no N different fermions in nature with the same charge and the same mass. The established equivalence by fine-tuning is however difficult to recreate in an analysis of standard QED with one flavor and plain perturbation theory, in fact it has never been accomplished. The study by means of an extra parameter therefore gives us a new perspective from which eventually we hope to learn other possibilities in standard QED.

The low energy approximation

In the FFT the photon is a two particle excitation that arises when analysing the theory at low energies μ, with μ low compared to the high energy cut-off level Λ. This cut-off can be chosen at a high but finite scale, just as in QED which is limited by the Landau pole Λ_L, see eq. (4.67).

If we set this cut-off level at say the Planck scale we have $\Lambda = \mathcal{O}(10^{19}\text{GeV})$ and even the highest energies reached at LEP which were of order $\mathcal{O}(10^2\text{GeV})$, are of a scale 10^{-17} lower. In the calculations where we neglected all terms of order $\mathcal{O}(\mu^2/\Lambda^2)$ this means that terms were neglected which contributed at an order 10^{-34} lower. We can thus safely state that neglecting these terms is justified.

The important point to notice here is that we have established an equivalence, an exact mapping between both theories at low energies. The underlying dynamics of the interacting fermions lie at high energies, but as we discussed in section 2.3.1 there is at energies of order $\mathcal{O}(10^2\text{GeV})$ the unification with the weak force into the GWS theory; above this energy QED is no longer a valid theory.

As we just explained, $\mathcal{O}(10^2\text{GeV})$ is very low compared to the Planck scale and so in other words, the equivalence of QED and the FFT is valid over the whole energy range of QED. In Suzuki's model [8, 9] the electroweak sector is dynamically generated without a scalar Higgs boson. The three weak mediators are massive bound states of fermions, while the photon is a massless fundamental particle. An interesting question is whether Suzuki's model could be reconciled with the subject discussed in this thesis. Can all four gauge particles of the electroweak force be dynamically generated? And would this still require fine-tuning or would the dynamics at higher energies naturally form the photon?

7.1.3 Renormalization group viewpoint

With the three renormalization flows of the parameters e^2, ξ and m_γ we can visualize the flow as in the three dimensonal parameter space in figure 7.1; we leave out the marginal fermion mass M.

The two irrelevant parameters e^2 and m_γ always flow towards the fixed point in the origin and in the limit of $\mu \to 0$ they renormalize to zero. Even though the two systems $\mathcal{L}_{QED}$ and $\mathcal{L}_{FFT}$ have completely different high energy behavior, we have shown that they both renormalize to the same fixed point. At lower

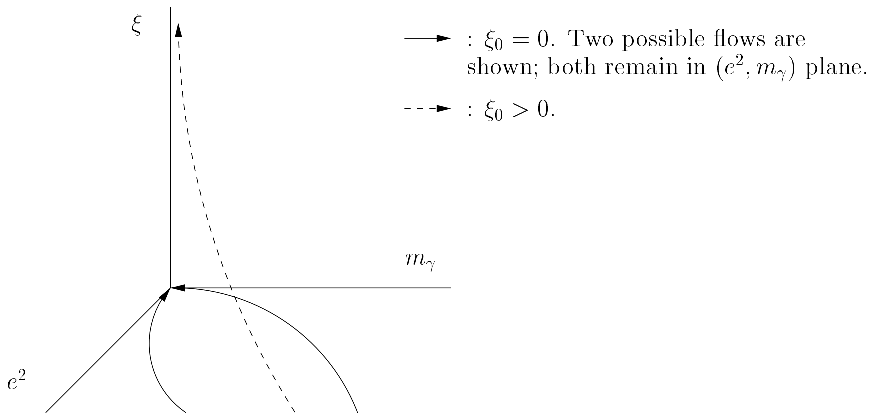


Figure 7.1: Possible renormalization flows in parameter space.

energies these parameters thus become weaker and weaker, and ultimately we have $\mathcal{L}_{QED} = \mathcal{L}_{FFT}$.

The (e^2, m_γ) plane is defined as the critical plane, see [15, 16]: from any point in this parameter plane the system renormalizes to the fixed point resulting in always the same low energy behavior. This explains how two systems, which may seem completely different at first, have exactly the same low energy physics.

The gauge fixing term ξ is a relevant parameter and if its initial value ξ_0 is non-zero, the flow diverges away from the fixed point. As this parameter has no consequences for any physical observable its numerical value is of no importance. What is important is that again in both theories its flow is the same and that the parameters e^2 and m_γ do reach their fixed point as the flow eventually approaches the ξ axis.

The requirement of fine-tuning

In chapter 2 we discussed many interactions forming a condensate in the strong coupling regime and in all these models the picture is that of a particle and an anti-particle being so strongly attracted to each other that they form a massless bound state. This picture also applies to the discussed non-abelian gauge theories of Eguchi [6, 7] and Suzuki [8, 9].

We require a precise fine-tuning of the free parameters in the FFT such that, wherever the renormalization flow starts, its end phase is as in figure 7.1 and with the fine-tuning an exact mapping of the low energy limit of the FFT is possible onto QED. If this tuning is not satisfied, the low energy dynamics would be anything *but* QED. So even though the starting point of the flow is not important, the high energy physics still determines theresulting effective physics at low energies.

With an exact mapping between both theories such that there is no difference at low energies, the equivalence raises another question: if there is no physical difference to be found between both models, what is the physical difference between a fundamental and a composite particle? We can draw a parallel with the idea of the photon being either a particle or a wave and perhaps the best answer is, that at a certain point there is no difference.

This last reasoning coincides of course with the outset of this thesis, i.e. to find a fermionic theory equivalent to QED from the viewpoint of universality: systems with completely different high energy physics can exhibit at low energies near their fixed point the same universal behavior.

7.2 Conclusion

We conclude that by a fine-tuning of the free parameters in the Four Fermi Theory with current-current interaction, the theory exhibits at low energy all properties of Quantum Electrodynamics. The low energy limit of the purely fermionic theory is thus an effective gauge theory and in the FFT the photon is a two particle excitation.

Fine-tuning implies that a delicate connection exists between the high and low energy dynamics. Aside from the photon in QED, the gluons and the weak mediators in the Standard Model have been shown by different authors to possibly be composite. It is extremely interesting whether a fully dynamically generated electroweak gauge theory is possible. Perhaps that such a model which would unite effective non-abelian and abelian gauge fields in the so far unreached Higgs sector, can unravel the connection between the low and high energy physics and shed new light on the esteemed gauge invariance.

Appendix A

Notation and Convention

A.1 Euclidean space

In this section we introduce the conventions for Euclidean space we will be using throughout this thesis. Let $\psi(x)$ be a field depending on position x; in Minkowski space a functional integral is defined by

$$\int \mathcal{D}\psi e^{iS[\psi(x)]} , \tag{A.1}$$

with $S[\psi]$ the action depending on the field ψ. A functional integral can be interpreted as integrating all possible paths of $\psi(x)$, with a given weight of $e^{iS[\psi(x)]}$. S is the Minkowskian four-dimensional integral of the Lagrangian density $\mathcal{L}$, which throughout this thesis we shall call the Lagrangian. For the transformation to Euclidean space the Wick rotation is performed: $t \to it_{\mathrm{E}}$. What happens is that

$$x_\mu x^\mu = x^0 x^0 - x^i x^i \qquad \to \qquad -x^0_{\mathrm{E}}\, x^0_{\mathrm{E}} - x^i x^i . \tag{A.2}$$

In Euclidean space the metric $\delta_{\mu\nu}$ is used instead of the Minkowski metric $\eta_{\mu\nu}$. This removes any difference between upper and lower indices, yet we will leave terms with upper and lower indices to emphasize the product between both terms. We thus define

$$x_{E\mu} x^\mu_E = x^0_E x^0_E + x^i_E x^i_E . \tag{A.3}$$

Not only x^0 is changed. The Dirac matrices must be changed simultaneously, see [26], according to

$$\gamma^0 \to \gamma^0_{\mathrm{E}} \qquad ; \qquad \gamma^i \to i\gamma^i_{\mathrm{E}} . \tag{A.4}$$

This transforms the anti-commutation relation for the γ matrices to:

$$\{\gamma^\mu, \gamma^\nu\}_{\mathrm{E}} = 2\,\mathbb{1}\,\delta^{\mu\nu} . \tag{A.5}$$

With the Euclidean metric and the new Dirac matrices, the most important inner products are transformed according to

$$\begin{aligned} x^2 &\to \; - x^2_{\mathrm{E}} ; \\ \partial^2 &\to \; - \partial^2_{\mathrm{E}} ; \\ \not{\partial} &\to \; - i\not{\partial}_{\mathrm{E}} . \end{aligned} \tag{A.6}$$

In the action S the time integral changes too and the functional integral becomes

$$\int \mathcal{D}\psi e^{-S[\psi(x)]} . \tag{A.7}$$

Throughout the thesis we leave out the subscript E as we will mostly be working in Euclidean space. In the few cases we do work in Minkowskian space the difference is made clear.

A.2 Fourier transformations

If we have a function $X(x,y)$ depending on position x and y, we can perform a Fourier transformation to the function $\tilde{X}(s)$. The two functions are connected by

$$X(x,y) = \int \frac{\mathrm{d}^4 s}{(2\pi)^4} e^{is\cdot(x-y)} \tilde{X}(s) . \tag{A.8}$$

We will however not write $\tilde{X}(s)$, yet $X(s)$, even though the function $\tilde{X}(s)$ depends on the momentum s and is totally different from the function $X(x,y)$. In practise this is never confusing, dependence on either $(x-y)$ or (s) makes clear which of the two functions is meant.

A.3 The quantization of the N fermion fields

We have extended our systems, both QED and the Four Fermion Theory (FFT), to systems of N fermions with all the same mass and electric charge yet with different flavor. The quantization of one fermion and corresponding anti-fermion field is standard, see [28, 29], and in this section we will extend these rules to N fields.

The field operators are defined by, with *no* summation over the I,

$$\overline{\psi}^I(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{\sqrt{2p_0}} \sum_s \left(b^I_{p,s} \overline{v}_{p,s} e^{-ip\cdot y} + a^{\dagger I}_{p,s} \overline{v}_{p,s} e^{ip\cdot y} \right) \tag{A.9}$$

$$\psi^I(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{\sqrt{2p_0}} \sum_s \left(a^I_{p,s} u_{p,s} e^{-ip\cdot y} + b^{\dagger I}_{p,s} v_{p,s} e^{ip\cdot y} \right) , \tag{A.10}$$

with $a^{\dagger^I}$ the creator operator for a fermion of flavor I and $b^{\dagger^I}$ the creator operator for an anti-fermion of flavor I:

$$a^{\dagger I}_{p,s}|0> = \sqrt{2p_0}\ |p,s,I> \tag{A.11}$$

$$b^{\dagger I}_{p,s}|0> = \sqrt{2p_0}\ |\overline{p,s,I}> \tag{A.12}$$

$$a^I_{p,s}|0> = b^I_{p,s}|0> = 0 . \tag{A.13}$$

The anti-commutation rules for the operators are extended to

$$\left\{ a^I_{p,s}, a^{\dagger I'}_{p',s'} \right\} = \left\{ b^I_{p,s}, b^{\dagger I'}_{p',s'} \right\} = (2\pi)^3\, 2p_0 \delta(\overline{p} - \overline{p'}) \delta_{ss'} \delta^{II'} \tag{A.14}$$

with all other anti-commutation rules equal to zero. As for the field operators we have

$$\left\{\psi_a^I(x), {\psi^\dagger}_b^{I'}(y)\right\} = \delta^{(4)}(x-y)\delta_{ab}\delta^{II'} \; ; \tag{A.15}$$

$$\left\{\psi_a^I(x), \psi_b^{I'}(y)\right\} = \left\{{\psi^\dagger}_a^I(x), {\psi^\dagger}_b^{I'}(y)\right\} = 0 \; . \tag{A.16}$$

The field operator $\psi^{I'}(x)$ can be seen to annihilate a fermion of flavor I to the vacuum:

$$< 0|\; \psi^{I'}(x)\; |p, s, I >= u_{p,s} e^{-ip\cdot x} \delta^{II'} \tag{A.17}$$

and similar we have for the operator $\overline{\psi}^{I'}(x)$ that

$$< p, s, I|\; \overline{\psi}^{I'}(x)\; |0 >= \overline{v}_{p,s} e^{ip\cdot x} \delta^{II'} \; . \tag{A.18}$$

Appendix B

Regularization

In this appendix we analyse the divergent one loop integral of the photon self energy not by means of the cut-off regularization scheme, yet by means of the dimensional regularization scheme. The two have different end results which we will briefly comment.

B.1 Dimensional regularization

In section 4.2.2 we computed the photon self energy $\Pi^{\mu\nu}$ and applied the cut-off regularization scheme. Our expression for the one loop integral $I^{\mu\nu}$ before the regularization in eq. (4.21) equals

$$4\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\int_0^1 \mathrm{d}x\frac{\frac{1}{2}\delta^{\mu\nu}l^2+\delta^{\mu\nu}\left(M^2-x(1-x)k^2\right)+2k^\mu k^\nu(1-x)x}{\left[l^2+M^2+x(1-x)k^2\right]^2}\ . \tag{B.1}$$

We will now apply the dimensional regularization scheme to this expression. In d dimensional Euclidean space we have the following two integral expressions:

$$\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\left(C^2+k^2-i\epsilon\right)^n}=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-d/2\right)}{\Gamma\left(n\right)}\left(\frac{1}{C^2}\right)^{n-d/2}\ ; \tag{B.2}$$

$$\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^\mu k^\nu}{\left(C^2+k^2-i\epsilon\right)^n}=\frac{\delta^{\mu\nu}}{2\left(4\pi\right)^{d/2}}\frac{\Gamma\left(n-1-d/2\right)}{\Gamma\left(n\right)}\left(\frac{1}{C^2}\right)^{n-1-d/2}\ , \tag{B.3}$$

see [28]. $\Gamma\left(\alpha\right)$ is equivalent to the function $(n-1)!$ with

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)\ , \tag{B.4}$$
$$\Gamma(1) = 1\ , \tag{B.5}$$

except α can be a non-integer. An integral representation of Γ is

$$\Gamma(\alpha)=\int_0^\infty \mathrm{d}t\ t^{\alpha-1}e^{-t}\ . \tag{B.6}$$

We first define $C^2 = \left[M^2 + x(1-x)k^2\right]$ and next apply eq. (B.2) and (B.3) to the one loop integral $I^{\mu\nu}$ in eq. (B.1). We obtain

$$\begin{aligned}
I^{\mu\nu}(k) =& \frac{4}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \left(\frac{1}{C^2}\right)^{2-\frac{d}{2}} \left\{ \delta^{\mu\nu} C^2 \frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(1)} + \right. \\
&\left. + \left[\delta^{\mu\nu}\left(C^2 - 2x(1-x)k^2\right) + 2k^\mu k^\nu (1-x)\, x\right] \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \right\} \qquad \text{(B.7)} \\
=& \frac{4}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \left(\frac{1}{C^2}\right)^{2-\frac{d}{2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \left\{ \delta^{\mu\nu} C^2 \frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(1)} \frac{\Gamma(2)}{\Gamma\left(2-\frac{d}{2}\right)} \right. \\
&\left. + \ \delta^{\mu\nu}\left(C^2 - 2x(1-x)k^2\right) + 2k^\mu k^\nu (1-x)\, x \right\} \qquad \text{(B.8)} \\
=& \frac{-4\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \left(\frac{1}{C^2}\right)^{2-\frac{d}{2}} \frac{\Gamma\left(3-\frac{d}{2}\right)}{2-\frac{d}{2}} 2x(1-x) \ . \qquad \text{(B.9)}
\end{aligned}$$

This final expression is gauge invariant making it is possible to extract a scalar part $I(k^2)$:

$$I^{\mu\nu}(k) = \left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right) I(k^2) \ ; \qquad \text{(B.10)}$$

$$I(k^2) = -\frac{4(4\pi)^{\epsilon/2}}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \left(\frac{1}{\left[M^2 + x(1-x)k^2\right]}\right)^{2-\frac{d}{2}} \frac{\Gamma\left(3-\frac{d}{2}\right)}{2-\frac{d}{2}} 2x(1-x) \ . \qquad \text{(B.11)}$$

The dimension cannot be set to 4 yet, there is a pole due to the $(2-d/2)$ term in the denominator. This singularity can however easily be isolated, while for $k^2 = 0$ the integral is divergent, for $k^2 > 0$ the pole is cancelled. This is more clear if we shift our dimension variable and set $d = 4 - \epsilon$. The limit is now set by $\epsilon \to 0$.

We split the integral into two parts:

$$\begin{aligned}
I(k^2) &= I(k^2=0) + \left(I(k^2) - I(k^2=0)\right) \\
&= I(0) + I_{\mathrm{Reg}}(k^2) \ . \qquad \text{(B.12)}
\end{aligned}$$

The isolated part $I(0)$ is still infinite due to the $\epsilon \to 0$ in the denominator and cannot be computed:

$$I(0) = -\frac{4(4\pi)^{\epsilon/2}}{(4\pi)^2} \int_0^1 \mathrm{d}x \left(\frac{1}{M^2}\right)^{\epsilon/2} \frac{\Gamma\left(1+\frac{\epsilon}{2}\right)}{\epsilon/2} 2x(1-x) \ . \qquad \text{(B.13)}$$

The regularized part however can now be computed by applying the following expansion:

$$\lim_{\epsilon\to 0} X^\epsilon = 1 + \epsilon \log X + \mathcal{O}(\epsilon^2) \ . \qquad \text{(B.14)}$$

The end result is, after we finally have set $\epsilon = 0$, the singularity free integral

$$\begin{aligned}
I_{\mathrm{Reg}}(k^2) &= -\frac{4\Gamma(1+\epsilon/2)}{(4\pi)^{2-\epsilon/2}(\epsilon/2)} \int_0^1 \mathrm{d}x \left(\frac{2x(1-x)}{\left[M^2 + k^2 x(1-x)\right]^{\epsilon/2}} - \frac{2x(1-x)}{M^\epsilon}\right) \\
&= -\frac{4}{(4\pi)^2} \int_0^1 \mathrm{d}x\ 2x\ (1-x) \ln\left[1 + \frac{k^2}{M^2}(1-x)x\right]^{-1} \ . \qquad \text{(B.15)}
\end{aligned}$$

With eq. (B.13) the infinity has been isolated and after renormalization will be absorbed into the *renormalization constant* Z_3, see section C.3.

B.2 Comparison to the cut-off regularization

When applying the cut-off regularization scheme the one loop integral resulted in the expression

$$\begin{aligned} I^{\mu\nu}(k) =& \frac{2}{(4\pi)^2}\delta^{\mu\nu}\left(\Lambda^2 - M^2\right) + \frac{1}{(4\pi)^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right) - \frac{1}{3(4\pi)^2}k^\mu k^\nu \\ &+ \frac{-8}{(4\pi)^2}\int_0^1 \mathrm{d}x \left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)(1-x)x\ln\frac{\Lambda^2}{M^2 + x(1-x)k^2} , \end{aligned} \tag{B.16}$$

see eq. (4.23). The divergent integral was regularized by cutting off the integral at a certain high energy Λ. The highest order term in eq. (B.16) is of order $\mathcal{O}(\Lambda^2)$ and the expression is therefore defined as quadratically divergent. The second term dependent of Λ^2 is of order $\mathcal{O}(\ln\Lambda^2)$ and defined as logarithmically divergent.

To determine the order of divergence of the one loop integral after dimensional regularization we must analyse the expression in eq. (B.9). In the limit of $\epsilon \to 0$ we can use the approximation

$$\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}\left(C^2\right)^{2-d/2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) - \ln C^2\right) , \tag{B.17}$$

with the Euler-Mascheroni constant $\gamma \approx 0.5772$ and again the transformation $\epsilon = 4 - d$. In [28] it becomes clear that the $1/\epsilon$ pole in the dimensional regularization scheme corresponds to a logaritmically divergence in the cut-off regularization scheme and the integral $I^{\mu\nu}(k)$ in eq. (B.9) can thus be written as

$$\begin{aligned} I^{\mu\nu} &= \frac{-4\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x \left(\frac{1}{C^2}\right)^{2-\frac{d}{2}} \frac{\Gamma\left(3-\frac{d}{2}\right)}{2-\frac{d}{2}} 2(1-x)\,x \\ &= -\frac{8}{(4\pi)^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)\int \mathrm{d}x\ (1-x)\ x\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) - \ln C^2\right) \end{aligned} \tag{B.18}$$

$$\to -\frac{8}{(4\pi)^2}\left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)\int \mathrm{d}x\ (1-x)\ x\left(\ln\Lambda^2 - \ln C^2\right) , \tag{B.19}$$

where in the last equation we absorbed the constant $-\gamma + \ln(4\pi)$ into the cut-off Λ. With the definition $C^2 = M^2 + x(1-x)k^2$ we can now easily see that the dimensional regularization scheme applied to the one loop integral results in no more than a logarithmically divergent expression, this in contrast to eq. (B.16) which not only has the same logarithmically divergent term, but also a quadratically divergent term.

One advantage of the dimensional regularization scheme is that it preserves the gauge invariance, see Collins [17]. This is not the case when using the cut-off procedure, the quadratically divergent term breaks the invariance. It can of course not be that two regularization schemes result in different values of the physical parameters. Both schemes therefore will demand a different approach of renormalization. In section C.3 we briefly show how the renormalization of the charge parameter after use of the dimensional regularization scheme is performed.

We have explicitly not used this approach. Although very useful when one assumes the invariance as a prerequisite, we have set out to show that the local gauge invariance of QED could perhaps be generated by some underlying mechanism. Applying the cut-off method we therefore obtain renormalization flows in which gauge invariant fixed points are non-trivial and require fine-tuning of the parameters.

The fact that the cut-off regularization scheme does not preserve gauge invariance during calculations is not a reason to assume the scheme to be wrong. Before dimensional regularization was developed, it was standard to use the Pauli-Villars scheme, which introduces an endless summation of virtual heavy photons to deal with the divergences. Although more elegant than the plain cut-off procedure it has the same physical implications, as shown by Jauch and Rohrlich [34] and its method, also by the use of non gauge invariant counterterms, was known to work, see 't Hooft [35].

Appendix C

Renormalization Theory

In this appendix we discuss the Renormalization Theory. First we give a brief summary on the Renormalization Group together with Wilson's approach and we analyse the possible consequences of a Renormalization flow. Then we show how after the dimensional regularization scheme we can interpret divergent terms by renormalizing a parameter and how physical parameters can be defined by their experimental value. Finally we state the difference between renormalizable and non-renormalizable theories.

C.1 The renormalization group

The renormalization group is a method for simplifying systems without loosing information or changing its properties and was first developed in the field of Statistical Physics. In any physical system there are numerous degrees of freedom and these are impossible to be handled simultaneously. One can reduce the number of degrees of freedom by looking at a smaller scale without loosing information: the properties of a gas of 10^{23} atoms or a gas of 10^4 atoms in a proportional smaller system are not different. But one cannot keep on going to smaller scales, at a certain point the properties do change. The minimum size with which one can describe the whole system is the *correlation length* ξ.

An ideal correlation length is only one or two atomic spacings. Systems with such a ξ have not too many degrees of freedom and can easily be analyzed with several well-known techniques, like perturbation expansion, virial expansion, etc. Some systems have much larger ξ and are not so straight-forward, or worse. Take a quantum field $\phi(x)$: at each point x the field ϕ is one degree of freedom, so no matter how small ξ, the number of degrees of freedom is infinite. In quantum field the correlation length is taken to be the biggest Compton wavelength. In QED, which has the infinite Compton wavelength of the massless photon, it is the electron's Compton wavelength.

It is impossible to handle all degrees of freedom in such cases. This is where the renormalization group can be of use. It turns out that in some cases not the specific degrees of freedom are of importance, but the fact that they are behaving as a whole. It is this cooperative behavior that characterizes a system. In contrast to the previous described simplification by looking at smaller scales, the renormalization group goes the other way: it starts at a certain length scale with a few parameters and then doubles the scale without doubling the parameters. A new set of parameters, equal in number to the old set, describes in a new effective theory the same system. Thus the density of parameters is lowered. These steps are repeated until the density of parameters is of order one per square ξ.

For the new set of parameters to describe the same system the range of interactions must be smaller than ξ, else it is impossible to build effective theories for these scales. But it is not the interaction that determines the behavior of the system: the characteristics of the system lie in *the way* the renormalization goes. The transformation to a bigger scale is performed repeatedly and it is this transformation which determines the nature of the system. This led, among other things, to the insight that systems in quantum field theory and statistical physics, at first sight two different types, thus turned out to have much in common, see K. Wilson [15] or R. Shankar [16].

C.2 Renormalization flow

As was said, the properties of the theory must not change under the renormalization. Assuming the properties are determined from a quantity H, which in many of our cases of course is the Lagrangian or action, this H has a number of parameters λ_i, $i = 1, 2, 3, \ldots$. Properties and characteristics are found when doing experiments and these are performed at a certain *scale* μ. In our case the energy scale, but it could also be temperature or length.

The renormalization group is about changing the scale μ and the parameters λ_i's in such a way that the quantity H stays the same. Changing $\mu \to \mu + \delta\mu$ the parameters must be changed, $\lambda_i \to \lambda_i + \delta\lambda_i$, such that the properties of $H[g_i, \mu]$ and $H[\lambda_i + \delta\lambda_i, \mu + \delta\mu]$ are equivalent. If this is done several times, we obtain a *flow of parameters*:

$$\mu \frac{d}{d\mu} \lambda_i \equiv \beta_i \ . \tag{C.1}$$

In Quantum Field Theory the scale μ is an energy scale and we define the flow as a flow of *decreasing* energy.

Starting at initial values $\lambda_i^{(0)}$'s, it can be that the β_i turn to zero. The parameters λ_i no longer change and we have a *fixed point* λ_i*. The behavior of the flow around the fixed point is of great importance. Expanding the β_i function

$$\beta_i = \alpha(\lambda_i - \lambda_i*) + \ldots , \tag{C.2}$$

there are three options for α:

$\alpha > 0$ If $\lambda_i < \lambda_i*$ the β_i function is < 0 and λ_i grows towards the fixed point when μ is decreased. If $\lambda_i > \lambda_i*$ the β_i function is > 0 and λ_i diminishes when μ is decreased, until it reaches the fixed point. Thus the flow is always towards the fixed point and λ_i is called an *irrelevant parameter.* When decreasing μ it flows toward the fixed point; in our case the scale is that of energy / momentum. Thus we say we have an *infra-red fixed point*, or IR fixed point.

$\alpha = 0$ The parameter λ_i is called *marginal* and its behavior is determined by the higher order terms of eq. (C.2). It can turn out to be *marginal irrelevant*, *strictly marginal* or *marginal relevant.*

$\alpha < 0$ The description is like that of the $\alpha > 0$ case, except the flow is now away from the fixed point and the parameter blows up. Therefore it is called *relevant*. The parameter blows up for μ decreasing. Turned around, for μ increasing the flow is towards the fixed point. This therefore lies in the region of high energy and we have an *ultra-violet fixed point*, or UV fixed point.

Relevant or irrelevant?

Beware that the renormalization scale μ in this case was the energy and the flow was one of decreasing energy. If the scale had been of length and the flow of μ again decreasing, the relevance of a parameter changes: a parameter that would grow and blow up at lower energies, now decreases. For the scale is length and a smaller length means a higher energy.

RG flows are used in QED to lower the energy scale. In the literature μ is often called Λ, but in quantum field theory divergences are dealt with by introducing a cut-off, also called Λ. This cut-off is continuously increased, rescaled and finally sent to infinity. This cut-off being raised is *not* the energy scale being lowered.

Of course a fixed point remains a fixed point, whether the scale is one of energy or of length. If a UV fixed point gives a parameter equal to zero, it is also called *asymptotically free*. Asymptotic in the sense that length goes to zero, and free for the coupling goes to zero.

C.3 Charge renormalization

Applying the Euler-Lagrange equation on a Lagrangian we can derive the equations of motion. The parameters in these equations of motion should represent physical parameters. If we start with the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 + \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu \ . \tag{C.3}$$

What is m? It is the electron mass parameter, but when calculating propagators and cross-sections we end up with divergent integrals, making it impossible for the parameter to be equal to any physical value. As was said, the Lagrangian is the formula from which physical equations of motion are derived. If these equations do not coincide with experimental results, the infinite m against a finite real mass, then the Lagrangian is not correct.

The parameter m is not the only one not coinciding with physical results, e has the same problem. We therefore rename all variables to emphasize that each is merely mathematical and not yet representing our physical value:

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}\right)_0^2 + \overline{\psi}_0\left(i\gamma^\mu\partial_\mu - m_0\right)\psi_0 - e_0\overline{\psi}_0\gamma^\mu\psi_0 A_{\mu,0} \ . \tag{C.4}$$

How the infinities are dealt with is now shown by analysing the expansion of the full photon propagator. Using eq. (3.16) and setting the gauge fixing parameter ξ_0 to 1, the photon propagator to zero loop order is

$$D_{\mu\nu} = \frac{-1}{k^2 + i\epsilon}\delta_{\mu\nu} \ . \tag{C.5}$$

We added an infinitesimal ϵ. With the photon self-energy

$$\Pi^{\mu\nu}(k) = \left(\delta^{\mu\nu}k^2 - k^\mu k^\nu\right)\Pi(k^2), \tag{C.6}$$

the infinite sum for the full propagator is now:

$$\begin{aligned}
D'_{\mu\nu} &= D_{\mu\nu} + D_{\mu\rho}(\Pi^{\rho\sigma}(k))D_{\sigma\nu} + D_{\mu\rho}(\Pi^{\rho\sigma}(k))D_{\sigma\eta}(\Pi^{\eta\lambda}(k))D_{\lambda\nu} + \dots \\
&= \frac{-\delta_{\mu\nu}}{k^2+i\epsilon} + \frac{-\delta_{\mu\nu}}{k^2+i\epsilon}\left(\frac{-k^2\Pi(k^2)}{k^2}\right) + \\
&\quad + \frac{-\delta_{\mu\nu}}{k^2+i\epsilon}\left(\frac{-k^2\Pi(k^2)}{k^2}\right)^2 + \dots \qquad \text{(C.7)} \\
&= \frac{-\delta_{\mu\nu}}{k^2+i\epsilon}\left(\frac{1}{1+\Pi(k^2)}\right), \qquad \text{(C.8)}
\end{aligned}$$

where we used in eq. (C.7) the fact that the $k^\mu k^\nu$-term always yield zero due to gauge invariance when contracted to the initial or final currents.

In section B.1 the one loop integral $I(k^2)$ was split in a finite part $I_{\text{Reg}}(k^2)$ and infinite part $I(0)$. We can apply the same isolation to the self energy and thus have

$$\frac{1}{1+\Pi(k^2)} = \frac{1}{1+\Pi(0)+\Pi_{\text{Reg}}(k^2)} = \frac{1}{1+\Pi(0)}\frac{1}{1+\frac{\Pi_{\text{Reg}}(k^2)}{1+\Pi(0)}} \equiv \frac{Z_3}{1+Z_3\Pi_{\text{Reg}}(k^2)}; \tag{C.9}$$

the full propagator is now

$$D'_{\mu\nu} = -\frac{\delta_{\mu\nu}}{k^2+i\epsilon}\frac{Z_3}{1+Z_3\Pi_{\text{Reg}}(k^2)}. \tag{C.10}$$

The term $Z_3 = (1+\Pi(0))^{-1}$ has absorbed the infinity. We now return to the mathematical parameter e_0. When defining our system we absorbed e_0 into the gauge field A_μ, see eq. (3.3). In standard QED with the Lagrangian (3.2) this is not the case and the self energy is proportional to e_0^2. At the end of each photon line we have a triple vertex of order e_0 and thus the propagator $D'_{\mu\nu}$ is multiplied with a factor e_0^2. Our propagator changes to

$$D'_{\mu\nu} = -\frac{\delta_{\mu\nu}}{k^2+i\epsilon}\,\frac{Z_3\,e_0^2}{1+Z_3\,e_0^2\,\Pi_{\text{Reg}}(k^2)}. \tag{C.11}$$

Shifting variable $e_0 \to \sqrt{Z_3}\,e_0$, this term turns out to be what physically is measured as the electron charge in experiments. Thus the measured charge

$$e = \sqrt{Z_3}\,e_0 \tag{C.12}$$

and the bare parameter $e_0 = e\,Z_3^{(-1/2)}$ has absorbed the infinity.

Incomplete renormalization

We have partly shown the renormalization of the electron charge parameter. For the exact and complete renormalization of this parameter we have to take into account corrections to the triple interaction term, which we will however not analyse. We merely wanted to show how the singularities isolated by the dimensional regularization scheme are dealt with and how thus the physical parameters become finite expressions.

C.4 Renormalizable and non-renormalizable

When dealing with divergent integrals we have introduced the cut-off Λ, a large but finite momentum. When calculating the values of physical parameters we have in the end two possibilities:

- All physical parameters are independent of the cut-off. We can set Λ to infinity and the theory is called *renormalizable*.
- Their is a dependence on the cut-off. We cannot set $\Lambda = \infty$ and the theory is called *non-renormalizable*.

Non-renormalizable theories therefore have an explicit dependence on the cut-off and cannot be dealt with by means of the dimensional regularization. An option is the cut-off regularization procedure.

A rule of thumb for determining the renormalizability of a theory is the so-called power counting argument: if the interaction coefficient is of dimension $[mass]^0$ or higher the interaction is renormalizable, if it is $[mass]^{-1}$ or lower it is not. A theory with multiple interactions is renormalizable when all its interactions are.

Another way in which the difference becomes clear is when calculating the counterterms. When computing expressions to all possible higher orders in an expansion a renormalizable theory needs a finite number of counterterms to cancel the divergent expressions. A non-renormalizable theory however needs an infinite number of counterterms.

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